NASA TECHNICAL
TRANSLATION

NASA TT F-118

NASA TT F-118

THEORY OF ANISOTROPIC SHELLS

by S. A. Ambartsumyan

State Publishing House for Physical
and Mathematical Literature
Moscow, 1961

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1964

THEORY OF ANISOTROPIC SHELLS

By S. A. Ambartsumyan

Translation of "Teoriya anizotropnykh obolochek"

State Publishing House for Physical and Mathematical Literature
Moscow, 1961

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $6.00

CONTENTS

Foreword v

CHAPTER I

FUNDAMENTAL EQUATIONS OF THE THEORY OF ELASTICITY OF AN ANISOTROPIC BODY IN CURVILINEAR COORDINATES

Section 1. Some Remarks on Curvilinear Coordinates in Space. 1
Section 2. Deformation Components and Differential Equations of Equilibrium in the Triorthogonal System of Curvilinear Coordinates 4
Section 3. Curvilinear Anisotropy. Generalized Hooke's Law 7
Section 4. Transformation of Elastic Constants with Rotation of the Coordinate System 12
Section 5. Elastic Constants for Certain Anisotropic Materials 14

CHAPTER II

FUNDAMENTAL EQUATIONS OF THE THEORY OF SHELLS CONSISTING OF AN ARBITRARY NUMBER OF ANISOTROPIC LAYERS

Section 1. Basic Concepts, Initial Relationships and Hypotheses. 18
Section 2. Displacements and Deformations 22
Section 3. Equations of Continuity of Deformations of the Coordinate Surface 25
Section 4. Stresses in Layers 26
Section 5. Conditions of Contact of Adjacent Layers 28
Section 6. Internal Forces and Moments 29
Section 7. Equilibrium Equations 31
Section 8. Potential Energy of Deformation 33
Section 9. Elasticity Relationships 35

Section 10. Boundary Conditions 38
Section 11. Additional Remarks Concerning the Conditions of Contact of Adjacent Layers and the Conditions at the Outer Surfaces of a Shell 40
Section 12. Special Cases of Anisotropy of the Material of the Shell Layers 43
Section 13. Shells Consisting of an Odd Number of Layers Symmetrically Arranged Relative to the Coordinate Surface . . . 46
Section 14. Single-Layer Anisotropic Shells 50
Section 15. Further Remarks Concerning Elasticity Relationships 55
Section 16. Calculation of Stiffnesses for Arbitrary Directions 58

CHAPTER III

MEMBRANE THEORY OF ANISOTROPIC SHELLS

Section 1. General Premises and Initial Relationships in the Membrane Theory of Single-Layer Isotropic Shells 61
Section 2. Boundary Conditions. 64
Section 3. Area of Applicability of the Membrane Theory 65
Section 4. Fundamental Equations of the Membrane Theory of Symmetrically loaded Shells of Revolution 66
Section 5. Examples of Calculation of Symmetrically Loaded Shells of Revolution 73
Section 6. Evaluation of Results Obtained in the Preceding Section . . . 86
Section 7. Continuation of Section 5 87
Section 8. An Arbitrarily Loaded Cylindrical Shell of Arbitrary Shape 95
Section 9. Some Remarks Concerning the Membrane Theory of Anisotropic Laminar Shells 104

CHAPTER IV

SYMMETRICALLY LOADED ANISOTROPIC SHELLS OF REVOLUTION

Section 1. Basic Premises. Initial Relationships and Equations. 109
Section 2. Equations of Solution and Design Formulas 113
Section 3. Shells of Revolution Consisting of an Odd Number of Layers Symmetrically Arranged Relative to the Median Surface of the Shell 117
Section 4. Single-Layer Shells of Revolution 120
Section 5. Reduction of the System of Equations in (3.16) and (3.17) to a Single Equation. A Particular Solution of the Inhomogeneous Equation 121
Section 6. Asymptotic Integration of the Equation of Solution (5.9) . . . 124

Section 7. Internal Forces, Moments, Stresses and Displacements 132
Section 8. Edge Effect in Anisotropic Shells 136
Section 9. Long Shells of Revolution 139
Section 10. Examples of Calculation of Long Shells of Revolution 143
Section 11. Solution of a Few Problems of Shells of Revolution of Zero Gaussian Curvature Consisting of an Arbitrary Number of Layers 166
Section 12. Anisotropic Cylindrical Shells of Revolution Reinforced by Lateral Ribs 177

CHAPTER V

ANISOTROPIC CYLINDRICAL SHELLS

Section 1. Basic Premises. Initial Relationships and Equations. . . . 195
Section 2. System of Differential Equations of Solution in Displacements 198
Section 3. Cylindrical Shells Consisting of an Arbitrary Number of Orthotropic Layers 201
Section 4. Engineering Theory of Cylindrical Shells Consisting of an Arbitrary Number of Anisotropic Layers 208
Section 5. Continuation of Section 4. 214
Section 6. Cylindrical Shells Consisting of an Odd Number of Homogeneous Anisotropic Layers Symmetrically Arranged Relative to the Median Surface 224
Section 7. Engineering Theory of Cylindrical Shells Consisting of an Odd Number of Homogeneous Anisotropic Layers Symmetrically Arranged Relative to the Median Surface . . 228
Section 8. Continuation of Section 7 232
Section 9. Integration of the Equations of Engineering Theory of a Cylindrical Shell by the Method of Double Trigometric Series 240
Section 10. Integration of Equations in the Engineering Theory of Cylindrical Shells by the Method of Single Trigonometric Series 265

CHAPTER VI

SHALLOW ANISOTROPIC SHELLS

Section 1. Basic Premises. Initial Relationships and Equations. . . . 277
Section 2. Equations of Solution and Design Formulas 280
Section 3. Shells Consisting of an Odd Number of Homogeneous Anisotropic Layers Symmetrically Arranged Relative to the Median Surface 285

Section 4. Extremely Shallow Shells. Basic Premises, Initial Relationships and Equations 288
Section 5. Equations of Solution and Design Formulas in the Theory of Extremely Shallow Shells Consisting of an Arbitrary Number of Anisotropic Layers 292
Section 6. Equations of Solution and Design Formulas in the Theory of Extremely Shallow Shells Consisting of an Odd Number of Homogeneous Anisotropic Layers Symmetrically Arranged Relative to the Median Surface 300
Section 7. Integration of the Equations of Solution in the Theory of Extremely Shallow Orthotropic Shells 306

CHAPTER VII

NEW THEORIES OF ANISOTROPIC SHELLS AND PLATES

Section 1. Basic Premises and Hypotheses 315
Section 2. Theory of Orthotropic Plates 316
Section 3. Theory of Bending of a Plate Possessing Cylindrical Anisotropy 328
Section 4. Approximate Theory of an Anisotropic Plate Considering Transverse Shear 332
Section 5. Another Approximate Theory of Anisotropic Plates 336
Section 6. Examples of Plate Calculations 342
Section 7. Theory of Extremely Shallow Orthotropic Shells 358
Section 8. Approximate Theory of an Extremely Shallow Shell Considering Transverse Shear 365
Section 9. Another Approximate Theory for an Extremely Shallow Shell 368
Section 10. Example of Calculation of a Shell 372
Section 11. Theory of Extremely Shallow Laminar Orthotropic Shells . . 376
Section 12. Examples of Calculation of Laminar Shells and Plates 386

FOREWORD

Shells are widely used as structural elements in modern construction engineering, aircraft construction, ship building, rocket construction, etc.

A careful study of the shells used in engineering leads to the conclusion that they are most often anisotropic (naturally or structurally) and in many cases are anisotropic and laminar.

The last few years have seen the appearance of a number of thorough monographs: V.Z. Vlasov, Obshchaya teoriya obolochek (General Theory of Shells); A.S. Vol'mir, Gibkiye plastinki i obolochki (Elastic plates and shells); A.L. Gol'denveyzer, Teoriya uprugikh tonkikh obolochek (Theory of thin elastic shells); A.I. Lur'ye, Statika tonkostennykh uprugikh obolochek (Statics of thin-walled elastic shells); Kh.M. Mushtar' and K.Z. Galimov, Nelineynaya teoriya uprugikh obolochek (Nonlinear theory of elastic shells); V.V. Novozhilov, Teoriya tonkikh obolochek (Theory of thin shells). These monographs shed light on many fundamental problems in the theory of shells in general. However, there is almost nothing therein dealing with the theory of anisotropic laminar shells.

Despite the large number of articles appearing in journals, there is as yet not one book devoted to the theory of anisotropic laminar shells. In the present book the author partially fills this gap.

The text is based on the author's investigations over the last few years. It consists of the following divisions: (a) fundamental equations of the theory of elasticity of an anisotropic body in curvilinear coordinates; (b) general theory of anisotropic laminar shells; (c) membrane theory of anisotropic shells; (d) theory of symmetrically loaded anisotropic shells of revolution; (e) anisotropic cylindrical shells; (f) shallow anisotropic shells; (g) new theories of anisotropic shells and plates.

In distinction from the first divisions of the book, which are based on the hypothesis of nondeformable normals as given for the stack of the shell as a whole, the last chapter attempts to construct a theory of essentially anisotropic shells and plates without the hypothesis of nondeformable normals.

The book does not deal with the undeniably important problems of nonlinear theory, the theories of stability and vibration, as well as temperature problems of anisotropic laminar shells. Nor does it deal with problems associated with plastic and elastic-plastic deformations of the material of the shell layers, since these problems have not been adequately investigated.

Within each chapter the formulas have a two-part enumeration. Where reference is made to the formulas of preceding chapters a three-part enumeration is used (the first digit referring to the chapter).

In conclusion I wish to express my deep gratitude to my colleagues at the Institute of Mathematics and Mechanics AN of the Armenian SSR, D.V. Peshtmaldzhyan, A.A. Khachatryan and L.A. Movsesyan, who were of great assistance in the preparation of this book.

S.A. Ambartsumyan

May 1959, Yerevan

CHAPTER I

FUNDAMENTAL EQUATIONS OF THE THEORY OF ELASTICITY OF AN ANISOTROPIC BODY IN CURVILINEAR COORDINATES

SECTION 1. SOME REMARKS ON CURVILINEAR COORDINATES IN SPACE

Here we shall discuss in brief and without proof those aspects of the theory of curvilinear coordinates in space which will be used in subsequent discussion.

As is known, the position of any point M in space may be uniquely defined by its radius vector r relative to a certain fixed point O. In rectangular Cartesian coordinates for r we have

$$r = xi + yj + zk, \tag{1.1}$$

where, as usual, i, j, k represent the corresponding unit vectors.

In problems of the theory of shells the location of any point M with radius-vector r is conveniently defined not by three cartesian coordinates x, y and z but by any three other numbers α, β, γ. The quantities α, β, γ, which uniquely define the location of point M in space are known as the curvilinear coordinates of point M. Each of these coordinates is a function of radius vector r or a function of the components x, y and z of this radius-vector in Cartesian coordinates:

$$\left.\begin{aligned} \alpha(r) &= \alpha(x, y, z), \\ \beta(r) &= \beta(x, y, z), \\ \gamma(r) &= \gamma(x, y, z). \end{aligned}\right\} \tag{1.2}$$

Conversely, since the radius vector r of any point M in space is completely defined when we are given α, β and γ, it is a function of these independent variables, and, consequently, the components of this radius vector will be functions of curvilinear coordinates

$$x = x(\alpha, \beta, \gamma), \quad y = y(\alpha, \beta, \gamma), \quad z = z(\alpha, \beta, \gamma). \tag{1.3}$$

In (1.2), assuming $\alpha(r) = \text{const}$, $\beta(r) = \text{const}$, and $\gamma(r) = \text{const}$, we obtain three families of surfaces. Through each point M in space there passes a surface of each of these families. These surfaces are known as the coordinate surfaces. The lines of intersection of the coordinate surfaces are referred to as coordinate lines.

It is evident that in our case there will be three such coordinate lines: coordinate line α (with current coordinate α), which is formed by the intersection of coordinate surfaces $\beta = \text{const}$ and $\gamma = \text{const}$; coordinate line β (with current coordinate β), which is formed by the intersection of coordinate surfaces $\alpha = \text{const}$ and $\gamma = \text{const}$; coordinate line γ (with current coordinate γ), which is formed by the intersection of coordinate surfaces $\beta = \text{const}$ and $\alpha = \text{const}$ (Figure 1).

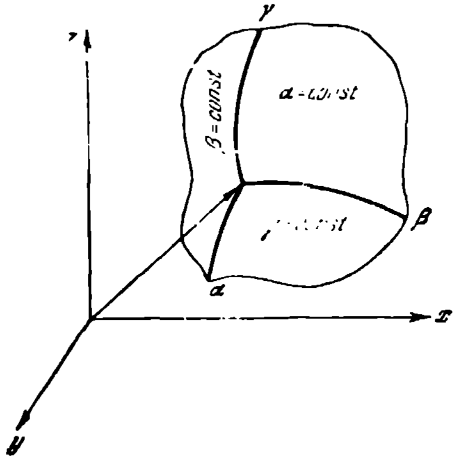

Figure 1

In subsequent discussion we shall be interested only in orthogonal curvilinear coordinates, that is, those curvilinear coordinates of which all coordinate lines α, β and γ are mutually perpendicular at each point M in space. In the given triorthogonal system of curvilinear coordinates we have for the square of a linear element in space

$$ds^2 = H_1^2 d\alpha^2 + H_2^2 d\beta^2 + H_3^2 d\gamma^2, \tag{1.4}$$

where

$$H_1 = H_1(\alpha, \beta, \gamma), \quad H_2 = H_2(\alpha, \beta, \gamma), \quad H_3 = H_3(\alpha, \beta, \gamma),$$

in the general case of orthogonal curvilinear coordinates, are functions of independent variables α, β and γ and are known as the Lame coefficients.

With the given relations of (1.3) for the given coordinate system the Lame coefficients are determined by means of the following formulas:

$$\left.\begin{aligned} H_1^2&=\left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2+\left(\frac{\partial z}{\partial\alpha}\right)^2;\\ H_2^2&=\left(\frac{\partial x}{\partial\beta}\right)^2+\left(\frac{\partial y}{\partial\beta}\right)^2+\left(\frac{\partial z}{\partial\beta}\right)^2;\\ H_3^2&=\left(\frac{\partial x}{\partial\gamma}\right)^2+\left(\frac{\partial y}{\partial\gamma}\right)^2+\left(\frac{\partial z}{\partial\gamma}\right)^2.\end{aligned}\right\}\qquad(1.5)$$

For example, in cylindrical coordinates with $\alpha=R$, $\beta=\varphi$, $\gamma=z$, when the relations in (1.3) have the form

$$x=R\cos\varphi,\quad y=R\sin\varphi,\quad z=z,$$

from (1.5) we obtain for the Lame coefficients

$$H_1=1,\quad H_2=r,\quad H_3=1.$$

In spherical coordinates for $\alpha=R$, $\beta=\theta$, $\gamma=\varphi$, when the relations in (1.3) have the form

$$x=R\sin\theta\cos\varphi,\quad y=R\sin\theta\sin\varphi,\quad z=R\cos\theta,$$

from (1.5) we have for the Lame coefficients

$$H_1=1,\quad H_2=R,\quad H_3=R\sin\theta.$$

Finally, in Cartesian rectangular coordinates x, y, z, as is known, the formula for the square of a linear element has the form

$$ds^2=dx^2+dy^2+dz^2,$$

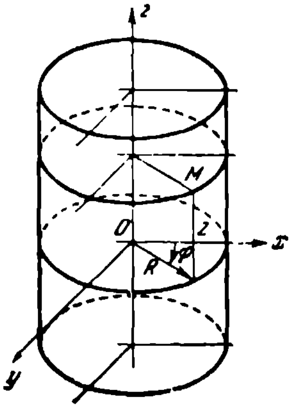

Figure 2

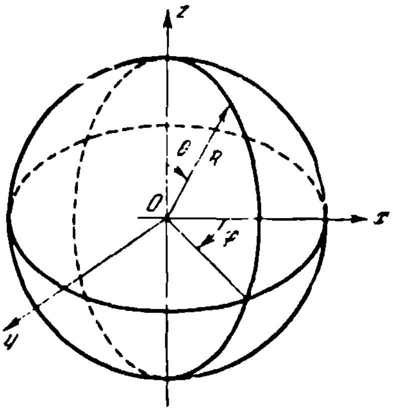

Figure 3

in virtue of which we have for the Lame coefficients

$$H_1 = H_2 = H_3 = 1.$$

Coefficients H_i, as functions of the three independent variables α, β and γ, in any system of curvilinear orthogonal coordinates must satisfy the following six differential equations:

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial\beta}\right)+\frac{1}{H_3^2}\frac{\partial H_1}{\partial\gamma}\frac{\partial H_2}{\partial\gamma}=0;\\
&\frac{\partial}{\partial\beta}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial\gamma}\right)+\frac{1}{H_1^2}\frac{\partial H_2}{\partial\alpha}\frac{\partial H_3}{\partial\alpha}=0;\\
&\frac{\partial}{\partial\gamma}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial\gamma}\right)+\frac{\partial}{\partial\alpha}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial\alpha}\right)+\frac{1}{H_2^2}\frac{\partial H_3}{\partial\beta}\frac{\partial H_1}{\partial\beta}=0;
\end{aligned}\right\} \qquad (1.6)$$

$$\left.\begin{aligned}
&\frac{\partial^2 H_1}{\partial\beta\,\partial\gamma}-\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}\frac{\partial H_1}{\partial\beta}-\frac{1}{H_3}\frac{\partial H_3}{\partial\beta}\frac{\partial H_1}{\partial\gamma}=0;\\
&\frac{\partial^2 H_2}{\partial\alpha\,\partial\gamma}-\frac{1}{H_3}\frac{\partial H_3}{\partial\alpha}\frac{\partial H_2}{\partial\gamma}-\frac{1}{H_1}\frac{\partial H_1}{\partial\gamma}\frac{\partial H_2}{\partial\alpha}=0;\\
&\frac{\partial^2 H_3}{\partial\alpha\,\partial\beta}-\frac{1}{H_1}\frac{\partial H_1}{\partial\beta}\frac{\partial H_3}{\partial\alpha}-\frac{1}{H_2}\frac{\partial H_2}{\partial\alpha}\frac{\partial H_3}{\partial\beta}=0.
\end{aligned}\right\} \qquad (1.7)$$

Proof of the validity of these equations may be found in texts on differential geometry.*

SECTION 2. DEFORMATION COMPONENTS AND DIFFERENTIAL EQUATIONS OF EQUILIBRIUM IN THE TRIORTHOGONAL SYSTEM OF CURVILINEAR COORDINATES

Let a solid body expressed in the triorthogonal system of curvilinear coordinates α, β, and γ undergo deformation under the influence of any forces. Then any point M belonging to the body and having the coordinates α, β and γ will acquire a displacement which may be represented by the following three projections of the total displacement vector onto the directions of the tangents to coordinate lines α, β and γ:

$$\left.\begin{aligned}
u_\alpha &= u_\alpha(\alpha,\ \beta,\ \gamma);\\
u_\beta &= u_\beta(\alpha,\ \beta,\ \gamma);\\
u_\gamma &= u_\gamma(\alpha,\ \beta,\ \gamma).
\end{aligned}\right\} \qquad (2.1)$$

---

*S. P. Finikov, Teoriya poverkhnostey (Theory of surfaces), ONTI, Gostekhizdat, 1934, pp. 185-186.

Henceforth, we shall refer to all these quantities as the displacements of point M. For the positive displacements let us use u_α, u_β, and u_γ, directed toward positive changes of the corresponding independent variables α, β and γ.

The deformed state of a solid three-dimensional body in the vicinity of point M is characterized by six deformation components. Of these components three, which are designated by e_α, e_β, e_γ, are the relative deformations of elongation in three mutually perpendicular directions α, β and γ, respectively, while the remaining three, designated by $e_{\alpha\beta}$, $e_{\beta\gamma}$, $e_{\gamma\alpha}$, are the corresponding deformations of shear occurring in three mutually perpendicular planes, which planes are tangent at point M to three mutually perpendicular coordinate surfaces $\gamma = \text{const}$, $\alpha = \text{const}$, $\beta = \text{const}$.

The deformation components e_α, e_β, ..., $e_{\gamma\alpha}$ are associated with displacements u_α, u_β, and u_γ of point M by means of the following formulas:

$$\left.\begin{aligned} e_\alpha &= \frac{1}{H_1}\frac{\partial u_\alpha}{\partial\alpha} + \frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta}u_\beta + \frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma}u_\gamma;\\ e_\beta &= \frac{1}{H_2}\frac{\partial u_\beta}{\partial\beta} + \frac{1}{H_2H_3}\frac{\partial H_2}{\partial\gamma}u_\gamma + \frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha}u_\alpha;\\ & \frac{1}{H_3}\frac{\partial u_\gamma}{\partial\gamma} + \frac{1}{H_3H_1}\frac{\partial H_3}{\partial\alpha}u_\alpha + \frac{1}{H_3H_2}\frac{\partial H_3}{\partial\beta}u_\beta; \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} e_{\alpha\beta} &= \frac{H_1}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_1}u_\alpha\right) + \frac{H_2}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_2}u_\beta\right);\\ e_{\beta\gamma} &= \frac{H_2}{H_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{H_2}u_\beta\right) + \frac{H_3}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_3}u_\gamma\right);\\ e_{\gamma\alpha} &= \frac{H_3}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_3}u_\gamma\right) + \frac{H_1}{H_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{H_1}u_\alpha\right). \end{aligned}\right\} \tag{2.3}$$

As is known, the stress at any point M of a solid three-dimensional body is characterized by a stress tensor which is defined by nine components. Of these components three are normal stresses which act in three mutually perpendicular directions of coordinate lines α, β and γ and six are shearing stresses acting in three mutually perpendicular planes which are tangent at point M to three mutually perpendicular coordinate surfaces $\alpha = \text{const}$, $\beta = \text{const}$, and $\gamma = \text{const}$. In virtue of the duadicity of the shearing stresses the number of independent stresses is not nine, but only six.

σ_α, σ_β, σ_γ are normal stresses with the subscripts indicating the direction of the external normal to that area to which the given stresses . ier.

$\tau_{\alpha\beta} = \tau_{\beta\alpha}$, $\tau_{\alpha\gamma} = \tau_{\gamma\alpha}$, $\tau_{\beta\gamma} = \tau_{\gamma\beta}$ are shearing stresses; the first subscript indicates the direction in which the given shearing stress acts and the second subscript indicates the direction of the external normal to the area to which the given stress is applied.

All stresses, both normal and shearing, are considered positive if, upon being applied to the areas with positive external normals, they act along the direction of the corresponding positive external normals (Figure 4).

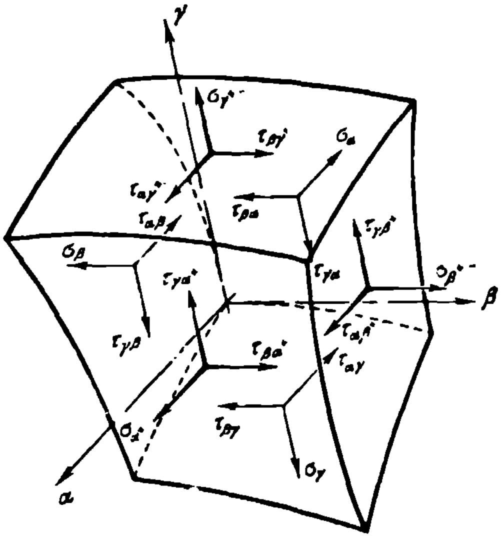

Figure 4.

If the solid three-dimensional body under consideration is in equilibrium, the equilibrium conditions of a differential element of the body in an arbitrarily chosen triorthogonal system of curvilinear coordinates are represented by the following three differential equations:

$$\begin{aligned} &\frac{\partial}{\partial\alpha}(H_2H_3\sigma_\alpha) + \frac{\partial}{\partial\beta}(H_1H_3\tau_{\alpha\beta}) + \frac{\partial}{\partial\gamma}(H_1H_2\tau_{\alpha\gamma}) - \\ &\quad - \sigma_\beta H_3 \frac{\partial H_2}{\partial\alpha} - \sigma_\gamma H_2 \frac{\partial H_3}{\partial\alpha} + \tau_{\beta\alpha} H_3 \frac{\partial H_1}{\partial\beta} + \\ &\qquad + \tau_{\gamma\alpha} H_2 \frac{\partial H_1}{\partial\gamma} + P_\alpha H_1H_2H_3 = 0; \end{aligned} \qquad (2.4')$$

$$\begin{aligned} &\frac{\partial}{\partial\beta}(H_3H_1\sigma_\beta) + \frac{\partial}{\partial\gamma}(H_2H_1\tau_{\beta\gamma}) + \frac{\partial}{\partial\alpha}(H_2H_3\tau_{\beta\alpha}) - \\ &\quad - \sigma_\gamma H_1 \frac{\partial H_3}{\partial\beta} - \sigma_\alpha H_3 \frac{\partial H_1}{\partial\beta} + \tau_{\gamma\beta} H_1 \frac{\partial H_2}{\partial\gamma} + \\ &\qquad + \tau_{\alpha\beta} H_3 \frac{\partial H_2}{\partial\alpha} + P_\beta H_1H_2H_3 = 0; \end{aligned} \qquad (2.4'')$$

$$\frac{\partial}{\partial\gamma}(H_1H_2\sigma_\gamma)+\frac{\partial}{\partial\alpha}(H_3H_2\tau_{\gamma\alpha})+\frac{\partial}{\partial\beta}(H_3H_1\tau_{\gamma\beta})-$$
$$-\sigma_\alpha H_2\frac{\partial H_1}{\partial\gamma}-\sigma_\beta H_1\frac{\partial H_2}{\partial\gamma}+\tau_{\alpha\gamma}H_2\frac{\partial H_3}{\partial\alpha}+$$
$$+\tau_{\beta\gamma}H_1\frac{\partial H_3}{\partial\beta}+P_\gamma H_1H_2H_3=0, \qquad (2.4''')$$

where $P_\alpha=P_\alpha(\alpha, \beta, \gamma)$, $P_\beta=P_\beta(\alpha, \beta, \gamma)$, and $P_\gamma=P_\gamma(\alpha, \beta, \gamma)$ are the corresponding projections of the volumetric force onto the directions of the tangents to the coordinate lines α, β and γ.

The reader will find a detailed discussion of the results presented in the first two sections of this chapter in the work by V.Z. Vlasov.* These subjects are also discussed in many courses in the theory of elasticity (for example, L.S. Leybenzon** or A.I. Lur'ye).***

SECTION 3. CURVILINEAR ANISOTROPY. GENERALIZED HOOKE'S LAW.****,*****

Let us assume that the discussed uniform elastic body is curvilinearly anisotropic (that is, from the standpoint of elastic properties the equivalent directions are not the parallel directions through various points of the body but directions which conform to other rules). In choosing a system of curvilinear coordinates so that at each point the elastically equivalent directions coincide with coordinate directions, we note that infinitely small elements defined at different points of the body by three pairs of coordinate surfaces, being antisotropic, possess identical elastic properties.

We shall assume that the discussed solid elastic body undergoes small deformations and obeys the generalized Hooke's Law. In the general case of a uniform curvilinearly anisotropic body the generalized Hooke's Law in the discussed system of triorthogonal coordinates α, β and γ has the following form:

---

*V.Z. Vlasov, Obshchaya teoriya obolochek, Gostekhizdat, 1949, pp. 181-182.

**L.S. Leybenzon, Kurs teorii uprugosti (Course in the theory of elasticity), Gostekhizdat, 1947, pp. 125, 138.

***A.I. Lur'ye, Prostranstvennyye zadachi teorii uprugosti (Spatial problems in the theory of elasticity), Gostekhizdata 1955, pp. 29-49.

****S.G. Lekhnitskiy, Teoriya uprugosti anizotropnogo tela (Theory of elasticity of an anisotropic body), Gostekhizdat, 1950, pp. 15-33.

*****S.G. Lekhnitskiy, Anizotropnyye plastinki (Anisotropic plates), Gostekhizdat, 1957, pp. 15-23.

$$\left.\begin{array}{l}
e_{\alpha}=a_{11}\sigma_{\alpha}+a_{12}\sigma_{\beta}+a_{13}\sigma_{\gamma}+a_{14}\tau_{\beta\gamma}+a_{15}\tau_{\alpha\gamma}+a_{16}\tau_{\alpha\beta};\\
e_{\beta}=a_{12}\sigma_{\alpha}+a_{22}\sigma_{\beta}+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;+a_{26}\tau_{\alpha\beta};\\
.\;.\\
e_{\alpha\beta}=a_{16}\sigma_{\alpha}+a_{26}\sigma_{\beta}+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;+a_{66}\tau_{\alpha\beta},
\end{array}\right\} \qquad (3.1)$$

where a_{jk} are the elastic constants (coefficients of deformation); there is a total of 21 independent elastic constants; in terms of engineering constants, they are as follows:

$$\text{I.}\quad a_{11}=\frac{1}{E_{\alpha}},\quad a_{22}=\frac{1}{E_{\beta}},\quad a_{33}=\frac{1}{E_{\gamma}}.$$

$$\text{II.}\quad a_{12}=-\frac{\nu_{\beta\alpha}}{E_{\alpha}}=-\frac{\nu_{\alpha\beta}}{E_{\beta}},\quad a_{23}=-\frac{\nu_{\gamma\beta}}{E_{\beta}}=-\frac{\nu_{\beta\gamma}}{E_{\gamma}},$$

$$a_{13}=-\frac{\nu_{\alpha\gamma}}{E_{\gamma}}=-\frac{\nu_{\gamma\alpha}}{E_{\alpha}}.$$

$$\text{III.}\quad a_{44}=\frac{1}{G_{\beta\gamma}},\quad a_{55}=\frac{1}{G_{\alpha\gamma}}\quad a_{66}=\frac{1}{G_{\alpha\beta}}.$$

$$\text{IV.}\quad a_{56}=\frac{\mu_{\alpha\beta,\gamma\alpha}}{G_{\alpha\gamma}}=\frac{\mu_{\gamma\alpha,\alpha\beta}}{G_{\alpha\beta}},\quad a_{46}=\frac{\mu_{\alpha\beta,\beta\gamma}}{G_{\beta\gamma}}=\frac{\mu_{\beta\gamma,\alpha\beta}}{G_{\alpha\beta}},$$

$$a_{45}=\frac{\mu_{\gamma\alpha,\beta\gamma}}{G_{\beta\gamma}}=\frac{\mu_{\beta\gamma,\gamma\alpha}}{G_{\alpha\gamma}}.$$

$$\text{V.}\quad a_{14}=\frac{\eta_{\beta\gamma,\alpha}}{E_{\alpha}}=\frac{\eta_{\alpha,\beta\gamma}}{G_{\beta\gamma}},\quad a_{25}=\frac{\eta_{\gamma\alpha,\beta}}{E_{\beta}}=\frac{\eta_{\beta,\gamma\alpha}}{G_{\alpha\gamma}},\quad a_{36}=\frac{\eta_{\alpha\beta,\gamma}}{E_{\gamma}}=\frac{\eta_{\gamma,\alpha\beta}}{G_{\alpha\beta}}.$$

$$\text{VI.}\quad a_{24}=\frac{\eta_{\beta\gamma,\beta}}{E_{\beta}}=\frac{\eta_{\beta,\beta\gamma}}{G_{\beta\gamma}},\quad a_{35}=\frac{\eta_{\gamma\alpha,\gamma}}{E_{\gamma}}=\frac{\eta_{\gamma,\gamma\alpha}}{G_{\alpha\gamma}},\quad a_{16}=\frac{\eta_{\alpha\beta,\alpha}}{E_{\alpha}}=\frac{\eta_{\alpha,\alpha\beta}}{G_{\alpha\beta}};$$

$$a_{34}=\frac{\eta_{\beta\gamma,\gamma}}{E_{\gamma}}=\frac{\eta_{\gamma,\beta\gamma}}{G_{\beta\gamma}},\quad a_{15}=\frac{\eta_{\gamma\alpha,\alpha}}{E_{\alpha}}=\frac{\eta_{\alpha,\gamma\alpha}}{G_{\alpha\gamma}},\quad a_{26}=\frac{\eta_{\alpha\beta,\beta}}{E_{\beta}}=\frac{\eta_{\beta,\alpha\beta}}{G_{\alpha\beta}},$$

where E_{α}, E_{β}, and E_{γ} are Young's moduli in directions α, β and γ, respectively; $G_{\beta\gamma}$, $G_{\alpha\gamma}$, $G_{\alpha\beta}$ are the shear moduli for the planes parallel at each point to the coordinate surfaces $\alpha=\text{const}$, $\beta=\text{const}$, and $\gamma=\text{const}$; $\nu_{\beta\alpha}$, $\nu_{\gamma\beta}$, $\nu_{\alpha\gamma}$, $\nu_{\alpha\beta}$, $\nu_{\beta\gamma}$, $\nu_{\gamma\alpha}$ are Poisson's ratios characterizing the transverse compression in the presence of tension in the direction of the coordinate axes (the first subscript indicates the direction of transverse compression and the second subscript indicates the direction of the force); $\mu_{\gamma\alpha,\beta\gamma}$, $\mu_{\alpha\beta,\beta\gamma}$, ..., $\mu_{\beta\gamma,\alpha\beta}$ are Chentsov coefficients characterizing the shear in planes tangent to the coordinate surfaces and caused by shearing stresses acting in planes tangent to the other coordinate surfaces. For example, $\mu_{\gamma\alpha,\beta\gamma}$ characterizes the shear in a plane tangent to the coordinate surface $\beta=\text{const}$, and caused by stress $\tau_{\beta\gamma}$. $\eta_{\beta\gamma,\alpha}$, $\eta_{\gamma\alpha,\alpha}$, ..., $\eta_{\beta\alpha,\gamma}$ are the coefficients of interaction of the first kind and $\eta_{\alpha,\beta\gamma}$, $\eta_{\beta,\beta\gamma}$, ..., $\eta_{\gamma,\alpha\beta}$ are the coefficients of interaction of the second kind. The first of these coefficients characterize the shear in planes tangent to the coordinate surfaces and caused by normal stresses; the second coefficients characterize elongation in the directions of the coordinates and caused by shearing stresses. The coefficients of elasticity a_{ik} given above are grouped according to the classification proposed by

P. Bekhterev* (the Roman numerals indicate the group numbers).

For the elastic potential referred to a unit volume of the body, we have

$$\begin{aligned}\overline{V} = {} & \frac{1}{2} a_{11}\sigma_\alpha^2 + \sigma_\alpha (a_{12}\sigma_\beta + a_{13}\sigma_\gamma + a_{14}\tau_{\beta\gamma} + a_{15}\tau_{\gamma\alpha} + a_{16}\tau_{\alpha\beta}) + \\ & + \frac{1}{2} a_{22}\sigma_\beta^2 + \sigma_\beta (a_{23}\sigma_\gamma + a_{24}\tau_{\beta\gamma} + a_{25}\tau_{\gamma\alpha} + a_{26}\tau_{\alpha\beta}) + \\ & + \frac{1}{2} a_{33}\sigma_\gamma^2 + \sigma_\gamma (a_{34}\tau_{\beta\gamma} + a_{35}\tau_{\gamma\alpha} + a_{36}\tau_{\alpha\beta}) + \\ & + \frac{1}{2} a_{44}\tau_{\beta\gamma}^2 + \tau_{\beta\gamma} (a_{45}\tau_{\gamma\alpha} + a_{46}\tau_{\alpha\beta}) + \\ & + \frac{1}{2} a_{55}\tau_{\gamma\alpha}^2 + \tau_{\gamma\alpha} a_{56}\tau_{\alpha\beta} + \\ & + \frac{1}{2} a_{66}\tau_{\alpha\beta}^2. \end{aligned} \tag{3.2}$$

On the basis of (3.1) $\overline{V}$ may also be represented in bilinear form

$$\overline{V} = \frac{1}{2}(\sigma_\alpha e_\alpha + \sigma_\beta e_\beta + \sigma_\gamma e_\gamma + \tau_{\beta\gamma} e_{\beta\gamma} + \tau_{\gamma\alpha} e_{\gamma\alpha} + \tau_{\alpha\beta} e_{\alpha\beta}). \tag{3.3}$$

The potential energy of deformation for the entire body is determined by integration of $\overline{V}$ over the entire volume of the body.

$$V = \int\int\int \overline{V} H_1 H_2 H_3 \, d\alpha \, d\beta \, d\gamma. \tag{3.4}$$

If there exists any symmetry in the internal structure of the material of an anisotropic body, then in its elastic properties there is detected a certain elastic symmetry, that is, there exist symmetrical directions relative to which the elastic properties of the material are identical; in this case the symmetrical directions are known as equivalent directions.

When an anisotropic body possesses elastic symmetry the equations of Hooke's Law in general form are simplified. Let us discuss a few of the more important cases of elastic symmetry.

1. Plane of Elastic Symmetry. Assume that at each point of the body there is a plane possessing the property that any two directions which are symmetrical relative to this plane are equivalent with respect to their elastic

---

*P. Bekhterev, Analiticheskoye issledovaniye obobshchennogo zakona Guka (Analytical investigation of generalized Hooke's Law), Author's lithograph, Leningrad, 1925.

properties. Assuming that coordinate γ at each point of a curvilinearly anisotropic body is perpendicular to the plane of elastic symmetry (that is, the plane of elastic symmetry at each point is parallel to the coordinate surface $\gamma = \text{const}$), we obtain the following equations for the generalized Hooke's Law:

$$\left.\begin{aligned}
e_\alpha &= a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma + a_{16}\tau_{\alpha\beta};\\
e_\beta &= a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{23}\sigma_\gamma + a_{26}\tau_{\alpha\beta};\\
e_\gamma &= a_{13}\sigma_\alpha + a_{23}\sigma_\beta + a_{33}\sigma_\gamma + a_{36}\tau_{\alpha\beta};\\
e_{\beta\gamma} &= a_{44}\tau_{\beta\gamma} + a_{45}\tau_{\alpha\gamma};\\
e_{\gamma\alpha} &= a_{45}\tau_{\beta\gamma} + a_{55}\tau_{\alpha\gamma};\\
e_{\alpha\beta} &= a_{16}\sigma_\alpha + a_{26}\sigma_\beta + a_{36}\sigma_\gamma + a_{66}\tau_{\alpha\beta}.
\end{aligned}\right\} \quad (3.5)$$

In this case the number of independent elastic constants a_{ik} is reduced to 13.

Equations (3.5) may also be represented in the following form:

$$\left.\begin{aligned}
e_\alpha &= \frac{1}{E_1}\sigma_\alpha - \frac{\nu_{12}}{E_2}\sigma_\beta - \frac{\nu_{13}}{E_3}\sigma_\gamma + \frac{\eta_{1,12}}{G_{12}}\tau_{\alpha\beta};\\
e_\beta &= -\frac{\nu_{21}}{E_1}\sigma_\alpha + \frac{1}{E_2}\sigma_\beta - \frac{\nu_{23}}{E_3}\sigma_\gamma + \frac{\eta_{2,12}}{G_{12}}\tau_{\alpha\beta};\\
e_\gamma &= -\frac{\nu_{31}}{E_1}\sigma_\alpha - \frac{\nu_{32}}{E_2}\sigma_\beta + \frac{1}{E_3}\sigma_\gamma + \frac{\eta_{3,12}}{G_{12}}\tau_{\alpha\beta};\\
e_{\beta\gamma} &= \frac{1}{G_{23}}\tau_{\beta\gamma} + \frac{\mu_{23,31}}{G_{13}}\tau_{\alpha\gamma};\\
e_{\alpha\gamma} &= \frac{\mu_{31,12}}{G_{23}}\tau_{\beta\gamma} + \frac{1}{G_{13}}\tau_{\alpha\gamma};\\
e_{\alpha\beta} &= \frac{\eta_{12,1}}{E_1}\sigma_\alpha + \frac{\eta_{12,2}}{E_2}\sigma_\beta + \frac{\eta_{12,3}}{E_3}\sigma_\gamma + \frac{1}{G_{12}}\tau_{\alpha\beta}
\end{aligned}\right\} \quad (3.6)$$

Here and henceforth the literal subscripts of the engineering constants are replaced by numeral subscripts; α is replaced by 1, β by 2 and γ by 3.

Directions perpendicular to the plane of elastic symmetry are known as the principal directions of elasticity. In the discussed case of elastic symmetry one principal direction passes through each point of the body.

2. Three Planes of Elastic Symmetry. Through each point of the body let there pass three mutually perpendicular planes of elastic symmetry. Assuming that at each point of a curvilinear anisotropic body these planes are perpendicular to the corresponding orthogonal coordinate directions α, β, γ (that is, all three planes of elastic symmetry at each point of the body are parallel to the coordinate surfaces $\alpha = \text{const}$, $\beta = \text{const}$, and $\gamma = \text{const}$), we obtain the following equations for the generalized Hooke's Law:

$$\left.\begin{aligned} e_\alpha &= a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma; & e_{\beta\gamma} &= a_{44}\tau_{\beta\gamma};\\ e_\beta &= a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{23}\sigma_\gamma; & e_{\gamma\alpha} &= a_{55}\tau_{\gamma\alpha};\\ e_\gamma &= a_{13}\sigma_\alpha + a_{23}\sigma_\beta + a_{33}\sigma_\gamma; & e_{\alpha\beta} &= a_{66}\tau_{\alpha\beta}. \end{aligned}\right\} \qquad (3.7)$$

In this case the number of independent elastic constants a_{ik} is nine

Equations (3.7) may also be presented in the following form:

$$\left.\begin{aligned} e_\alpha &= \frac{1}{E_1}\sigma_\alpha - \frac{\nu_{12}}{E_2}\sigma_\beta - \frac{\nu_{13}}{E_3}\sigma_\gamma; & e_{\beta\gamma} &= \frac{1}{G_{23}}\tau_{\beta\gamma};\\ e_\beta &= -\frac{\nu_{21}}{E_1}\sigma_\alpha + \frac{1}{E_2}\sigma_\beta - \frac{\nu_{23}}{E_3}\sigma_\gamma; & e_{\gamma\alpha} &= \frac{1}{G_{13}}\tau_{\gamma\alpha};\\ e_\gamma &= -\frac{\nu_{31}}{E_1}\sigma_\alpha - \frac{\nu_{32}}{E_2}\sigma_\beta + \frac{1}{E_3}\sigma_\gamma; & e_{\alpha\beta} &= \frac{1}{G_{12}}\tau_{\alpha\beta}. \end{aligned}\right\} \qquad (3.8)$$

Here, in virtue of the symmetry of equations (3.7), we have the relations

$$E_2\nu_{21} = E_1\nu_{12}, \quad E_3\nu_{32} = E_2\nu_{23}, \quad E_1\nu_{13} = E_3\nu_{31}. \qquad (3.9)$$

A body in which at each point there are three mutually perpendicular planes of elastic symmetry is known as an orthogonally anisotropic or orthotropic body.

3. Plane of Isotropy. Through each point of a body let there pass a plane in which all directions are elastically equivalent. Assuming that in a curvilinearly anisotropic body coordinate γ is at each point perpendicular to the plane of isotropy (that is, the plane of isotropy is at each point parallel to the coordinate surface $\gamma = \text{const}$), we obtain the following equations for Hooke's Law in general form:

$$\left.\begin{aligned} e_\alpha &= a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma; & e_{\beta\gamma} &= a_{44}\tau_{\beta\gamma};\\ e_\beta &= a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{13}\sigma_\gamma; & e_{\gamma\alpha} &= a_{44}\tau_{\gamma\alpha};\\ e_\gamma &= a_{13}(\sigma_\alpha + \sigma_\beta) + a_{33}\sigma_\gamma; & e_{\alpha\beta} &= 2(a_{11} - a_{12})\tau_{\alpha\beta}. \end{aligned}\right\} \qquad (3.10)$$

In this case the number of independent elastic constants a_{ik} is five.

Equations (3.10) may also be given in the following form:

$$\left.\begin{aligned} e_\alpha &= \frac{1}{E}(\sigma_\alpha - \nu\sigma_\beta) - \frac{\nu'}{E'}\sigma_\gamma; & e_{\beta\gamma} &= \frac{1}{G'}\tau_{\beta\gamma};\\ e_\beta &= \frac{1}{E}(\sigma_\beta - \nu\sigma_\alpha) - \frac{\nu'}{E'}\sigma_\gamma; & e_{\gamma\alpha} &= \frac{1}{G'}\tau_{\gamma\alpha};\\ e_\gamma &= -\frac{\nu'}{E'}(\sigma_\alpha + \sigma_\beta) + \frac{1}{E'}\sigma_\gamma; & e_{\alpha\beta} &= \frac{2(1+\nu)}{E}\tau_{\alpha\beta}. \end{aligned}\right\} \qquad (3.11)$$

Here E is Young's modulus for directions in the plane of isotropy; E' is Young's modulus for directions perpendicular to the plane of isotropy; ν is the Poisson's ratio characterizing contraction in the plane of isotropy in the presence of tension in this same plane; ν' is the Poisson's ratio characterizing contraction in the plane of isotropy in the presence of tension in the direction perpendicular to this plane; G' is the shear modulus for planes normal to the plane of isotropy; $G = \frac{E}{2(1+\nu)}$ is the shear modulus for planes parallel to the plane of isotropy.

A body possessing the mentioned elastic properties is known as a transversely isotropic body.

In the discussed case of elastic symmetry the direction perpendicular to the plane of isotropy and all directions in this plane are principal directions.

4. Complete Symmetry — Isotropic Body. Here all directions are equivalent and any plane at any point of the body is a plane of elastic symmetry. In this case the equations for the generalized Hooke's Law have the form

$$\left.\begin{aligned} e_\alpha &= \frac{1}{E}[\sigma_\alpha - \nu(\sigma_\beta + \sigma_\gamma)]; & e_{\beta\gamma} &= \frac{1}{G}\tau_{\beta\gamma};\\ e_\beta &= \frac{1}{E}[\sigma_\beta - \nu(\sigma_\gamma + \sigma_\alpha)]; & e_{\gamma\alpha} &= \frac{1}{G}\tau_{\gamma\alpha},\\ e_\gamma &= \frac{1}{E}[\sigma_\gamma - \nu(\sigma_\alpha + \sigma_\beta)]; & e_{\alpha\beta} &= \frac{1}{G}\tau_{\alpha\beta} \end{aligned}\right\} \qquad (3.12)$$

Here E is Young's modulus; ν is the Poisson's ratio; $G = \frac{E}{2(1+\nu)}$ is the shear modulus. The number of independent elastic constants is two.

From (3.2) it is easy to derive the expressions for potential energy $\overline{V}$ in any special case of anisotropy. In view of their elementary nature we shall not present these constructions here.

SECTION 4. TRANSFORMATION OF ELASTIC CONSTANTS WITH ROTATION OF THE COORDINATE SYSTEM*, **

In the theory of elasticity of an anisotropic body we encounter problems in which the solutions require the values of elastic constants for a certain

---

*S.G. Lekhnitskiy, see pp. 33-48 of work cited in footnote **** of p. 7.

S.G. Lekhnitskiy, see pp. 44-48 of work cited in footnote *** of p. 7.

system of coordinates α', β' and γ' when the elastic constants for another system of coordinates α, β and γ are known.

Let us discuss the generalized plane stress of an anisotropic plate. At each point of the material of the plate let there be only one plane of elastic symmetry parallel to the median plane of the plate, which coincides with the combined coordinate planes $\alpha\beta$ and $\alpha'\beta'$. Let the coordinate system (α, β, γ) and $(\alpha', \beta', \gamma')$ be obtained one from the other by rotation through a certain angle φ around the common axis γ.

Proceeding in the known manner,* we obtain the following transformation formulas of the elastic constants:

$$a'_{11} = a_{11}\cos^4\varphi + (2a_{12} + a_{66})\sin^2\varphi\cos^2\varphi + a_{22}\sin^4\varphi + (a_{16}\cos^2\varphi + a_{26}\sin^2\varphi)\sin 2\varphi, \tag{4.1}$$

$$a'_{22} = a_{11}\sin^4\varphi + (2a_{12} + a_{66})\sin^2\varphi\cos^2\varphi + a_{22}\cos^4\varphi - (a_{16}\sin^2\varphi + a_{26}\cos^2\varphi)\sin 2\varphi, \tag{4.2}$$

$$a'_{12} = (a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\varphi\cos^2\varphi + a_{12} + \frac{1}{2}(a_{26} - a_{16})\sin 2\varphi\cos 2\varphi, \tag{4.3}$$

$$a'_{66} = 4(a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\varphi\cos^2\varphi + a_{66} + 2(a_{26} - a_{16})\sin 2\varphi\cos 2\varphi, \tag{4.4}$$

$$a'_{16} = \left[a_{22}\sin^2\varphi - a_{11}\cos^2\varphi + \frac{1}{2}(2a_{12} + a_{66})\cos 2\varphi\right]\sin 2\varphi + a_{16}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) + a_{26}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi), \tag{4.5}$$

$$a'_{26} = \left[a_{22}\cos^2\varphi - a_{11}\sin^2\varphi - \frac{1}{2}(2a_{12} + a_{66})\cos 2\varphi\right]\sin 2\varphi + a_{16}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi) + a_{26}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi). \tag{4.6}$$

Let us give the following invariant relations:

$$\begin{gathered} a'_{11} + a'_{22} + 2a'_{12} = a_{11} + a_{22} + 2a_{12}, \\ a'_{66} - 4a'_{12} = a_{66} - 4a_{12}. \end{gathered} \tag{4.7}$$

In the special case where we have an orthotropic plate for which the principal directions of elasticity coincide with the directions of coordinate lines α, β, and for them the principal elastic constants are known, from (4.1)-(4.6) by virtue of (3.8), we obtain for the elastic constants a'_{jk}

$$a'_{11} = \frac{\cos^4\varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\sin^2\varphi\cos^2\varphi + \frac{\sin^4\varphi}{E_2}, \tag{4.8}$$

$$a'_{22} = \frac{\sin^4\varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_2}{E_2}\right)\sin^2\varphi\cos^2\varphi + \frac{\cos^4\varphi}{E_2}, \tag{4.9}$$

$$a'_{12} = \frac{1}{4}\left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G}\right)\sin^2 2\varphi - \frac{\nu_1}{E_1}, \tag{4.10}$$

$$a_{66} = \left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G}\right)\sin^2 2\varphi + \frac{1}{G}, \tag{4.11}$$

$$a'_{16} = \left[2\left(\frac{\sin^2\varphi}{E_2} - \frac{\cos^2\varphi}{E_1}\right) + \right.$$
$$\left. + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)(\cos^2\varphi - \sin^2\varphi)\right]\sin\varphi\cos\varphi, \tag{4.12}$$

$$a'_{26} = \left[2\left(\frac{\cos^2\varphi}{E_2} - \frac{\sin^2\varphi}{E_1}\right) - \right.$$
$$\left. - \left(\frac{1}{G} - \frac{2\nu_2}{E_2}\right)(\cos^2\varphi - \sin^2\varphi)\right]\sin\varphi\cos\varphi. \tag{4.13}$$

From (4.7) we find the corresponding invariant expressions for an orthotropic material

$$\frac{1}{E'_1} + \frac{1}{E'_2} - \frac{2\nu'_1}{E'_1} = \frac{1}{E_1} + \frac{1}{E_2} - \frac{2\nu_1}{E_1},$$
$$\frac{1}{G'} + \frac{4\nu'_1}{E'_1} = \frac{1}{G} + \frac{4\nu_1}{E_1}. \tag{4.14}$$

In formulas (4.8)-(4.14) the following new notation is introduced:

$$\nu_{12} = \nu_2, \quad \nu_{21} = \nu_1, \quad (E_1\nu_2 = E_2\nu_1), \quad G_{12} = G. \tag{4.15}$$

The notation given in (4.15) will also be used subsequently.

SECTION 5. ELASTIC CONSTANTS FOR CERTAIN ANISOTROPIC MATERIALS

Of the various types of anisotropic bodies we shall discuss only noncrystalline anisotropic materials, which are widely used in modern engineering for shells. These materials include, for example, natural wood, concrete, rolled sheet metal, paper, delta wood and plywood, fabrics, and others. In the first approximation we shall assume that the discussed materials are homogeneous, orthotropic and subjected to a generalized plane stress. We shall assume that the principal plane xOy is the center plane of an elementary plate cut from the discussed anisotropic material.

1. Natural Wood. Due to the presence of annual layers, natural wood is regarded as an anisotropic body. Disregarding nonuniformity and curvature of the layers in natural wood, it is easy to detect three planes of elastic symmetry: one of them (yOz) is normal to the grain of the wood, the second (xOy) is parallel to the plane of the annual layers and the third (xOy) is orthogonal to the first two (Figure 5).

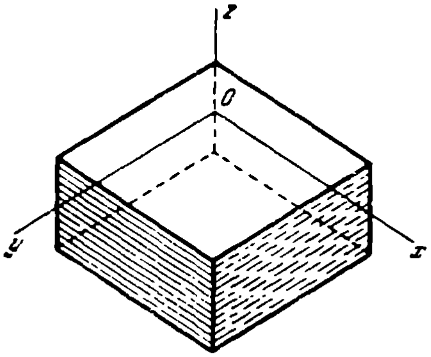

Figure 5

By way of example let us present the numerical values of the elastic constants for pine:*

$$\left.\begin{array}{ll} E_1 = E_x = 1 \cdot 10^5\,\text{kg/см}^2; & E_2 = E_y = 0{,}042 \cdot 10^5\,\text{kg/см}^2; \\ \nu_{xy} = \nu_2 = 0{,}01; & G_{12} = G_{xy} = 0{,}075 \cdot 10^5\,\text{kg/см}^2. \end{array}\right\} \quad (5.1)$$

A detailed discussion of the physical characteristics of wood may be found in the work by A.N. Mitinskiy.**

2. Delta Wood. Delta wood is a laminated wood material obtained by hot pressing of a wood stack consisting of a large number of layers (plies) impregnated with resin. At intervals of every ten plies with longitudinally directed grain one layer with transversely directed grain is inserted; delta wood is essentially laminar and anisotropic. However, to the first approximation we shall assume that delta wood is a homogeneous orthotropic material. Assuming that the x axis coincides with the direction of the prevailing grain (Figure 6), for the averaged elastic constants of delta wood we have the following numerical values:***

$$\left.\begin{array}{ll} E_1 = 3{,}05 \cdot 10^5\,\text{kg/см}^2; & E_2 = 0{,}467 \cdot 10^5\,\text{kg/см}^2; \\ \nu_2 = 0{,}02; & G_{12} = 0{,}22 \cdot 10^5\,\text{kg/см}^2. \end{array}\right\} \quad (5.2)$$

---

*A. L. Rabinovich, Concerning elastic constants and strength of anisotropic materials, Trudy TsAGI, No. 582, 1946.

**A. N. Mitinskiy, Elastic constants of wood as an orthotropic material, Trudy Lesotekhnicheskoy akademii im. S. M. Kirova, No. 63, 1948.

***A. L. Rabinovich, see footnote * on this page.

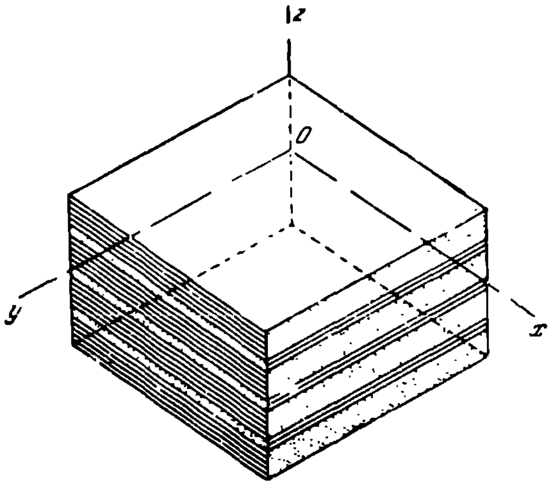

Figure 6

3. Plywood. Let us discuss birch plywood made of an odd number of layers (plies) bound by bakelite film. The plies are symmetrically arranged relative to the center ply and the directions of the grain of two adjacent layers are mutually perpendicular (Figure 7). A sheet of plywood is laminar and anisotropic. However, in the first approximation we may assume that in the case of plywood we also have a homogeneous orthotropic material.

For the averaged elastic constants of three-layer plywood with thicknesses of 1.0, 1.5, 2.0, 2.5, 3.0, 4.0 and 5.0 mm we have the following numerical values:*

$$\left.\begin{array}{ll} E_1 = 1{,}2 \cdot 10^5 \text{kg}/c\mathcal{M}^2; & E_2 = 0{,}6 \cdot 10^5 \text{kg}/c\mathcal{M}^2; \\ \nu_2 = 0{,}036; & G_{12} = 0{,}07 \cdot 10^5 \text{kg}/c\mathcal{M}^2. \end{array}\right\} \quad (5.3)$$

Here it is assumed that the x and y axes are parallel and perpendicular, respectively, to the grains of the outer layers of the plywood sheet.

Shells made of delta wood or of plywood will be dealt with in subsequent discussion as laminated structures of orthotropic layers of wood (plies).

4. Concrete. By virtue of the technological characteristics in manufacture and placement of concrete this material acquires the properties of

---

*Spravochnik aviakonstruktora, t. III, Prochnost' samoleta (Aircraft designers handbook, Vol. III, Aircraft strength), Izd. TsAGI, 1939, p. 325.

anisotropy. We present the numerical values of the modulus of elasticity of anisotropic (orthotropic) lightweight concrete:*

$$E_1 = 1{,}08 \cdot 10^5\ \text{kg/cm}^2;\quad E_2 = 0{,}81 \cdot 10^5\ \text{kg/cm}^2. \tag{5.4}$$

5. Paper. The technological aspects of paper manufacture are such that paper generally possesses the properties of anisotropy.

The numerical values of the elastic constants of semi-Whatman paper** (produced by the Goznak plant) having a density of 160 g/m^3 are:

$$\left.\begin{aligned} &E_1 = 30{,}1 \cdot 10^3\ \text{kg/cm}^2; && E_2 = 22{,}6 \cdot 10^3\ \text{kg/cm}^2;\\ &\nu_2 = 0{,}23; && G_{12} = 9{,}96 \cdot 10^3\ \text{kg/cm}^2. \end{aligned}\right\} \tag{5.5}$$

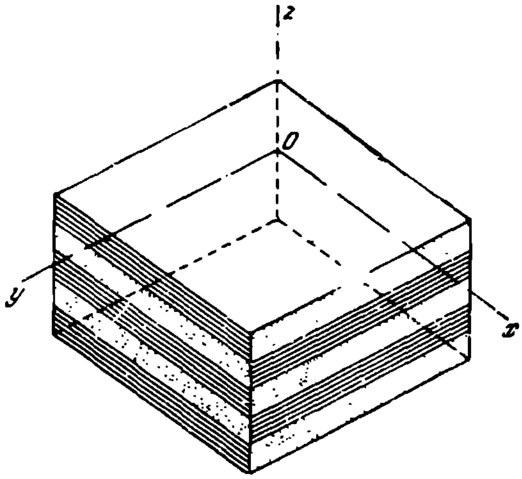

Figure 7.

---

*K. S. Karapetyan, Influence of anisotropy on creep of concrete, Izvestiya AN ArmSSR, FM nauki, Vol. X, No. 6, 1957.

**V. M. Chebanov, Investigation of stability of thin-walled shells by means of paper models, Inzhenernyy sbornik, Vol. XXII, 1955.

CHAPTER II

FUNDAMENTAL EQUATIONS OF THE THEORY OF SHELLS CONSISTING OF AN ARBITRARY NUMBER OF ANISOTROPIC LAYERS

SECTION 1. BASIC CONCEPTS, INITIAL RELATIONSHIPS AND HYPOTHESES

Let us examine a multi-layer thin shell of uniform thickness h consisting of an arbitrary number of homogeneous anisotropic layers of uniform thickness t_i (Figure 8).

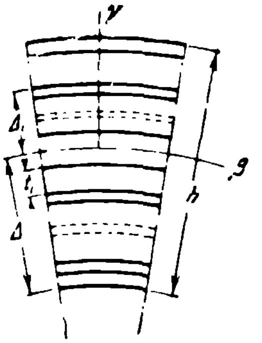

Figure 8

It is assumed that at each point of each layer of the shell there is only one plane of axial symmetry parallel to the coordinate surface of the shell;* the coordinate surface is parallel to the outer surfaces of the shell and lies within any i-th layer. In special cases, the coordinate surface of the shell may also be any of the surfaces of contact of the layers or any of the boundary surfaces of the shell*,**,***,****.

---

*Here and henceforth we shall assume that the term "coordinate surface of the shell" refers to the surface $\gamma = 0$.

The α and β be curvilinear orthogonal coordinates coinciding with the lines of principal curvature of the coordinate surface of the shell while γ, being normal to the coordinate lines $\alpha = \text{const}$, and $\beta = \text{const}$, is rectilinear and represents the distance along the normal from point (α, β) of the coordinate surface to point (α, β, γ) of the shell (Figure 9). We assume that all layers of the shell remain elastic in the presence of deformation, that is, they obey the generalized Hooke's Law (1.3.5) and function simultaneously without slip.

In the chosen triorthogonal system of coordinates for the Lame coefficients (1.1.5) we will have*****

$$H_1 = A(1 + k_1\gamma), \quad H_2 = B(1 + k_2\gamma),$$
$$H_3 = 1, \tag{1.1}$$

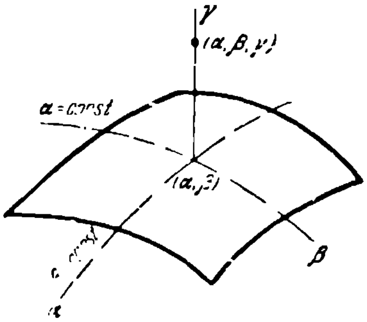

Figure 9

where $A = A(\alpha, \beta)$ and $B = B(\alpha, \beta)$ are coefficients of the first quadratic form of the coordinate surface, $k_1 = k_1(\alpha, \beta)$, $k_2 = k_2(\alpha, \beta)$ — are the principal curvatures of the coordinate surface of the shell at lines $\beta = \text{const}$, and $\alpha = \text{const}$, respectively.

---

*D. Yu. Panov, Heat resistance of bimetallic shells (toward a theory of a heat switch). PMM, Vol. XI, No. 6, 1947.

**S. A. Ambartsumyan, Anisotropic laminar shells (problems of strength). Dissertation, 1951.

***S. A. Ambartsumyan, Concerning design calculations for laminar anisotropic shells. Izvestiya AN Arm SSR (FMYeiT nauki), Vol. VI, No. 3, 1953.

****E. I. Grigolyuk, Thin bimetallic shells and plates. Inzhenernyy sbornik, Vol. XVII, 1953.

*****V. Z. Vlasov, Obshchaya teoriya obolocnek, Gostekhizdat, 1949, pp. 212-214.

Inserting the values of H_j from (1.1) into (1 4.3) and assuming $\gamma = 0$, we obtain the well-known Gauss-Codazzi relationships for the coordinate surface of a shell

$$\left.\begin{aligned} &\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right)=-k_1k_2AB;\\ &\frac{\partial}{\partial\beta}(Ak_1)=k_2\frac{\partial A}{\partial\beta},\quad \frac{\partial}{\partial\alpha}(Bk_2)=k_1\frac{\partial B}{\partial\alpha}.\end{aligned}\right\} \tag{1.2}$$

In virtue of relationships (1.2) we also have

$$\frac{1}{H_1}\frac{\partial H_2}{\partial\alpha}=\frac{1}{A}\frac{\partial B}{\partial\alpha},\quad \frac{1}{H_2}\frac{\partial H_1}{\partial\beta}=\frac{1}{B}\frac{\partial A}{\partial\beta}, \tag{1.3}$$

which will be used in subsequent discussion.

For the deformation components of the i-th layer of the shell of (1.2.2) and (1.2.3) we obtain (for the sake of simplicity we have omitted the i subscripts of H_1 and H_2)

$$e_\alpha^i=\frac{1}{H_1}\frac{\partial u_\alpha^i}{\partial\alpha}+\frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta}u_\beta^i+\frac{1}{H_1}\frac{\partial H_1}{\partial\gamma}u_\gamma^i; \tag{1.4}$$

$$e_\beta^i=\frac{1}{H_2}\frac{\partial u_\beta^i}{\partial\beta}+\frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha}u_\alpha^i+\frac{1}{H_2}\frac{\partial H_2}{\partial\gamma}u_\gamma^i; \tag{1.5}$$

$$e_\gamma^i=\frac{\partial u_\gamma^i}{\partial\gamma}; \tag{1.6}$$

$$e_{\alpha\beta}^i=\frac{H_1}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_1}u_\alpha^i\right)+\frac{H_2}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_2}u_\beta^i\right); \tag{1.7}$$

$$e_{\beta\gamma}^i=H_2\frac{\partial}{\partial\gamma}\left(\frac{1}{H_2}u_\beta^i\right)+\frac{1}{H_2}\frac{\partial}{\partial\beta}u_\gamma^i; \tag{1.8}$$

$$e_{\alpha\gamma}^i=H_1\frac{\partial}{\partial\gamma}\left(\frac{1}{H_1}u_\alpha^i\right)+\frac{1}{H_1}\frac{\partial}{\partial\alpha}u_\gamma^i. \tag{1.9}$$

The equilibrium equations in (1.2.4) are also simplified, which in the chosen system of coordinates for the 8-th layer of the shell have the following form:

$$\begin{aligned} &\frac{\partial}{\partial\alpha}(H_2\sigma_\alpha^i)-\sigma_\beta^i\frac{\partial H_2}{\partial\alpha}+\frac{1}{H_1}\frac{\partial}{\partial\beta}(H_1^2\tau_{\alpha\beta}^i)+\\ &\qquad+\frac{1}{H_1}\frac{\partial}{\partial\gamma}(H_1^2H_2\tau_{\alpha\gamma}^i)+H_1H_2P_\alpha^i=0; \end{aligned} \tag{1.10}$$

$$\frac{\partial}{\partial\beta}(H_1\sigma_\beta^i)-\sigma_\alpha^i\frac{\partial H_1}{\partial\beta}+\frac{1}{H_2}\frac{\partial}{\partial\alpha}(H_2^2\tau_{\beta\alpha}^i)+$$
$$+\frac{1}{H_2}\frac{\partial}{\partial\gamma}(H_1H_2^2\tau_{\beta\gamma}^i)+H_1H_2P_\beta^i=0; \tag{1.11}$$

$$\frac{\partial}{\partial\gamma}(H_1H_2\sigma_\gamma^i)-\sigma_\alpha^iH_2\frac{\partial H_1}{\partial\gamma}-\sigma_\beta^iH_1\frac{\partial H_2}{\partial\gamma}+$$
$$+\frac{\partial}{\partial\alpha}(H_2\tau_{\alpha\gamma}^i)+\frac{\partial}{\partial\beta}(H_1\tau_{\beta\gamma}^i)+H_1H_2P_\gamma^i=0. \tag{1.12}$$

The initial equations of the generalized Hooke's Law (1.3.5) for the i-th layer are written as follows:

$$\left.\begin{aligned}
e_\alpha^i&=a_{11}^i\sigma_\alpha^i+a_{12}^i\sigma_\beta^i+a_{13}^i\sigma_\gamma^i+a_{16}^i\tau_{\alpha\beta}^i;\\
e_\beta^i&=a_{12}^i\sigma_\alpha^i+a_{22}^i\sigma_\beta^i+a_{23}^i\sigma_\gamma^i+a_{26}^i\tau_{\alpha\beta}^i;\\
e_\gamma^i&=a_{13}^i\sigma_\alpha^i+a_{23}^i\sigma_\beta^i+a_{33}^i\sigma_\gamma^i+a_{36}^i\tau_{\alpha\beta}^i;\\
e_{\beta\gamma}^i&=a_{44}^i\tau_{\beta\gamma}^i+a_{45}^i\tau_{\alpha\gamma}^i;\\
e_{\gamma\alpha}^i&=a_{45}^i\tau_{\beta\gamma}^i+a_{55}^i\tau_{\alpha\gamma}^i;\\
e_{\alpha\beta}^i&=a_{16}^i\sigma_\alpha^i+a_{26}^i\sigma_\beta^i+a_{36}^i\sigma_\gamma^i+a_{66}^i\tau_{\alpha\beta}^i.
\end{aligned}\right\} \tag{1.13}$$

The basic premise for construction of a theory of thin anisotropic laminar shells is the well-known hypothesis of nondeformable normals. It is formulated as follows: after deformation a rectilinear element normal to the coordinate surface of a shell remains rectilinear, normal to the deformed coordinate surface of the shell and preserves its length. Usually this geometric assumption is supplemented by the static assumption that the normal stresses σ_γ^i at areas parallel to the coordinate surface of a thin shell may be disregarded in comparison with other stresses.

As is known, the classical formulation adopted here for the hypothesis of nondeformable normals is not, strictly speaking, a fortunate choice and requires refinement*,**, the more so since we are discussing anisotropic laminar shells.***

In applying the hypothesis of nondeformable normals we introduce into the theory of shells a certain unavoidable error which, as investigations reveal*,**,***, will be of the order of h/R_i, however, there are cases in which this error is considerably greater.**** In the present work we shall assume that the error introduced into the theory of shells by the hypothesis of nondeformable normals is at least as great as h/R_i.

---

*A.L. Gol'denveyzer, Teoriya uprugikh tonkikh cbolochek, Gostekhizdat, 1953, pp. 64-66.

**V.V. Novozhilov, Teoriya tonkikh obolochek, Sudpromgiz, 1951. p. 33.

***S.A. Ambartsumyan, Toward calculation of two-layer orthotropic shells, Izvestiya OTN, AN SSSR, Nc. 7, 1957.

Finally, with the usual engineering assumption of a relative error of 5%, those shells will be considered thin in which***** $\max(hk_j) \leqslant \frac{1}{20}$ and at the same time $\max(h/a) \leqslant \varepsilon$, where a is the minimum linear dimension of the shell at the coordinate surface and ϵ is a small quantity (for example, for an isotropic shell ~ 0.1), which will be established subsequently. The second condition (borrowed from the theory of plates) is a necessary condition, since if a thin shell is defined only from the point of view of the ratio of the thickness of the shell to the minimum radius of curvature of the coordinate surface (the first condition), from the point of view of the theory of plates (second condition) it may be thick and the adopted basic assumption becomes unacceptable.

The definition of a thin shell as given here is somewhat conditional in nature, since if the thickness of the shell is regarded from the standpoint of the possibility of applying the hypothesis of nondeformable normals, the geometric definition given here for a thin shell will, in the case of anisotropic laminar shells, require substantial modification.

We shall return to the problem of formulation of the hypothesis of nondeformable normals and, accordingly, to the definition of a thin anisotropic shell (particularly in the last chapter).

SECTION 2. DISPLACEMENTS AND DEFORMATIONS

The geometric hypothesis of nondeformable normals, given for the entire stack of the shell as a whole, relieves us to the necessity of investigating the displacements and deformations of each layer individually.

---

*B.G. Galerkin, Equilibrium of a spherical elastic shell, PMM, Vol VI, No. 6, 1942.

**A.I. Lur'ye, Equilibrium of a symmetrically loaded, spherical, elastic shell, PMM, Vol. VII, No. 6, 1943.

***V.V. Novozhilov, R.M. Finkel'shteyn, concerning error in the Kirkhoff's hypothesis in the theory of shells, PMM, Vol. VII, No. 5, 1943.

****A. L. Gol'denveyzer, see p. 66 and 236 of the work cited in reference 1 above.

*****V.V. Novozhilov, see p. 6 of the work cited in reference 2 above.

Having the deformations of elongation and shear as well as the parameters characterizing the change in curvature and torsion of the coordinate surface of the shell, we may determine the deformations and displacements of any layer of the shell by elementary means. Moreover, as is easily shown, all the characteristics of deformation and displacement of each layer are obtained from the deformations and displacements of a certain homogeneous shell. Using the fundamental hypothesis, we may consider* that of the six relationships in (1.3.5) the third, fourth and fifth may be replaced by the following approximate equalities·

$$e_\gamma = 0, \quad e_{\beta\gamma} = 0, \quad e_{\alpha\gamma} = 0 \qquad (2.1)$$

or, for an individual layer of the shell

$$e^i_\gamma = 0, \quad e^i_{\beta\gamma} = 0, \quad e^i_{\alpha\gamma} = 0, \qquad (2.2)$$

which are formally equivalent to the assumption that the deformation of the shell as a whole occurs without shear deformations $e_{\beta\gamma}$, and $e_{\alpha\gamma}$ in the planes of normal sections and without elongation e_γ in the direction of the thickness of the shell.

In virtue of (2.1) and (2.2), from (1.6) we have

$$\frac{\partial u^i_\gamma}{\partial \gamma} = 0, \quad u^i_\gamma = w^i(\alpha, \beta) = w(\alpha, \beta), \qquad (2.3)$$

that is, normal displacement u^i_γ of any point of any layer of the shell does not depend on coordinate γ. Normal displacement of all points of a given normal element have constant value and are equal to the normal displacement $w = w(\alpha, \beta)$ of that point of the coordinate surface of the shell, which point is formed upon intersection of the given normal with the coordinate surface of the shell.

On the basis of (2.1) and (2.2), from (1.7) and (1.8), after the known transformations**, for tangential displacements of the i-th layer of the shell we obtain

$$u^i_\alpha = (1 + k_1\gamma)u - \frac{\gamma}{A}\frac{\partial w}{\partial \alpha}; \qquad (2.4)$$

$$u^i_\beta = (1 + k_2\gamma)v - \frac{\gamma}{B}\frac{\partial w}{\partial \beta}. \qquad (2.5)$$

---

*A. L. Gol'denveyzer, see footnote * on p. 21.

V. Z. Vlasov, see pp. 218-220 of work cited in footnote *** on p. 19.

where $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$ are the tangential displacements of the corresponding point of the coordinate surface of the envelope, that is, the displacements of this point are in the directions of the positive tangents to the lines of curvature of the coordinate surface $\beta=\text{const}$, and $\alpha = \text{const}$. respectively. Thus, formulas (2.3)-(2.5) establish the geometric model of the deformed state of the shell.

On the basis of (2.3)-(2.5) deformations e^i_α, e^i_β and $e^i_{\alpha\beta}$ may be represented in the form of power series of variable γ, wherein, in virtue of the fundamental hypothesis, we may limit ourselves to the first two terms of the expansions, namely,

$$e^i_\alpha = \varepsilon_1 + \gamma\varkappa_1; \tag{2.6}$$

$$e^i_\beta = \varepsilon_2 + \gamma\varkappa_2; \tag{2.7}$$

$$e^i_{\alpha\beta} = \omega + \gamma\tau. \tag{2.8}$$

Inserting the values of u^i_α, u^i_β and u^i_γ respectively, from (2.4), (2.5) and (2.3) into expressions (1.4), (1.5), (1.6) and equating the results to (2.6), (2.7) and (2.8), respectively, for the coefficients of expansions (2.6)-(2.8) we obtain

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_1 w; \tag{2.9}$$

$$\varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + k_2 w; \tag{2.10}$$

$$\omega = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right); \tag{2.11}$$

$$\varkappa_1 = -\frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} + \frac{\partial k_1}{\partial \alpha}\frac{u}{A} + \frac{\partial k_1}{\partial \beta}\frac{v}{B} - k_1 w; \tag{2.12}$$

$$\varkappa_2 = -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{\partial k_2}{\partial \alpha}\frac{u}{A} + \frac{\partial k_2}{\partial \beta}\frac{v}{B} - k_2^2 w; \tag{2.13}$$

$$\begin{aligned}\tau = -\frac{2}{AB}\left(\frac{\partial^2 w}{\partial \alpha\, \partial \beta} - \frac{1}{A}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha} - \frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta}\right) + \\ + (k_1 - k_2)\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right],\end{aligned} \tag{2.14}$$

which are the same as the corresponding coefficients of the expansions derived in the case of a homogeneous isotropic shell.*

---

*V.Z. Vlasov, see p. 221 of the work cited in footnote ***** on p. 19.

The expansion coefficients ε_1, ε_2, ω, which are known as the components of tangential deformation, represent the relative deformations of elongation and shear of the coordinate surface of a shell. The expansion coefficients $\varkappa_1$, $\varkappa_2$, and which are known as the changes in curvature of the coordinate surface of the shell, represent the variation in deformations e_α, e_β, $e_{\alpha\beta}$ over the thickness of the shell (linearly). $\varkappa_1$ and $\varkappa_2$ describe the bending deformation, τ the torsional deformation.

Instead of expressions (2.12)-(2.14), most authors use the following relationships for changes in curvature:

$$\varkappa_1 = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} - k_1 u\right) - \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} - k_2 v\right); \qquad (2.15)$$

$$\varkappa_2 = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} - k_2 v\right) - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} - k_1 u\right); \qquad (2.16)$$

$$\tau = -\frac{2}{AB}\left(\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right) +$$

$$+ 2k_1\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u\right) + 2k_2\left(\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right), \qquad (2.17)$$

which, strictly speaking, cannot be identified with changes in curvature of the coordinate surface of the shell, since in deriving relationships (2.15)-(2.17) it is assumed that the tangential deformations of the central surface do not correspondingly affect the changes in curvature of the coordinate surface of the shell.

Analyzing the above derived variants of (2.12)-(2.14) and (2.15)-(2.17), for evaluation of the change in curvatures it is not difficult to see* that they differ only within the limits of accuracy of the initial assumptions of the theory of thin-walled shells. For the purpose of simplification of calculations in each special case in the present work we shall use either the first or second variant of the relationships for the change in curvature.

SECTION 3. EQUATIONS OF CONTINUITY OF DEFORMATIONS OF THE COORDINATE SURFACE

Among the six parameters ε_1, ε_2, ω, $\varkappa_1$, $\varkappa_2$, τ, which characterize deformation of the coordinate surface of a shell there are three differential relationships which are valid for any values of displacements u, v and w.

---

*V.V. Novozhilov, see p. 27 of work cited in footnote ** on p. 21.

Eliminating displacements $u, v, w,$ from equations (2.9)-(2.11) and (2.15)-(2.17) and using the Gauss-Codazzi equations (1.2), we obtain*,**,*** three differential relationships in terms of the six components of deformation of the coordinate surface of the shell:

$$B\frac{\partial\varkappa_2}{\partial\alpha}+\frac{\partial B}{\partial\alpha}(\varkappa_2-\varkappa_1)-\frac{A}{2}\frac{\partial\tau}{\partial\beta}-\frac{\partial A}{\partial\beta}\tau+k_2\frac{\partial A}{\partial\beta}\omega+$$
$$+k_1\left[A\frac{\partial\omega}{\partial\beta}+\frac{\partial A}{\partial\beta}\omega-B\frac{\partial\varepsilon_2}{\partial\alpha}-\frac{\partial B}{\partial\alpha}(\varepsilon_2-\varepsilon_1)\right]=0; \qquad (3.1)$$

$$A\frac{\partial\varkappa_1}{\partial\beta}+\frac{\partial A}{\partial\beta}(\varkappa_1-\varkappa_2)-\frac{B}{2}\frac{\partial\tau}{\partial\alpha}-\frac{\partial B}{\partial\alpha}\tau+k_1\frac{\partial B}{\partial\alpha}\omega+$$
$$+k_2\left[B\frac{\partial\omega}{\partial\alpha}+\frac{\partial B}{\partial\alpha}\omega-A\frac{\partial\varepsilon_1}{\partial\beta}-\frac{\partial A}{\partial\beta}(\varepsilon_1-\varepsilon_2)\right]=0; \qquad (3.2)$$

$$k_2\varkappa_1+k_1\varkappa_2+\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\frac{1}{A}\left[B\frac{\partial\varepsilon_2}{\partial\alpha}+\frac{\partial B}{\partial\alpha}(\varepsilon_2-\varepsilon_1)-\right.\right.$$
$$\left.-\frac{A}{2}\frac{\partial\omega}{\partial\beta}-\frac{\partial A}{\partial\beta}\omega\right]+\frac{\partial}{\partial\beta}\frac{1}{B}\left[A\frac{\partial\varepsilon_1}{\partial\beta}+\right.$$
$$\left.\left.+\frac{\partial A}{\partial\beta}(\varepsilon_1-\varepsilon_2)-\frac{B}{2}\frac{\partial\omega}{\partial\alpha}-\frac{\partial B}{\partial\alpha}\omega\right]\right\}=0. \qquad (3.3)$$

The functions $\varepsilon_1 \ldots \tau$, satisfying these equations characterize that deformed state of the shell in which the coordinate surface remains continuous, without undergoing rupture. For this reason (3.1)-(3.3) are known as the conditions of nondiscontinuity of the coordinate surface.

These conditions for a shell of arbitrary shape were first derived by A. L. Gol'denveyzer.****

SECTION 4. STRESSES IN LAYERS

Using the fundamental hypothesis and disregarding stresses σ_γ^i, we obtain from the generalized Hooke's Law (1.13)

$$\left.\begin{aligned} e_\alpha^i&=a_{11}^i\sigma_\alpha^i+a_{12}^i\sigma_\beta^i+a_{16}^i\tau_{\alpha\beta}^i;\\ e_\beta^i&=a_{12}^i\sigma_\alpha^i+a_{22}^i\sigma_\beta^i+a_{26}^i\tau_{\alpha\beta}^i;\\ e_{\alpha\beta}^i&=a_{16}^i\sigma_\alpha^i+a_{26}^i\sigma_\beta^i+a_{66}^i\tau_{\alpha\beta}^i.\end{aligned}\right\} \qquad (4.1)$$

---

*V.Z. Vlasov, see pp. 287-292 of work cited in footnote ***** on p. 19.

**V.V. Novozhilov, see pp. 28-29 of work cited in footnote ** on p. 21.

***A.L. Gol'denveyzer, see pp. 55-59 of work cited in footnote * on p. 21.

**** A.L. Gol'denveyzer, Additions and corrections to Love's theory of thin shells, Sb. Plastinki i obolochki (Symposium on plates and shells), Gosstroyizdat, 1939.

Solving these equations for the components of the stress tensor and considering (2.6)-(2.8), we obtain*

$$\sigma_\alpha^i = B_{11}^i e_1 + B_{12}^i e_2 + B_{16}^i \omega + \gamma \left(B_{11}^i \varkappa_1 + B_{12}^i \varkappa_2 + B_{16}^i \tau\right); \tag{4.2}$$

$$\sigma_\beta^i = B_{22}^i e_2 + B_{12}^i e_1 + B_{26}^i \omega + \gamma \left(B_{22}^i \varkappa_2 + B_{12}^i \varkappa_1 + B_{26}^i \tau\right); \tag{4.3}$$

$$\tau_{\alpha\beta}^i = B_{16}^i e_1 + B_{26}^i e_2 + B_{66}^i \omega + \gamma \left(B_{16}^i \varkappa_1 + B_{26}^i \varkappa_2 + B_{66}^i \tau\right). \tag{4.4}$$

where for coefficients B_{jk} we have

$$\left.\begin{aligned} B_{11}^i &= \frac{a_{22}^i a_{66}^i - [a_{26}^i]^2}{\Omega_i}; & B_{22}^i &= \frac{a_{11}^i a_{66}^i - [a_{16}^i]^2}{\Omega_i}; \\ B_{12}^i &= \frac{a_{16}^i a_{26}^i - a_{12}^i a_{66}^i}{\Omega_i}; & B_{66}^i &= \frac{a_{11}^i a_{22}^i - [a_{12}^i]^2}{\Omega_i}; \\ B_{16}^i &= \frac{a_{12}^i a_{26}^i - a_{22}^i a_{16}^i}{\Omega_i}; & B_{26}^i &= \frac{a_{12}^i a_{16}^i - a_{11}^i a_{26}^i}{\Omega_i}; \end{aligned}\right\} \tag{4.5}$$

$$\Omega_i = \left(a_{11}^i a_{22}^i - [a_{12}^i]^2\right) a_{66}^i + 2 a_{12}^i a_{16}^i a_{26}^i - a_{11} [a_{26}^i]^2 - a_{22}^i [a_{16}^i]^2. \tag{4.6}$$

Having the values of stresses σ_α^i, σ_β^i, and $\tau_{\alpha\beta}^i$, the remaining three components of the stress tensor are easily determined from the equilibrium equations (1.10)-(1.12). Restricting ourselves to the general form, for the mentioned stresses we obtain

$$\begin{aligned} \tau_{\alpha\gamma}^i = -\frac{1}{H_1^2 H_2} \Big\{ \int \Big[H_1 \frac{\partial}{\partial \alpha} (H_2 \sigma_\alpha^i) - H_1 \frac{\partial H_2}{\partial \alpha} \sigma_\beta^i + \\ + \frac{\partial}{\partial \beta} (H_1^2 \tau_{\alpha\beta}^i) \Big] d\gamma + \varphi_i(\alpha, \beta) \Big\}; \end{aligned} \tag{4.7}$$

$$\begin{aligned} \tau_{\beta\gamma}^i = -\frac{1}{H_2^2 H_1} \Big\{ \int \Big[H_2 \frac{\partial}{\partial \beta} (H_1 \sigma_\beta^i) - H_1 \frac{\partial H_1}{\partial \beta} \sigma_\alpha^i + \\ + \frac{\partial}{\partial \alpha} (H_2^2 \tau_{\alpha\beta}^i) \Big] d\gamma + \psi_i(\alpha, \beta) \Big\}; \end{aligned} \tag{4.8}$$

$$\begin{aligned} \sigma_\gamma^i = \frac{1}{H_1 H_2} \Big\{ \int \Big[H_2 \frac{\partial H_1}{\partial \gamma} \sigma_\alpha^i + H_1 \frac{\partial H_2}{\partial \gamma} \sigma_\beta^i - \\ - \frac{\partial}{\partial \alpha} (H_2 \tau_{\alpha\gamma}^i) - \frac{\partial}{\partial \beta} (H_1 \tau_{\beta\gamma}^i) \Big] d\gamma + \chi_i(\alpha, \beta) \Big\}. \end{aligned} \tag{4.9}$$

Here $\varphi_i(\alpha, \beta)$, $\psi_i(\alpha, \beta)$ and $\chi_i(\alpha, \beta)$ are functions resulting from integration for γ. They may be determined from the conditions at the contact surfaces of adjacent layers and at the external surfaces of the shell *,**.

---

*S.G. Lekhnitskiy, Anizotropnyye plastinki, Gostekhizdat, 1957, pp. 243-244.

SECTION 5. CONDITIONS OF CONTACT OF ADJACENT LAYERS

Previously, in formulating the basic concepts, it was stated that the layers of the shell function concurrently without slippage. In virtue of this the stresses and displacements of individual layers at the bonded surface must satisfy the following contact conditions (Figures 8 and 10):

When $\gamma = \Delta_i$

$$u^i_\alpha = u^{i+1}_\alpha, \quad u^i_\beta = u^{i+1}_\beta, \quad u^i_\gamma = u^{i+1}_\gamma; \tag{5.1}$$

$$\tau^i_{\alpha\gamma} = \tau^{i+1}_{\alpha\gamma}, \quad \tau^i_{\beta\gamma} = \tau^{i+1}_{\beta\gamma}, \quad \sigma^i_\gamma = \sigma^{i+1}_\gamma. \tag{5.2}$$

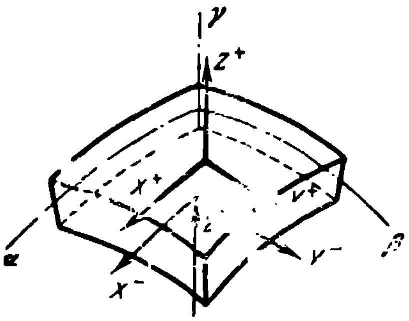

Figure 10

In addition, the stresses of the extreme layers must also satisfy the following conditions at the outer surfaces of the shell:

When $\gamma = -\Delta$

$$\sigma^1_\gamma = -Z^-, \quad \tau^1_{\alpha\gamma} = -X^-, \quad \tau^1_{\beta\gamma} = -Y^-; \tag{5.3}$$

when $\gamma = (h - \Delta)$

---

*S.G. Lekhnitskiy, Bending of heterogeneous, anisotropic, thin plates of symmetrical structure, PMM, Vol. V, No. 1, 1941.

**S.A. Ambartsumyan, Some fundamental equations in the theory of a laminar, thin shell, DAN ArmSSR, VIII, No. 5, 1948.

$$\sigma_{\gamma}^{m+n} = Z^{+}, \quad \tau_{\alpha\gamma}^{m+n} = X^{+},$$
$$\tau_{\beta\gamma}^{m+n} = Y^{+}. \qquad (5.4)$$

where $X^{-}, \ldots, Z^{+}$ are the corresponding components of the force vectors of the surface loads; 1 and (m + n) are the numbers of the outer layers (Figure 11).

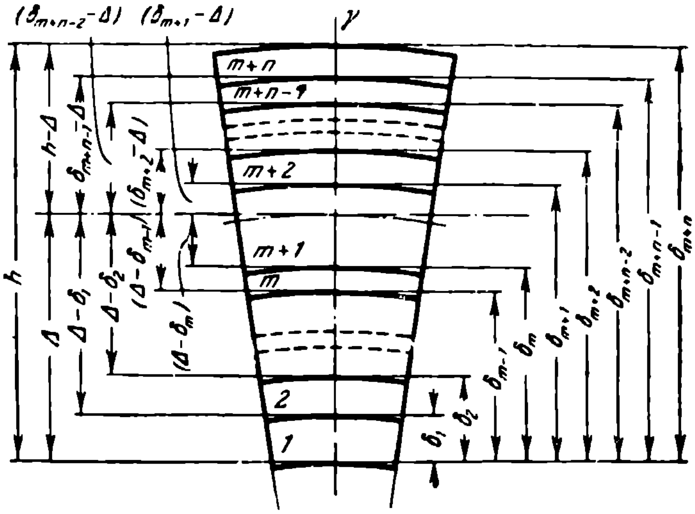

Figure 11

In virtue of the fact that the hypothesis of nondeformable normals was given for the entire stack of the shell, the contact conditions of (5.1) are automatically satisfied. This may be seen from an examination of the initial relationships (2.3)-(2.5).

SECTION 6. INTERNAL FORCES AND MOMENTS

Section 4 of the present chapter established the laws of change in stresses directed along the thickness of the shell; however, as in the theory of homogeneous isotropic shells, instead of the stresses it is convenient to introduce their statically equivalent internal forces and moments acting on the areas of the principal normal cross-sections of the shell.

From the conditions of static equivalence for internal tangential (T_1, T_2, S_{12}, S_{21}) and transverse (N_1, N_2) forces as well as for the bending (M_1, M_2) and torsional (H_{12}, H_{21}) moments referred to a unit length of arc of the corresponding coordinate lines, we have (Figures 8, 11, 12, 13)

$$T_1=\frac{1}{B}\sum_{s=1}^{m+1}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\sigma_\alpha^s H_2\,d\gamma,\quad T_2=\frac{1}{A}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\sigma_\beta^s H_1\,d\gamma; \tag{6.1}$$

$$S_{12}=\frac{1}{B}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\tau_{\alpha\beta}^s H_2\,d\gamma,\quad S_{21}=\frac{1}{A}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\tau_{\beta\alpha}^s H_1\,d\gamma; \tag{6.2}$$

$$N_1=\frac{1}{B}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\tau_{\alpha\gamma}^s H_2\,d\gamma,\quad N_2=\frac{1}{A}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\tau_{\beta\gamma}^s H_1\,d\gamma; \tag{6.3}$$

$$M_1=\frac{1}{B}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\sigma_\alpha^s\gamma H_2\,d\gamma,\quad M_2=\frac{1}{A}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\sigma_\beta^s\gamma H_1\,d\gamma; \tag{6.4}$$

$$H_{12}=\frac{1}{B}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\tau_{\alpha\beta}^s\gamma H_2\,d\gamma,\quad H_{21}=\frac{1}{A}\sum_{s=1}^{m+n}\int\limits_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}\tau_{\beta\alpha}^s\gamma H_1\,d\gamma. \tag{6.5}$$

Here (m + n) is the number of layers of the shell, m is the number of layers below the coordinate surface of the shell, n is the number of remaining layers. (If the coordinate surface of the shell is located within any layer, n is understood to be the number of layers above the coordinate surface plus one, and if the coordinate surface of the shell coincides with any surface of contact, n is understood to be the number of layers above the coordinate surface).

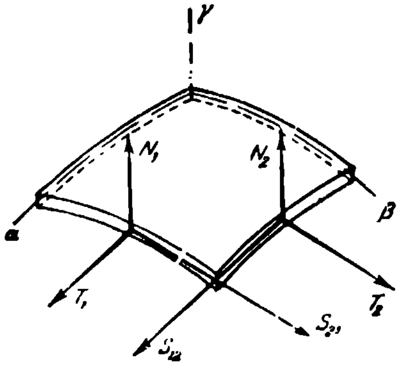

Figure 12

Tangential forces T_1, T_2 and S_{12}, S_{21} are, respectively, the normal and shear forces. The normal forces are considered positive if they are tensile forces. The shear forces are considered positive if, being applied to areas with positive external normals, they act in the directions of the corresponding positive tangents to the coordinate lines $\alpha = \text{const}$, $\beta = \text{const}$. Transverse forces N_1 and N_2 which, upon being applied to areas with positive external normals, are directed along the positive normal to the coordinate surface are assumed to be positive. The positive directions of internal forces are shown in Figure 12.

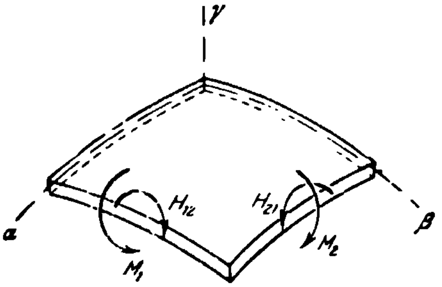

Figure 13

Bending moments M_1 and M_2 are positive if they increase the curvature of the coordinate surface of a shell of positive Gaussian curvature. The torsional moments H_{12} and H_{21} are positive if the vectors of these moments, when laid off according to the right hand screw rule, have directions opposed to the directions of the tangents to coordinate lines $\beta = \text{const.}$ and $\alpha = \text{const.}$ respectively. The positive directions of the moments are shown in Figure 13.

Replacing the stresses with their statically equivalent stresses, henceforth, instead of an arbitrary three-dimensional element of the shell, we may examine the corresponding two-dimensional element of the coordinate surface of a shell under the action of internal forces and moments. Herein the examined element of the coordinate surface of the shell will be considered to possess the derived physical-mechanical characteristics of the corresponding three-dimensional element of the shell.

SECTION 7. EQUILIBRIUM EQUATIONS

In Section 1 of the present chapter we presented the equilibrium equations for a volumetric element of the i-th layer of a shell; however, in subsequent discussion, as in the case of a homogeneous isotropic shell, we shall also be interested in the integral equilibrium conditions of the shell.

Using four sections normal to the coordinate surface of the shell we distinguish a small curvilinear rectangle whose sides coincide with the coordinate lines $\alpha = \text{const.}$ and $\beta = \text{const.}$

The indicated shell element will be acted upon by internal forces in the form of stresses acting on the edges of the shell, external forces in the form of volumetric forces applied to each layer of the discussed shell element, and surface forces applied to the corresponding external and internal surfaces delimiting the discussed shell element.

As was stated in the previous section, all the internal forces may be reduced to their statically equivalent forces and moments which will be applied to the contour lines of the corresponding element of the coordinate surface of the shell. In similar fashion, the external forces may also be reduced to their statically equivalent forces applied to the surface of the corresponding element of the coordinate surface of the shell. Moreover, as usual, we shall disregard moments which appear upon achieving the indicated process of reduction.

In the case of multi-layer shells, for the given external forces applied to the coordinate surface of the shell we have

$$X = X^{+}\left(1+\frac{h-\Delta}{R_1}\right)\left(1+\frac{h-\Delta}{R_2}\right)+X^{-}\left(1-\frac{\Delta}{R_1}\right)\left(1-\frac{\Delta}{R_2}\right)+ \\ +\frac{1}{AB}\sum_{s=1}^{m+n}\int_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)} P_\alpha^s H_1 H_2\, d\gamma; \tag{7.1}$$

$$Y = Y^{+}\left(1+\frac{h-\Delta}{R_1}\right)\left(1+\frac{h-\Delta}{R_2}\right)+Y^{-}\left(1-\frac{\Delta}{R_1}\right)\left(1-\frac{\Delta}{R_2}\right)+ \\ +\frac{1}{AB}\sum_{s=1}^{m+n}\int_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)} P_\beta^s H_1 H_2\, d\gamma; \tag{7.2}$$

$$Z = Z^{+}\left(1+\frac{h-\Delta}{R_1}\right)\left(1+\frac{h-\Delta}{R_2}\right)+Z^{-}\left(1-\frac{\Delta}{R_1}\right)\left(1-\frac{\Delta}{R_2}\right)+ \\ +\frac{1}{AB}\sum_{s=1}^{m+n}\int_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)} P_\gamma^s H_1 H_2\, d\gamma. \tag{7.3}$$

Thus, the problem of equilibrium of a small element of the shell has been reduced to the problem of equilibrium of the corresponding small element of the coordinate surface of the shell under the action of internal (6.1)-(6.5) and external (7.1)-(7.3) forces (Figure 14).

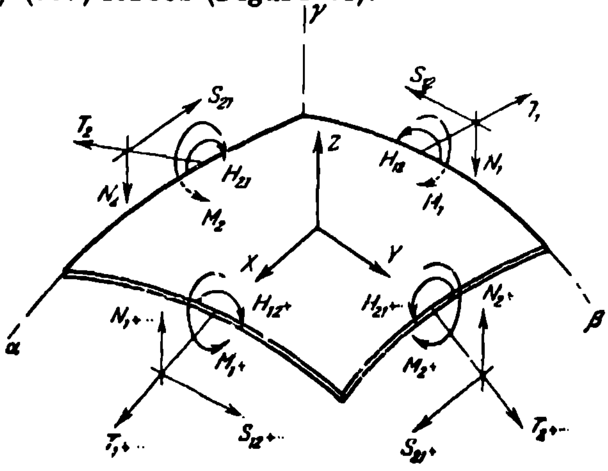

Figure 14

Henceforth, without further reference thereto, within the accuracy of the fundamental hypothesis in expressions (7.1)-(7.3), we shall disregard quantities of the order of h/R_j in comparison with unity.

The equilibrium equations of an anisotropic laminar shell, as is known, cannot differ from the corresponding equations for a homogeneous isotropic shell. In this connection, without going into a detailed derivation, we present the final expressions for the equilibrium equations

$$\left.\begin{aligned} &\frac{\partial BT_1}{\partial \alpha} - \frac{\partial B}{\partial \alpha} T_2 + \frac{\partial AS_{21}}{\partial \beta} + \frac{\partial A}{\partial \beta} S_{12} + ABk_1N_1 = -ABX; \\ &\frac{\partial AT_2}{\partial \beta} - \frac{\partial A}{\partial \beta} T_1 + \frac{\partial BS_{12}}{\partial \alpha} + \frac{\partial B}{\partial \alpha} S_{21} + ABk_2N_2 = -ABY; \\ &-(k_1T_1 + k_2T_2) + \frac{1}{AB}\left(\frac{\partial BN_1}{\partial \alpha} + \frac{\partial AN_2}{\partial \beta}\right) = -Z; \\ &\frac{\partial BM_1}{\partial \alpha} + \frac{\partial AH_{21}}{\partial \beta} + \frac{\partial A}{\partial \beta} H_{12} - \frac{\partial B}{\partial \alpha} M_2 = ABN_1; \\ &\frac{\partial AM_2}{\partial \beta} + \frac{\partial BH_{12}}{\partial \alpha} + \frac{\partial B}{\partial \alpha} H_{21} - \frac{\partial A}{\partial \beta} M_1 = ABN_2; \\ &S_{12} - S_{21} + \frac{H_{12}}{R_1} - \frac{H_{21}}{R_2} = 0. \end{aligned}\right\} \tag{7.4}$$

The derivation of equations (7.4) may be found in the monographs by A. L. Gol'denveyzer* and V.V. Novozhilov. **

The sixth equilibrium equation

$$S_{12} - S_{21} + \frac{H_{12}}{R_1} - \frac{H_{21}}{R_2} = 0 \tag{7.5}$$

is an identity. This is easily seen by inserting into (7.5) the values of S_{ik} and H_{ik} from (6.2) and (6.5) respectively.

SECTION 8. POTENTIAL ENERGY OF DEFORMATION

By virtue of the fundamental hypothesis, from (1.3.3) and (1.3.4) we obtain for the potential energy of deformation of a multi-layer shell

---

*A. L. Gol'denveyzer, see pp. 33-36 of work cited in footnote * on p. 21.

**V.V. Novozhilov, see pp. 34-38 of work cited in footnote ** on p. 21.

$$V = \frac{1}{2}\int\int\left[\sum_{s=1}^{m+n}\int_{(\delta_{s-1}-\Delta)}^{(\delta_s-\Delta)}(\sigma_\alpha^s e_\alpha^s + \sigma_\beta^s e_\beta^s + \right.$$

$$\left. + \tau_{\alpha\beta}^s e_{\alpha\beta}^s)(1+k_1\gamma)(1+k_2\gamma)\,d\gamma\right] AB\,d\alpha\,d\beta. \qquad (8.1)$$

Inserting the values of stresses from (4.2)-(4.4) and of deformations from (2.6)-(2.8) into (8.1) and integrating with respect to variable γ, we obtain the following expression for the potential energy of deformation of an anisotropic laminar shell:

$$V = \frac{1}{2}\int\int[C_{11}\varepsilon_1^2 + 2C_{12}\varepsilon_1\varepsilon_2 + C_{22}\varepsilon_2^2 + C_{66}\omega^2 +$$

$$+ 2C_{16}\omega\varepsilon_1 + 2C_{26}\omega\varepsilon_2]\,AB\,d\alpha\,d\beta + \int\int[K_{11}\varepsilon_1\varkappa_1 +$$

$$+ K_{12}(\varepsilon_1\varkappa_2 + \varepsilon_2\varkappa_1) + K_{22}\varepsilon_2\varkappa_2 + K_{66}\omega\tau +$$

$$+ K_{16}(\varepsilon_1\tau + \omega\varkappa_1) + K_{26}(\varepsilon_2\tau + \omega\varkappa_2)]\,AB\,d\alpha\,d\beta +$$

$$+ \frac{1}{2}\int\int[D_{11}\varkappa_1^2 + 2D_{12}\varkappa_1\varkappa_2 + D_{22}\varkappa_2^2 + D_{66}\tau^2 +$$

$$+ 2D_{16}\tau\varkappa_1 + 2D_{26}\tau\varkappa_2]\,AB\,d\alpha\,d\beta. \qquad (8.2)$$

where coefficients C_{jk}, K_{jk} and D_{jk} have the following form:

$$C_{jk} = \sum_{s=1}^{m+n} B_{jk}^s(\delta_s - \delta_{s-1}); \qquad (8.3)$$

$$K_{jk} = \frac{1}{2}\sum_{s=1}^{m+n} B_{jk}^s[(\delta_s^2 - \delta_{s-1}^2) - 2\Delta(\delta_s - \delta_{s-1})]; \qquad (8.4)$$

$$D_{jk} = \frac{1}{3}\sum_{s=1}^{m+n} B_{jk}^s[(\delta_s^3 - \delta_{s-1}^3) - 3\Delta(\delta_s^2 - \delta_{s-1}^2) + 3\Delta^2(\delta_s - \delta_{s-1})] \qquad (8.5)$$

In the special case in which the coordinate surface of the shell coincides with the lower surface of the shell, Δ becomes zero (Figures 8 and 11), in conjunction with which the expressions for C_{jk}, K_{jk} and D_{jk} are considerably simplified. Assuming in formulas (8.3)-(8.5) that $\Delta = 0$, we obtain*

$$C_{jk} = \sum_{s=1}^{m+n} B_{jk}^s(\delta_s - \delta_{s-1}); \qquad (8.6)$$

---

*S.A. Ambartsumyan, see footnote *** p. 19.

$$K_{jk}=\frac{1}{2}\sum_{s=1}^{m+n}B_{jk}^{s}(\delta_s^2-\delta_{s-1}^2); \tag{8.7}$$

$$D_{jk}=\frac{1}{3}\sum_{s=1}^{m+n}B_{jk}^{s}(\delta_s^3-\delta_{s-1}^3). \tag{8.8}$$

In all three components of formula (8.2) both integrals extend over the entire coordinate surface of the shell. In (8.2) the first component is the potential energy of elongation and shear, the third component is the potential energy of bending and torsion and, finally, the second component is the potential energy which the shell acquires due to interaction of bending, torsion, elongation and shear.

By analogy with the theory of bending of isotropic shells and anisotropic plates for coefficients C_{jk}, K_{jk} D_{jk} let us introduce the concept of stiffness. Let C_{11} and C_{22} be the tension and compression stiffnesses along coordinate lines $\beta = \text{const}$, and $\alpha = \text{const}$; let C_{66} be the shear stiffness in the plane tangent to the coordinate stiffness, characterizing the influence of elongation along the coordinate lines $\beta = \text{const}$, and $\alpha = \text{const}$ on shear in the tangential plane. Let D_{11} and D_{22} represent the bending stiffnesses about the axes tangent to the coordinate lines $\alpha = \text{const}$, and $\beta = \text{const}$; let D_{66} represent the torsional stiffness; let D_{16} and D_{26} represent the secondary stiffnesses of bending and torsion characterizing the influence of bending about the axes tangent to coordinate lines $\alpha = \text{const}$, and $\beta = \text{const}$, on torsion. Let K_{11} and K_{12} represent the stiffnesses of interaction of tension and bending characterizing the interaction of tension and bending along lines $\beta = \text{const}$, and $\alpha = \text{const}$; let K_{66} represent the stiffness of interaction of torsion and shear; let K_{16} and K_{66} represent the secondary stiffnesses of interaction characterizing the mutual effect of bending on shear and tension on torsion.

Formula (8.2) did not take into account those components of the potential energy of deformation which in comparison with the other components were of the orderof h/R.

SECTION 9. ELASTICITY RELATIONSHIPS

In the theory of shells the equations establishing the relationship between internal stresses and deformations of the coordinate surface of a shell are known as elasticity relationships. These relationships may be derived by elementary means. Inserting the values of stresses from (4.2)-(4.4) into formulas (6.1)-(6.5) and integrating, thereby retaining only the lowest powers of γ (the zero and first powers), we obtain the simplest elasticity relationships for the general case of a multi-layer anisotropic shell

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{16}\omega + K_{11}\varkappa_1 + K_{12}\varkappa_2 + K_{16}\tau; \tag{9.1}$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + C_{26}\omega + K_{22}\varkappa_2 + K_{12}\varkappa_1 + K_{26}\tau; \tag{9.2}$$

$$S_{12} = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + K_{66}\tau + K_{16}\varkappa_1 + K_{26}\varkappa_2; \tag{9.3}$$

$$S_{21} = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + K_{66}\tau + K_{16}\varkappa_1 + K_{26}\varkappa_2; \tag{9.4}$$

$$M_1 = D_{11}\varkappa_1 + D_{12}\varkappa_2 + D_{16}\tau + K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{16}\omega; \tag{9.5}$$

$$M_2 = D_{22}\varkappa_2 + D_{12}\varkappa_1 + D_{26}\tau + K_{22}\varepsilon_2 + K_{12}\varepsilon_1 + K_{26}\omega; \tag{9.6}$$

$$H_{12} = D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2 + K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2; \tag{9.7}$$

$$H_{21} = D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2 + K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2, \tag{9.8}$$

where for stiffnesses C_{jk}, K_{jk} and D_{jk} we have the known expressions (8.3)-(8.8).

Here and in subsequent text the formulas for determining N_1 and N_2 will not be derived, since in the presence of the well-known elasticity relationships for M_j and H_{jk} the transverse forces N_1 and N_2 may be derived from the fourth and fifth equilibrium equations of (7.4).

In examining the elasticity relationships of (9.3), (9.4), (9.7), (9.8) and equation (7.5) it is not difficult to see that between them there is an internal contradiction, namely: the mentioned relationships do not satisfy the sixth equilibrium equation, that is, the sixth equation is not an identity.*

However, we may derive other sufficiently simple elasticity relationships which do not contain the above-mentioned contradiction. In obtaining relationships (9.1)-(9.8) we have everywhere disregarded quantities of the order of $k_j h$ in comparison with unity. However, if we deviate from strict logic and in all formulas in (6.1)-(6.5), with the exception of (6.2), neglect quantities of the order of $k_j h$ in comparison with unity, while in formulas (6.2) we leave quantities of this same order of magnitude, we obtain new elasticity relationships which identically satisfy the sixth equilibrium equation

---

*A. L. Gol'denveyzer, Applicability of general theorems of elasticity theory to thin shells, PMM, Vol. VIII, No. 1, 1944.

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{16}\omega + K_{11}\varkappa_1 + K_{12}\varkappa_2 + K_{16}\tau; \tag{9.9}$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + C_{26}\omega + K_{22}\varkappa_2 + K_{12}\varkappa_1 + K_{26}\tau; \tag{9.10}$$

$$\begin{aligned} S_{12} = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + K_{66}\tau + K_{16}\varkappa_1 + K_{26}\varkappa_2 + \\ + k_2(K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2 + D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2); \end{aligned} \tag{9.11}$$

$$\begin{aligned} S_{21} = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + K_{66}\tau + K_{16}\varkappa_1 + K_{26}\varkappa_2 + \\ + k_1(K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2 + D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2); \end{aligned} \tag{9.12}$$

$$M_1 = D_{11}\varkappa_1 + D_{12}\varkappa_2 + D_{16}\tau + K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{16}\omega; \tag{9.13}$$

$$M_2 = D_{22}\varkappa_2 + D_{12}\varkappa_1 + D_{26}\tau + K_{22}\varepsilon_2 + K_{12}\varepsilon_1 + K_{26}\omega; \tag{9.14}$$

$$H_{12} = D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2 + K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2; \tag{9.15}$$

$$H_{21} = D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2 + K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2. \tag{9.16}$$

where, as before, for C_{jk}, K_{jk} and D_{jk} we have formulas (8.3)-(8.8).

Elasticity relationships (9.9)-(9.16) may be obtained if we use the expression for potential energy (8.2) and apply the method proposed by V.V. Novozhilov* for the case of a homogeneous isotropic shell; this is not derived here, since the resulting elasticity relationships are in complete agreement with relationships (9.9)-(9.16).

By adopting a purely formal approach to the problem it would be possible to achieve further refinement of the elasticity relationships;** however, we shall not do this here, since, in accordance with the well-founded remark of A. L. Gol'denveyzer,*** an attempt to refine the elasticity relationships without proceeding beyond the bounds of the hypothesis of nondeformable normals has no chance of success and hence the best of all of the approximate variants of the elasticity relationships must be considered to be that which does not contain formal contradictions and leads to the simplest calculations.

Finally, we will note that, as in the theory of homogeneous isotropic shells, there is not and, regrettably, cannot be a single approach to the problem of choosing one or another elasticity relationship. In this connection, in subsequent discussion, in showing a certain preference for relationships (9.9)-(9.16) containing no formal contradictions, we shall, wherever possible, find wide use for the simplest relationships in (9.9)-(9.8).

---

*V.V. Novozhilov, see pp. 45-48 of work cited in footnote ** p. 21.

**A.I. Lur'ye, General theory of elastic thin shells, PMM, Vol. IV, No. 2, 1940.

***A. L. Gol'denveyzer, see p. 71 of work cited in footnote * p. 21.

SECTION 10. BOUNDARY CONDITIONS

As is known, the arbitrary functions contained in the general integral of differential equations in the theory of shells must be determined from the boundary conditions. The boundary conditions of an anisotropic laminar shell, when the hypothesis of nondeformable normals is given for the entire stack of the shell as a whole, do not differ from the corresponding boundary conditions given for a homogeneous isotropic shell, given the hypothesis of nondeformable normals.

In actual designs we may encounter extremely varied types of shell supports and this great variety of structural solutions for the supports of real shells, regrettably, cannot be represented in the form of mathematical models — boundary conditions. In this connection we shall here present only a few of the possible variants of boundary conditions.

In subsequent discussion we shall be interested in those shells which have an edge or edges coinciding with the lines of curvature of the coordinate surface of the shell.

For the sake of brevity, the boundary conditions will be given only for the edge defined by coordinate line $\alpha = \alpha_0 = \text{const}$.

I. Homogeneous boundary conditions:

a) Free edge

$$T_1 = 0, \quad S_{12} + \frac{H_{12}}{R_2} = 0, \quad N_1 + \frac{1}{B}\frac{\partial H_{12}}{\partial \beta} = 0, \quad M_1 = 0; \quad (10.1) \tag{10.1}$$

b) Hinged edge

$$M_1 = 0, \quad u = 0, \quad v = 0, \quad w = 0; \tag{10.2}$$

c) Hinged edge free in the tangential direction

$$T_1 = 0, \quad M_1 = 0, \quad w = 0, \quad v = 0, \tag{10.3}$$

or

$$S_{12} + \frac{H_{12}}{R_2} = 0, \quad M_1 = 0, \quad w = 0, \quad u = 0; \tag{10.4}$$

d) Hinged edge free in the normal direction

$$M_1 = 0, \quad N_1 + \frac{1}{B}\frac{\partial H_{12}}{\partial \beta} = 0, \quad u = 0, \quad v = 0; \tag{10.5}$$

e) Absolutely fixed edge

$$u=0,\quad v=0,\quad w=0,\quad \varphi=-\frac{1}{A}\frac{\partial w}{\partial \alpha}+\frac{u}{R_1}=0. \tag{10.6}$$

In the last relationship φ is the angle of rotation of the normal of the coordinate surface of the shell around the tangent to the line $\alpha=\alpha_0=\text{const}$.

II. Inhomogeneous boundary conditions:

a) Loaded edge

$$\left.\begin{array}{l} T_1=T_1^*,\quad S_{12}+\frac{H_{12}}{R_2}=S_{12}^*, \\ N_1+\frac{1}{B}\frac{\partial H_{12}}{\partial \beta}=N_1^*,\quad M_1=M_1^*, \end{array}\right\} \tag{10.7}$$

where T_1^*, S_{12}^*, N_1^*, M_1^* are the forces applied to the edge under consideration; in the special case some of these forces may be equal to 0;

b) Hinged, fixed-bearing, moment-loaded edge

$$M_1=M_1^*,\quad u=0,\quad v=0,\quad w=0; \tag{10.8}$$

c) Hinged edge free in the tangential direction, loaded by moment and tangential force (in the special case one of the loads may be equal to zero)

$$T_1=T_1^*,\quad M_1=M_1^*,\quad w=0,\quad v=0, \tag{10.9}$$

or

$$S_{12}+\frac{H_{12}}{R_2}=S^*,\quad M_1=M_1^*,\quad w=0,\quad u=0; \tag{10.10}$$

d) Hinged edge free in the normal direction and loaded by moment and transverse force (in the special case one of the loads may be equal to zero).

$$M_1=M_1^*,\quad N_1+\frac{1}{B}\frac{\partial H_{12}}{\partial \beta}=N_1^*,\quad u=0,\quad v=0; \tag{10.11}$$

e) Displaced edge

$$u=u^*,\quad v=v^*,\quad w=w^*,\quad \varphi=\varphi^*, \tag{10.12}$$

where u^*, v^*, $w^*$and φ^* are the given displacements and rotation of the discussed edge of the shell. In the special case certain of these displacements may be equal to zero.

Nonhomogeneous boundary conditions may also be combined; in this case it is necessary to see that the given forces and displacements are not incompatible.

In similar manner we may also write the boundary conditions for the edge defined by the coordinate line $\beta = \beta_0 = \text{const}$.

In the case where the shell is completely lacking a boundary contour (completely closed shell) or the boundary contour is determined only from the lines of one coordinate (partially closed shell), the boundary conditions for the direction of the closed coordinates no longer have the usual expressions and are replaced by conditions of periodicity with a period insuring nonambiguity of displacements at any point of the discussed closed line of the coordinates.

The results derived here may be generalized for the case of junction of the edge of the discussed shell with any elastic structural element (beam, plate, shell, etc.)

The reader will find these boundary conditions discussed in detail in the cited works of A. L. Gol'denveyser and V.V. Novozhilov as well as in the course on the mathematical theory of elasticity by A. Love.*

SECTION 11. ADDITIONAL REMARKS CONCERNING THE CONDITIONS OF CONTACT OF ADJACENT LAYERS AND THE CONDITIONS AT THE OUTER SURFACES OF A SHELL

In Section 5 of the present chapter, in presenting the conditions for contact of adjacent layers and conditions at the outer surfaces of a shell we limited our remarks to the observation that the geometric conditions of contact in (5.1) are automatically satisfied. Let us show that the conditions of contact in (5.2) and the conditions at the surfaces (5.3) and (5.4) reduce to the first three equilibrium equations in (7.4). For the sake of brevity and without disturbing the general validity, we shall assume that coordinate surface ($\gamma = 0$) coincides with the lower surface of the shell, that is, $\Delta = 0$ (Figure 8), while the load components of the outer surfaceX^+, X^-, Y^+, Y^-, $Z^+$and $Z^-$are equal to zero.

From (4.7) and after certain transformations we obtain for the i-th layer of shell the following value of shearing stress $\tau_{\alpha\gamma}$:

---

*A. Love, Mathematical theory of elasticity, 1935 (Russian translation from fourth English edition 1927) p. 563-564.

$$\begin{aligned}\tau^i_{\alpha\gamma} = -\frac{1}{H_1^2 H_2}\Bigg[& A\frac{\partial}{\partial\alpha}\int_{\delta_{i-1}}^{\gamma} H_2\sigma^i_\alpha\, d\gamma + Ak_1\frac{\partial}{\partial\alpha}\int_{\delta_{i-1}}^{\gamma}\gamma H_2\sigma^i_\alpha\, d\gamma - \\ & -\frac{\partial B}{\partial\alpha}\int_{\delta_{i-1}}^{\gamma} H_1\sigma^i_\beta\, d\gamma - k_1\frac{\partial B}{\partial\alpha}\int_{\delta_{i-1}}^{\gamma}\gamma H_1\sigma^i_\beta\, d\gamma + \frac{\partial A}{\partial\beta}\int_{\delta_{i-1}}^{\gamma} H_1\tau^i_{\alpha\beta}\, d\gamma + \\ & + k_2\frac{\partial A}{\partial\beta}\int_{\delta_{i-1}}^{\gamma}\gamma H_1\tau^i_{\alpha\beta}\, d\gamma + A\frac{\partial}{\partial\beta}\int_{\delta_{i-1}}^{\gamma} H_1\tau^i_{\alpha\beta}\, d\gamma + \\ & + k_1 A\frac{\partial}{\partial\beta}\int_{\delta_{i-1}}^{\gamma}\gamma H_1\tau^i_{\alpha\beta}\, d\gamma\Bigg] + \frac{1}{H_1^2 H_2}\varphi_i(\alpha,\beta). \end{aligned} \qquad (11.1)$$

From the conditions at the surface (5.4) we have $\tau^p_{\alpha\gamma} = 0$ with $\gamma = \delta_p$, where p = m + n is the number of layers (Figure 11). By virtue of this from (11.1) we obtain

$$\begin{aligned} 0 = -\Big[& A\frac{\partial}{\partial\alpha}(BT_1^p) + Ak_1\frac{\partial}{\partial\alpha}(BM_1^p) - \frac{\partial B}{\partial\alpha}AT_2^p - \\ & - k_1\frac{\partial B}{\partial\alpha}AM_2^p + \frac{\partial A}{\partial\beta}AS_{21}^p + k_2\frac{\partial A}{\partial\beta}AH_{21}^p + A\frac{\partial}{\partial\beta}(AS_{21}^p) + \\ & + k_1 A\frac{\partial}{\partial\alpha}(AH_{21}^p)\Big] + \varphi_p(\alpha,\beta). \end{aligned} \qquad (11.2)a$$

From the contact conditions adjacent layers (5.2) we have $\tau^p_{\alpha\gamma} = \tau^{p-1}_{\alpha\gamma}$ when $\gamma = \delta_{p-1}$. By virtue of this from (11.1) we obtain

$$\begin{aligned} \varphi_p(\alpha,\beta) = -\Big[& A\frac{\partial}{\partial\alpha}(BT_1^{p-1}) + Ak_1\frac{\partial}{\partial\alpha}(BM_1^{p-1}) - \frac{\partial B}{\partial\alpha}AT_2^{p-1} - \\ & - k_1\frac{\partial B}{\partial\alpha}AM_2^{p-1} + \frac{\partial A}{\partial\beta}AS_{21}^{p-1} + k_2\frac{\partial A}{\partial\beta}AH_{21}^{p-1} + \\ & + A\frac{\partial}{\partial\beta}(AS_{21}^{p-1}) + k_1 A\frac{\partial}{\partial\alpha}(AH_{21}^{p-1})\Big] + \varphi_{p-1}(\alpha,\beta). \end{aligned} \qquad (11.2)b$$

From the contact conditions (5.2) for the i-th and (i-1)-th layers we have $\tau^i_{\alpha\gamma} = \tau^{i-1}_{\alpha\gamma}$ with $\gamma = \delta_{i-1}$. Then we obtain

$$\varphi_i(\alpha,\ \beta) = -\Big[A\frac{\partial}{\partial\alpha}(BT_1^{i-1}) + Ak_1\frac{\partial}{\partial\alpha}(BM_1^{i-1}) - \frac{\partial B}{\partial\alpha}AT_2^{i-1} - \\ - k_1\frac{\partial B}{\partial\alpha}AM_2^{i-1} + \frac{\partial A}{\partial\beta}AS_{21}^{i-1} + k_2\frac{\partial A}{\partial\beta}AH_{21}^{i-1} + \\ + A\frac{\partial}{\partial\beta}(AS_{21}^{i-1}) + k_1A\frac{\partial}{\partial\alpha}(AH_{21}^{i-1})\Big] + \varphi_{i-1}(\alpha,\ \beta). \qquad (11.2)\text{c}$$

From contact conditions (5.2) we have $\tau_{\alpha\gamma}^{(2)} = \tau_{\alpha\gamma}^{(1)}$ with $\gamma = \delta_1$. Then we obtain:

$$\varphi_2(\alpha,\ \beta) = -\Big[A\frac{\partial}{\partial\alpha}(BT_1^{(1)}) + Ak_1\frac{\partial}{\partial\alpha}(BM_1^{(1)}) - \frac{\partial B}{\partial\alpha}AT_2^{(1)} - \\ - k_1\frac{\partial B}{\partial\alpha}AM_2^{(1)} + \frac{\partial A}{\partial\beta}AS_{21}^{(1)} + k_2\frac{\partial A}{\partial\beta}AH_{21}^{(1)} + \\ + A\frac{\partial}{\partial\beta}(AS_{21}^{(1)}) + k_1A\frac{\partial}{\partial\alpha}(AH_{21}^{(1)})\Big] + \varphi_1(\alpha,\ \beta). \qquad (11.2)\text{d}$$

From the conditions at the surface in (5.3) we have $\tau_{\alpha\gamma}^{(1)} = 0$ with $\gamma = \delta_0 = 0$.

Then we obtain

$$\varphi_1(\alpha,\ \beta) = 0. \qquad (11.2)\text{e}$$

As is easily seen, in formulas (11.2) the quantities T_1^i, T_2^i, S_{12}^i, S_{21}^i, M_1^i, M_2^i, H_{12}^i and H_{21}^i are the internal forces and moments of the individual layers.

By way of example let us present the expressions for one force and one moment for the i-th layer:

$$T_1^i = \frac{1}{B}\int\limits_{\delta_{i-1}}^{\delta_i} H_2\sigma_\alpha^i\,d\gamma, \quad M_1^i = \frac{1}{B}\int\limits_{\delta_{i-1}}^{\delta_i}\gamma H_2\sigma_\alpha^i\,d\gamma \qquad (11.3)$$

From an examination of formulas (11.2) it is evident that by successive substitution we may determine all the values of the integrand $\varphi_i(\alpha,\ \beta)$ and $\varphi_p(\alpha,\ \beta)$ inclusively, which is determined from (11.2b). Inserting the values of $\varphi_p(\alpha,\ \beta)$ into (11.2a), we obtain

$$\sum_{s=1}^{p}\Big[A\frac{\partial}{\partial\alpha}(BT_1^s) + Ak_1\frac{\partial}{\partial\alpha}(BM_1^s) - \frac{\partial B}{\partial\alpha}AT_2^s - k_1\frac{\partial B}{\partial\alpha}AM_2^s + \\ + \frac{\partial A}{\partial\beta}AS_{21}^s + k_2\frac{\partial A}{\partial\beta}AH_{21}^s + A\frac{\partial}{\partial\beta}(AS_{21}^s) + k_1A\frac{\partial}{\partial\alpha}(AH_{21}^s)\Big] = 0, \qquad (11.4)$$

whence, by virtue of (6.1)-(6.5) and (11.3), we obtain

$$\begin{aligned}\frac{\partial}{\partial\alpha}(BT_1)-T_2\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\beta}(AS_{21})+S_{21}\frac{\partial A}{\partial\beta}+\\+k_1\frac{\partial}{\partial\alpha}(BM_1)+k_1\frac{\partial}{\partial\beta}(AH_{21})+k_2H_{21}\frac{\partial A}{\partial\beta}-k_1M_2\frac{\partial B}{\partial\alpha}=0.\end{aligned}\qquad(11.5)$$

On the basis of the fourth and sixth equilibrium equations in (7.4), equation (11.5) takes the following form:

$$\frac{\partial BT_1}{\partial\alpha}-\frac{\partial B}{\partial\alpha}T_2+\frac{\partial AS_{21}}{\partial\beta}+\frac{\partial A}{\partial\beta}S_{12}+ABk_1N_1=0.\qquad(11.6)$$

Comparing (11.6) with the first equilibrium equation in (7.4), we notice complete agreement of these two equations. In similar manner it is easily shown that by satisfying the remaining conditions at the outer surfaces in (5.3) and (5.4) and the contact conditions for adjacent layers in (5.2) we arrive at the second and third equilibrium equations in (7.4).

Thus, fulfillment of all the conditions at the outer and contact surfaces of an arbitrarily laminar anisotropic shell reduces to equilibrium equations for an element of the given shell. Similar results in a few special cases of a plate* and a shell** have been derived elsewhere.

SECTION 12. SPECIAL CASES OF ANISOTROPY OF THE MATERIAL OF THE SHELL LAYERS

Let us investigate the changes in the fundamental relationships in the theory of a laminar anisotropic shell in the special cases of anisotropy of the material of the shell layers.

1. Shells Consisting of Orthotropic Layers. Let the shell consist of orthotropic layers and the principal direction of elasticity at each point of each layer coincide with the directions of coordinate lines α, β and γ, that is, at each point of each layer one of the planes of elastic symmetry is parallel to the coordinate surface of the shell and the other two planes are perpendicular to the coordinate lines $\alpha=\text{const.}$ and $\beta=\text{const.}$

In this case, for the desired elastic constants of the i-th layer of the shell, on the basis of (1.3.7), (1.3.8) and (1.4.5), we have

$$\left.\begin{aligned}&a^i_{11}=\frac{1}{E^i_1},\quad a^i_{22}=\frac{1}{E^i_2},\quad a^i_{66}=\frac{1}{G^i_{12}},\\&a^i_{12}=-\frac{\nu^i_2}{E^i_2}=-\frac{\nu^i_1}{E^i_1},\quad a^i_{16}=0,\quad a^i_{26}=0.\end{aligned}\right\}\qquad(12.1)$$

---

*S.G. Lekhnitskiy, see footnote *, p. 34.

**S.A. Ambartsumyan, see footnote **, p. 34.

By virtue of this, from (4.5) and (4.6), for coefficients B_{jk} we have

$$\left.\begin{aligned} B^l_{11} &= \frac{E^l_1}{1-\nu^l_1\nu^l_2}, \quad B_{22} = \frac{E^l_2}{1-\nu^l_1\nu^l_2}, \quad B^l_{66} = G^l_{12}, \\ B^l_{12} &= \frac{\nu^l_2 E^l_1}{1-\nu^l_1\nu^l_2} = \frac{\nu^l_1 E^l_2}{1-\nu^l_1\nu^l_2}, \quad B^l_{16} = B^l_{26} = 0. \end{aligned}\right\} \tag{12.2}$$

From (4.2)-(4.4), for the principal directions in the layers of the shell we have

$$\left.\begin{aligned} \sigma^l_\alpha &= \frac{E^l_1}{1-\nu^l_1\nu^l_2}\left[\varepsilon_1 + \nu^l_2\varepsilon_2 + \gamma\left(\varkappa_1 + \nu^l_2\varkappa_2\right)\right]; \\ \sigma^l_\beta &= \frac{E^l_2}{1-\nu^l_1\nu^l_2}\left[\varepsilon_2 + \nu^l_1\varepsilon_1 + \gamma\left(\varkappa_2 + \nu^l_1\varkappa_1\right)\right]; \end{aligned}\right\} \tag{12.3}$$

$$\tau^l_{\alpha\beta} = G^l_{12}(\omega + \gamma\tau).$$

The formulas for stiffness in (8.3)-(8.8) remain unchanged, hence the values of B_{jk} must be taken from (12.2) and we find that

$$C_{16} = C_{26} = 0, \quad K_{16} = K_{26} = 0, \quad D_{16} = D_{26} = 0. \tag{12.4}$$

From (9.1)-(9.8) and (9.9)-(9.16) we obtain the following two groups of elasticity relationships:

a) the first group is derived from (9.1)-(9.8)

$$\left.\begin{aligned} T_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + K_{11}\varkappa_1 + K_{12}\varkappa_2; \\ T_2 &= C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + K_{22}\varkappa_2 + K_{12}\varkappa_1; \\ S &= S_{12} = S_{21} = C_{66}\omega + K_{66}\tau; \\ M_1 &= D_{11}\varkappa_1 + D_{12}\varkappa_2 + K_{11}\varepsilon_1 + K_{12}\varepsilon_2; \\ M_2 &= D_{22}\varkappa_2 + D_{12}\varkappa_1 + K_{22}\varepsilon_2 + K_{12}\varepsilon_1; \\ H &= H_{12} = H_{21} = D_{66}\tau + K_{66}\omega; \end{aligned}\right\} \tag{12.5}$$

b) the second group is derived from (9.9)-(9.16); it differs from the first group only in the expressions for S_{12} and S_{21}, wherein we have

$$\left.\begin{aligned} S_{12} &= C_{66}\omega + K_{66}\tau + k_2(K_{66}\omega + D_{66}\tau); \\ S_{21} &= C_{66}\omega + K_{66}\tau + k_1(K_{66}\omega + D_{66}\tau). \end{aligned}\right\} \tag{12.6}$$

From (3.2) we obtain the following expression for the potential energy of deformation:

$$V=\int\int\Big\{\frac{1}{2}(C_{11}\varepsilon_1^2+2C_{12}\varepsilon_1\varepsilon_2+C_{22}\varepsilon_2^2+C_{66}\omega^2)+$$
$$+[K_{11}\varepsilon_1\varkappa_1+K_{12}(\varepsilon_1\varkappa_2+\varepsilon_2\varkappa_1)+K_{22}\varepsilon_2\varkappa_2+K_{66}\omega\tau]+$$
$$+\frac{1}{2}(D_{11}\varkappa_1^2+2D_{12}\varkappa_1\varkappa_2+D_{22}\varkappa_2^2+D_{66}\tau^2)\Big\}AB\,d\alpha\,d\beta. \qquad (12.7)$$

The expressions for deformation and changes in curvature and torsion (2.9)-(2.17) as well as the equilibrium equations in (7.4) and the equations of compatibility of deformations (3.1)-(3.3), as is easily seen, remain unchanged.

2. Shells Consisting of Isotropic Layers. Let the shell consist of different isotropic layers. In this case, on the basis of (1.3.12), we have for the elastic constants of the i-th layer of the shell

$$\left.\begin{aligned}&a_{11}^i=a_{22}^i=\frac{1}{E^i},\quad a_{66}^i=\frac{1}{G^i}=\frac{2(1+\nu^i)}{E^i},\\&a_{12}=-\frac{\nu^i}{E^i},\quad a_{16}=a_{26}=0.\end{aligned}\right\} \qquad (12.8)$$

By virtue of this, from (4.5) and (4.6), for coefficients B_{jk} we have

$$\left.\begin{aligned}&B_{11}^i=B_{22}^i=\frac{E^i}{1-(\nu^i)^2},\quad B_{66}^i=\frac{E^i}{2(1+\nu^i)},\\&B_{12}^i=\frac{\nu^iE^i}{1-(\nu^i)^2},\quad B_{16}^i=B_{26}^i=0.\end{aligned}\right\} \qquad (12.9)$$

From (12.3), for the primary stresses in the layers of the shell we obtain

$$\left.\begin{aligned}&\sigma_\alpha^i=\frac{E^i}{1-(\nu^i)^2}[\varepsilon_1+\nu^i\varepsilon_2+\gamma(\varkappa_1+\nu^i\varkappa_2)];\\&\sigma_\beta^i=\frac{E_i}{1-(\nu^i)^2}[\varepsilon_2+\nu^i\varepsilon_1+\gamma(\varkappa_2+\nu^i\varkappa_1)];\\&\tau_{\alpha\beta}^i=\frac{E^i}{2(1+\nu^i)}(\omega+\gamma\tau).\end{aligned}\right\} \qquad (12.10)$$

The formulas for determining stiffnesses (8.3)-(8.8) remain unchanged and the values of B_{jk} must be taken from (12.9), whence along with (12.4) we also have

$$C=C_{11}=C_{22},\quad K_{11}=K_{22}=K,\quad D=D_{11}=D_{22}. \qquad (12.11)$$

The formulas for internal forces (12.5) and (12.6) as well as the expressions for potential energy (12.7) remain unchanged and we need only consider the new values of stiffness.

As in the case of a shell consisting of orthotropic layers, we find there is no change in these expressions for deformation and changes in curvature and torsion (2.9)-(2.17), equilibrium equations (7.4) and the equations for compatibility of deformations (3.1)-(3.3).

3. Shells Consisting of Transversely Isotropic Layers. Let us discuss three cases.

a) Let the principal directions of elasticity at each point of each layer coincide with the directions of coordinate lines $\beta = \text{const}$. In this case, as is easily seen, it is necessary to use the relationships given for the case of a shell consisting of orthotropic layers; hence, on the basis of (1.3.10) and (1.3.11), for each layer we must assume

$$\left.\begin{array}{c} E_1^i = (E')^i, \quad E_2^i = E^i, \quad G_{12}^i = (G')^i, \\ \nu_1 = \nu', \quad \nu_2 = \dfrac{E}{E'}\nu'. \end{array}\right\} \qquad (12.12)$$

b) Let the principal directions of elasticity at each point of each layer coincide with the directions of coordinate lines $\alpha = \text{const}$. In this case it is also necessary to use the relationships given for the case of a shell consisting of orthotropic layers. Hence for each layer we must assume

$$\left.\begin{array}{c} E_1^i = E^i, \quad E_2^i = (E')^i, \quad G_{12}^i = (G')^i, \\ \nu_2 = \nu', \quad \nu_1 = \dfrac{E}{E'}\nu'. \end{array}\right\} \qquad (12.13)$$

c) The principal directions of elasticity at each point of each layer are normal to the coordinate surface of the shell; that is, at each point of each layer the planes of isotropy of the material of the layer are parallel to the coordinate surface of the shell. In this case, as is easily seen, it is necessary to use the relationship given for the case of a shell consisting of different isotropic layers; hence for each layer we must assume

$$E_1^i = E_2^i = E^i, \quad G_{12} = G, \quad \nu_1 = \nu_2 = \nu. \qquad (12.14)$$

SECTION 13. SHELLS CONSISTING OF AN ODD NUMBER OF LAYERS SYMMETRICALLY ARRANGED RELATIVE TO THE COORDINATE SURFACE

Let us discuss shells consisting of an odd number (2m + 1) of homogeneous anisotropic layers (Figure 15).

Layers symmetrically arranged relative to the coordinate surface of the shell have identical thicknesses and identical physical-mechanical properties.

The coordinate surface is median surface both of the middle layer and of the entire shell as a whole.

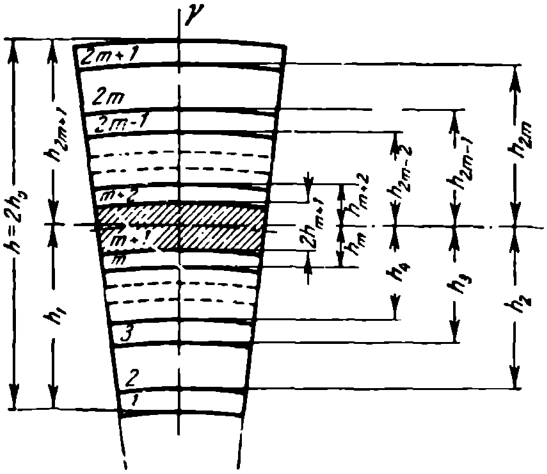

Figure 15

As in the general case of an anisotropic laminar shell, it is here assumed that all the layers function simultaneously without slippage and at each point of each layer there is only one plane of elastic symmetry parallel to the median surface of the shell*,**,***,****.

By virtue of symmetry we have (see Figures 8 and 12)

$$\left.\begin{aligned} m+n=2m+1,\quad \Delta-\delta_m=\delta_{m+1}-\Delta=h_{m+1},\\ h_1=h_{2m+1}=\Delta,\quad \delta_s-\delta_{s-1}=h_s-h_{s+1}.\end{aligned}\right\} \qquad (13.1)$$

Whereupon from (8.3)-(8.5) we obtain for the stiffness of a symmetrically assembled shell

---

*S.A. Ambartsumyan, see footnote **, on p. 28.

**S.A. Ambartsumyan, Calculation of laminar shells of revolution, DAN ArmSSR, Vol. XI, No. 2, 1949.

***S.A. Ambartsumyan, Calculation of shallow cylindrical shells consisting of anisotropic layers, Izvestiya AN Arm SSR (FMYeiT nauki), Vol. IV, No. 5, 1951.

****S.A. Ambartsumyan, Thermal stresses in laminar anisotropic shells, Izvestiya AN Arm SSR (FMYeit Nauki), Vol. V, No. 6, 1952.

$$C_{jk}=2\left[B_{jk}^{m+1}h_{m+1}+\sum_{s=1}^{m}B_{jk}^{s}(h_s-h_{s+1})\right]; \tag{13.2}$$

$$D_{jk}=\frac{2}{3}\left[B_{jk}^{m+1}h_{m+1}^3+\sum_{s=1}^{m}B_{jk}^{s}(h_s^3-h_{s+1}^3)\right]; \tag{13.3}$$

$$K_{jk}=0. \tag{13.4}$$

From (9.1)-(9.8) and (9.9)-(9.16) and by virtue of (13.2)-(13.4) we obtain the following two groups of elasticity relationships:

a) The first group is derived from (9.1)-(9.8)

$$\left.\begin{aligned}
&T_1=C_{11}\varepsilon_1+C_{12}\varepsilon_2+C_{16}\omega; \quad T_2=C_{22}\varepsilon_2+C_{12}\varepsilon_1+C_{26}\omega;\\
&S=S_{12}=S_{21}=C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2;\\
&M_1=D_{11}\varkappa_1+D_{12}\varkappa_2+D_{16}\tau; \quad M_2=D_{22}\varkappa_2+D_{12}\varkappa_1+D_{26}\tau;\\
&H=H_{12}=H_{21}=D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2;
\end{aligned}\right\} \tag{13.5}$$

b) The second group is derived from (9.9)-(9.16); it differs from the first group only in the expressions for S_{12} and S_{21}, for which we obtain

$$\left.\begin{aligned}
&S_{12}=C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2+k_2(D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2);\\
&S_{21}=C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2+k_1(D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2).
\end{aligned}\right\} \tag{13.6}$$

From (8.2) for the potential energy of deformation we obtain

$$\begin{aligned}
V=\frac{1}{2}\int\int(C_{11}\varepsilon_1^2+2C_{12}\varepsilon_1\varepsilon_2+C_{22}\varepsilon_2^2+C_{66}\omega^2+\\
+2C_{16}\omega\varepsilon_1+2C_{26}\omega\varepsilon_2)AB\,d\alpha\,d\beta+\frac{1}{2}\int\int(D_{11}\varkappa_1^2+\\
+2D_{12}\varkappa_1\varkappa_2+D_{22}\varkappa_2^2+D_{66}\tau^2+2D_{16}\tau\varkappa_1+2D_{26}\tau\varkappa_2)AB\,d\alpha\,d\beta.
\end{aligned} \tag{13.7}$$

The formulas for stresses in the layers (4.2)-(4.4), the expressions for deformations and changes in curvature (2.9)-(2.17), as well as the equilibrium equations (7.4) and the equations of compatibility deformations (3.1)-(3.3) remain unchanged.

Let us discuss the special cases of anisotropy of the material of the layers of the shell.

1. Shells Consisting of Orthotropic Layers. Let a shell consist of orthotropic layers so that at each point of each layer one of the planes of axial symmetry is parallel to the median surface of the shell and the other

two planes of elastic symmetry are perpendicular to coordinate lines $\alpha = \text{const}$. and $\beta = \text{const}$.

In this case for the desired elastic constants of the i-th layer of the shell we have (12.1) and (12.2). The formulas for stresses (12.3) and stiffnesses (13.2)-(13.3) remain unchanged, hence the values of B_{jk} must be taken from (12.2), wherein along with (13.4) we also have (12.4).

Considering (12.4), from (13.5) and (13.6) we obtain the following two groups of elasticity relationships:

a) The first group from (13.5)

$$\left.\begin{aligned} T_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2; & T_2 &= C_{22}\varepsilon_2 + C_{12}\varepsilon_1; \\ M_1 &= D_{11}\varkappa_1 + D_{12}\varkappa_2; & M_2 &= D_{22}\varkappa_2 + D_{12}\varkappa_1; \\ S &= S_{12} = S_{21} = C_{66}\omega; & H &= H_{12} = H_{21} = D_{66}\tau; \end{aligned}\right\} \qquad (13.8)$$

b) The second group; here, as before, we present only the values of S_{12} and S_{21}, which differ from the corresponding values of the first group

$$S_{12} = C_{66}\omega + k_2 D_{66}\tau. \quad S_{21} = C_{66}\omega + k_1 D_{66}\tau. \qquad (13.9)$$

From (13.7) for the potential energy of deformation we obtain

$$\begin{aligned} V = \frac{1}{2} \int \int \big[& (C_{11}\varepsilon_1^2 + 2C_{12}\varepsilon_1\varepsilon_2 + C_{22}\varepsilon_2^2 + C_{66}\omega^2) + \\ & + (D_{11}\varkappa_1^2 + 2D_{12}\varkappa_1\varkappa_2 + D_{22}\varkappa_2^2 + D_{66}\tau^2)\big] AB\, d\alpha\, d\beta. \end{aligned} \qquad (13.10)$$

2. Shells Consisting of Isotropic Layers.

In this case, for the desired elastic constants of the i-th layer of the shell we have (12.9).

The formulas for stresses (12.10) and stiffnesses (13.2), (13.3) remain unchanged; hence the values of B_{jk} must be taken from (12.9) and along with (13.4) we also have (12.11).

The formulas for internal forces (13.8), (13.9) and the expression for potential energy (13.10) remain unchanged; it is necessary only to consider the new values of the stiffnesses.

3. Shells Consisting of Transversely Isotropic Layers.

Here, as in item 3 of the previous section, we discuss three cases of arrangement of the transverse isotropic material in shell layers.

Of these cases, as before, the first two coincide with the cases of those symmetrically assembled orthotropic shells for which we have (12.2) and (12.13) while the third coincides with the case of the symmetrically assembled isotropic shell for which we have (12.14).

SECTION 14. SINGLE-LAYER ANISOTROPIC SHELLS

Let us investigate a single-layer anisotropic shell with constant thickness h. Let the material of the shell at each point have only one plane of elastic symmetry parallel to the median surface of the shell. Let the coordinate surface of the shell coincide with the median surface.*,**

The initial relationships for this case are easily obtained from the corresponding relationships in the preceding section. For this purpose it is necessary to assume (Figure 15)

$$2h_{m+1}=h, \quad B_{jk}^{m+1}=B_{jk}, \quad B_{jk}^{s}=0 \tag{14.1}$$

or

$$2h_{1}=h, \quad B_{jk}^{m+1}=B_{jk}^{s}=B_{jk}, \tag{14.2}$$

where for coefficients B_{jk} from (4.5) we have

$$\left.\begin{aligned} B_{11}&=\frac{a_{22}a_{66}-a_{26}^{2}}{\Omega}; & B_{22}&=\frac{a_{11}a_{66}-a_{16}^{2}}{\Omega};\\ B_{12}&=\frac{a_{16}a_{26}-a_{12}a_{66}}{\Omega}; & B_{66}&=\frac{a_{11}a_{22}-a_{12}^{2}}{\Omega};\\ B_{16}&=\frac{a_{12}a_{26}-a_{22}a_{16}}{\Omega}; & B_{26}&=\frac{a_{12}a_{16}-a_{11}a_{26}}{\Omega}; \end{aligned}\right\} \tag{14.3}$$

$$\Omega=\left(a_{12}a_{22}-a_{12}^{2}\right)a_{66}+2a_{12}a_{16}a_{26}-a_{11}a_{26}^{2}-a_{22}a_{16}^{2}.$$

By virtue of (14.1) or (14.2), from (13.2) and (13.3) for stiffnesses C_{jk} and D_{jk} we obtain

---

*S.A. Ambartsumyan, Some problems in the theory of anisotropic shells, Izvestiya AN Arm SSR (FMYeiT nauki), No. 9, 1947.

**S.A. Ambartsumyan, Toward a theory of shallow anisotropic shells, PMM, Vol. XII, No. 1, 1948.

$$C_{jk} = hB_{jk}, \quad D_{jk} = \frac{h^3}{12} B_{jk}. \tag{14.4}$$

By virtue of (14.4), from (13.5) and (13.6) we obtain the following two groups of elasticity relationships:

a) The first group from (13.5)

$$\left.\begin{aligned} T_1 &= h(B_{11}\varepsilon_1 + B_{12}\varepsilon_2 + B_{16}\omega); \quad T_2 = h(B_{22}\varepsilon_2 + B_{12}\varepsilon_1 + B_{26}\omega); \\ S &= S_{12} = S_{21} = h(B_{66}\omega + B_{16}\varepsilon_1 + B_{26}\varepsilon_2); \\ M_1 &= \frac{h^3}{12}(B_{11}\varkappa_1 + B_{12}\varkappa_2 + B_{16}\tau), \quad M_2 = \frac{h^3}{12}(B_{22}\varkappa_2 + B_{12}\varkappa_1 + B_{26}\tau); \\ H &= H_{12} = H_{21} = \frac{h^3}{12}(B_{66}\tau + B_{16}\varkappa_1 + B_{26}\varkappa_2); \end{aligned}\right\} \tag{14.5}$$

b) The second group from (13.5) and (13.6); it differs from the first group only in the expressions for S_{12} and S_{21}, which we present as

$$\left.\begin{aligned} S_{12} &= h(B_{66}\omega + B_{16}\varepsilon_1 + B_{26}\varepsilon_2) + k_2 \frac{h^3}{12}(B_{66}\tau + B_{16}\varkappa_1 + B_{26}\varkappa_2); \\ S_{21} &= h(B_{66}\omega + B_{16}\varepsilon_1 + B_{26}\varepsilon_2) + k_1 \frac{h^3}{12}(B_{66}\tau + B_{16}\varkappa_1 + B_{26}\varkappa_2). \end{aligned}\right\} \tag{14.6}$$

From (13.7), for the potential energy of deformation we obtain

$$\begin{aligned} V = \frac{h}{2} \int\int (B_{11}\varepsilon_1^2 + 2B_{12}\varepsilon_1\varepsilon_2 + B_{22}\varepsilon_2^2 + B_{66}\omega^2 + \\ + 2B_{16}\omega\varepsilon_1 + 2B_{26}\omega\varepsilon_2) AB\, d\alpha\, d\beta + \frac{h^3}{24} \int\int (B_{11}\varkappa_1^2 + 2B_{12}\varkappa_1\varkappa_2 + \\ + B_{22}\varkappa_2^2 + B_{66}\tau^2 + 2B_{16}\tau\varkappa_1 + 2B_{26}\tau\varkappa_2) AB\, d\alpha\, d\beta. \end{aligned} \tag{14.7}$$

The formulas for stresses (4.2)-(4.4) (without i subscripts), the expression for deformations and changes in curvature (10.9)-(10.17), as well as the equilibrium equations (7.4) and the equations of compatibility of deformation (3.1)-(3.3) remain unchanged.

Let us examine the special cases of anisotropy of the shell material.

1. Shells of Orthotropic Materials. Let the shell be made of an orthotropic material so that at each point of the shell one of the planes of elastic symmetry is parallel to the median surface while the remaining two planes of elastic symmetry are perpendicular to coordinate lines $\alpha = \text{const.}$ and $\beta = \text{const.}$ In this case, for the elastic constants we have

$$\left.\begin{aligned} a_{11} &= \frac{1}{E_1}, \quad a_{22} = \frac{1}{E_2}, \quad a_{66} = \frac{1}{G_{12}}, \\ a_{12} &= -\frac{\nu_2}{E_2} = -\frac{\nu_1}{E_1}, \quad a_{16} = 0, \quad a_{26} = 0; \end{aligned}\right\} \tag{14.8}$$

$$\left.\begin{aligned} &B_{11}=\frac{E_1}{1-\nu_1\nu_2}, \quad B_{22}=\frac{E_2}{1-\nu_1\nu_2}, \quad B_{66}=G_{12}, \\ &B_{12}=\frac{\nu_2E_1}{1-\nu_1\nu_2}=\frac{\nu_1E_2}{1-\nu_1\nu_2}, \quad B_{16}=B_{26}=0. \end{aligned}\right\} \tag{14.9}$$

For the stresses we have

$$\left.\begin{aligned} &\sigma_\alpha=\frac{E_1}{1-\nu_1\nu_2}[\varepsilon_1+\nu_2\varepsilon_2+\gamma(\varkappa_1+\nu_2\varkappa_2)]; \\ &\sigma_\beta=\frac{E_2}{1-\nu_1\nu_2}[\varepsilon_2+\nu_1\varepsilon_1+\gamma(\varkappa_2+\nu_1\varkappa_1)]; \\ &\tau_{\alpha\beta}=G_{12}(\omega+\gamma\tau). \end{aligned}\right\} \tag{14.10}$$

From (14.9) and (14.4), for the stiffnesses we obtain

$$\left.\begin{aligned} &C_{11}=\frac{E_1h}{1-\nu_1\nu_2}, \quad C_{22}=\frac{E_2h}{1-\nu_1\nu_2}, \quad C_{66}=G_{12}h, \\ &C_{12}=\frac{\nu_1E_2h}{1-\nu_1\nu_2}=\frac{\nu_2E_1h}{1-\nu_1\nu_2}; \end{aligned}\right\} \tag{14.11}$$

$$\left.\begin{aligned} &D_{11}=\frac{E_1h^3}{12(1-\nu_1\nu_2)}, \quad D_{22}=\frac{E_2h^3}{12(1-\nu_1\nu_2)}, \quad D_{66}=G_{12}\frac{h^3}{12}, \\ &D_{12}=\frac{\nu_1E_2h^3}{12(1-\nu_1\nu_2)}=\frac{\nu_2E_1h^3}{12(1-\nu_1\nu_2)}. \end{aligned}\right\} \tag{14.12}$$

From (14.5) and (14.6) we obtain two groups of elasticity relationships:

a) the first group from (14.5)

$$\begin{aligned} &T_1=\frac{E_1h}{1-\nu_1\nu_2}(\varepsilon_1+\nu_2\varepsilon_2); \quad T_2=\frac{E_2h}{1-\nu_1\nu_2}(\varepsilon_2+\nu_1\varepsilon_1); \\ &S=S_{12}=S_{21}=G_{12}h\omega; \\ &M_1=\frac{E_1h^3}{12(1-\nu_1\nu_2)}(\varkappa_1+\nu_2\varkappa_2), \quad M_2=\frac{E_2h^3}{12(1-\nu_1\nu_2)}(\varkappa_2+\nu_1\varkappa_1); \\ &H=H_{12}=H_{21}=G_{12}\frac{h^3}{12}\tau; \end{aligned} \tag{14.13}$$

b) the second group from (14.5) and (14.6); it differs from the first group only in the expressions for S_{12} and S_{21}, which we present as

$$S_{12}=G_{12}h\left(\omega+k_2\frac{h^2}{12}\tau\right), \quad S_{21}=G_{12}h\left(\omega+k_1\frac{h^2}{12}\tau\right). \tag{14.14}$$

From (14.7), for the potential energy of deformation we obtain

$$V=\frac{h}{2}\int\int\Big(\frac{E_1}{1-\nu_1\nu_2}\varepsilon_1^2+2\frac{\nu_1E_2}{1-\nu_1\nu_2}\varepsilon_1\varepsilon_2+\frac{E_2}{1-\nu_1\nu_2}\varepsilon_2^2+$$
$$+G_{12}\omega^2\Big)AB\,d\alpha\,d\beta+\frac{h^3}{24}\int\int\Big(\frac{E_1}{1-\nu_1\nu_2}\varkappa_1^2+2\frac{\nu_1E_2}{1-\nu_1\nu_2}\varkappa_1\varkappa_2+$$
$$+\frac{E_2}{1-\nu_1\nu_2}\varkappa_2^2+G_{12}\tau^2\Big)AB\,d\alpha\,d\beta. \qquad (14.15)$$

2. Shells of Isotropic Materials. In this case, for the elastic constants we have

$$a_{11}=a_{22}=\frac{1}{E},\qquad a_{66}=\frac{1}{G}=\frac{2(1+\nu)}{E},\qquad a_{12}=-\frac{\nu}{E}, \qquad (14.16)$$
$$B_{11}=B_{22}=\frac{E}{1-\nu^2},\qquad B_{66}=\frac{E}{2(1+\nu)},\qquad B_{12}=\frac{\nu E}{1-\nu^2}. \qquad (14.17)$$

From (14.10), for the stresses we obtain

$$\left.\begin{aligned}\sigma_\alpha&=\frac{E}{1-\nu^2}[\varepsilon_1+\nu\varepsilon_2+\gamma(\varkappa_1+\nu\varkappa_2)],\\ \sigma_\beta&=\frac{E}{1-\nu^2}[\varepsilon_2+\nu\varepsilon_1+\gamma(\varkappa_2+\nu\varkappa_1)];\\ \tau_{\alpha\beta}&=\frac{E}{2(1+\nu)}(\omega+\gamma\tau).\end{aligned}\right\} \qquad (14.18)$$

From (14.17) and (14.4) for the stiffnesses we obtain

$$C=C_{11}=C_{22}=\frac{Eh}{1-\nu^2},\qquad C_{66}=\frac{Eh}{2(1+\nu)},\qquad C_{12}=\frac{\nu Eh}{1-\nu^2}; \qquad (14.19)$$
$$D=D_{11}=D_{22}=\frac{Eh^3}{12(1-\nu^2)},\qquad D_{66}=\frac{Eh^3}{24(1+\nu)},\qquad D_{12}=\frac{\nu Eh^3}{12(1-\nu^2)}. \qquad (14.20)$$

Moreover, as usual, we have two groups of elasticity relationships:

a) the first group from (14.5)

$$\left.\begin{aligned}T_1&=\frac{Eh}{1-\nu^2}(\varepsilon_1+\nu\varepsilon_2),\quad T_2=\frac{Eh}{1-\nu^2}(\varepsilon_2+\nu\varepsilon_1);\\ S&=S_{12}=S_{21}=\frac{Eh}{2(1+\nu)}\omega;\\ M_1&=\frac{Eh^3}{12(1-\nu^2)}(\varkappa_1+\nu\varkappa_2),\quad M_2=\frac{Eh^3}{12(1-\nu^2)}(\varkappa_2+\nu\varkappa_1);\\ H&=H_{12}=H_{21}=\frac{Eh^3}{24(1+\nu)}\tau;\end{aligned}\right\} \qquad (14.21)$$

b) the second group from (14.5) and (14.6); it differs from the first group only in the expressions for S_{12} and S_{21}, which we present as

$$\left.\begin{aligned} S_{12} &= \frac{Eh}{2(1+\nu)}\left(\omega + \frac{h^2}{12}k_2\tau\right); \\ S_{21} &= \frac{Eh}{2(1+\nu)}\left(\omega + \frac{h^2}{12}k_1\tau\right). \end{aligned}\right\} \tag{14.22}$$

The elasticity relationships in the second group were first derived independently by L.I. Balabukh* and V.V. Novozhilov.**

From (13.7), for the potential energy of deformation we obtain

$$\begin{aligned} V = {} & \frac{Eh}{2(1-\nu^2)}\int\int\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\nu)\left(\varepsilon_1\varepsilon_2 - \frac{\omega^2}{4}\right)\right]AB\,d\alpha\,d\beta + \\ & + \frac{Eh^3}{24(1-\nu^2)}\int\int\left[(\varkappa_1+\varkappa_2)^2 - 2(1-\nu)\left(\varkappa_1\varkappa_2 - \frac{\tau^2}{4}\right)\right]AB\,d\alpha\,d\beta. \end{aligned} \tag{14.23}$$

The results derived in this item relate to the theory of homogeneous isotropic shells.

3. Shells of a Transversely Isotropic Material. Here, as before, we examine three cases of arrangement of a transversely isotropic material over the shell.

a) Let the principal directions of elasticity at each point of the shell coincide with the directions of coordinate lines $\beta = \text{const}$. In this case it is necessary to use the relationships given for an orthotropic shell and to assume

$$\left.\begin{aligned} & E_1 = E', \quad E_2 = E, \quad G_{12} = G', \\ & \nu_1 = \nu', \quad \nu_2 = \frac{E}{E'}\nu'. \end{aligned}\right\} \tag{14.24}$$

b) Let the principal directions of elasticity at each point of the shell coincide with the directions of coordinate lines $\alpha = \text{const}$. In this case it is also necessary to use the relationships given for an orthotropic shell and to assume

$$\left.\begin{aligned} & E_1 = E, \quad E_2 = E', \quad G_{12} = G', \\ & \nu_2 = \nu_1, \quad \nu_1 = \frac{E}{E'}\nu'. \end{aligned}\right\} \tag{14.25}$$

---

*L.I. Balabukh, Bending and torsion of a conical shell, Trudy TsAGI, No. 577, 1946.

**V.V. Novozhilov, New Method of calculation of thin shells, Izvestiya OTN AN SSSR, No. 1, 1946.

c) Let the principal directions of elasticity at each point be normal to the median surface of the shell. In this case it is necessary to use the relationships given for an isotropic shell and to assume

$$E_1 = E_2 = E, \quad G_{12} = G, \quad \nu_1 = \nu_2 = \nu. \tag{14.26}$$

SECTION 15. FURTHER REMARKS CONCERNING ELASTICITY RELATIONSHIPS

Examining elasticity relationships (9.1)-(9.16), we note that they differ fundamentally from the corresponding elasticity relationships for homogeneous anisotropic shells (14.5), (14.6) or shells consisting of an odd number of anisotropic layers symmetrically arranged relative to the coordinate surface of the shell (13.5), (13.6). This difference is expressed, on the one hand, by the presence in (9.1)-(9.4) and (9.9)-(9.12) of terms accounting for the influence of changes in curvature of the coordinate surface of the shell on the tangential forces and, on the other hand, by the presence in (9.5)-(9.8) and (9.12)-(9.16) of terms taking into account the influence of deformations of elongation and shear of the coordinate surface of the shell on the bending and torsional moments. Thus, the elasticity relationships in the general case of a multi-layer shell differ from the corresponding relationships for a homogeneous single-layer or symmetrically assembled multi-layer shell only in the terms which contain the interaction stiffnesses K_{jk}. On the basis of the hypothesis of nondeformable normals it is easily established that, independent of the location of the coordinate surface of the shell, all internal forces and moments in the general case depend both on the tensile and shear strains, and on the parameters of variation in curvature of the coordinate surface of the shell. In this connection the initial location of the coordinate surface of the shell is of no importance. Hence it is of interest to determine that location of the coordinate surface of the shell for which all interaction stiffnesses K_{jk} become zero and for a multi-layer shell we obtain the simpler elasticity relationship of type (13.5) and (13.6). Assuming that $K_{jk} = 0$, from (8.4) we obtain:

$$\Delta = \Delta_{11} = \frac{\sum\limits_{s=1}^{m+n} B_{11}^{s}(\delta_s^2 - \delta_{s-1}^2)}{2\sum\limits_{s=1}^{m+n} B_{11}^{s}(\delta_s - \delta_{s-1})}, \quad K_{11} = 0; \tag{15.1}$$

$$\Delta = \Delta_{22} = \frac{\sum\limits_{s=1}^{m+n} B_{22}^{s}(\delta_s^2 - \delta_{s-1}^2)}{2\sum\limits_{s=1}^{m+n} B_{22}^{s}(\delta_s - \delta_{s-1})} \qquad K_{22} = 0; \tag{15.2}$$

$$\Delta = \Delta_{66} = \frac{\sum\limits_{s=1}^{m+n} B_{66}^{s}(\delta_s^2 - \delta_{s-1}^2)}{2\sum\limits_{s=1}^{m+n} B_{66}^{s}(\delta_s - \delta_{s-1})}, \qquad K_{66} = 0; \tag{15.3}$$

$$\Delta = \Delta_{12} = \frac{\sum\limits_{s=1}^{m+n} B_{12}^{s}(\delta_s^2 - \delta_{s-1}^2)}{2\sum\limits_{s=1}^{m+n} B_{12}^{s}(\delta_s - \delta_{s-1})}, \qquad K_{12} = 0; \tag{15.4}$$

$$\Delta = \Delta_{16} = \frac{\sum\limits_{s=1}^{m+n} B_{16}^{s}(\delta_s^2 - \delta_{s-1}^2)}{2\sum\limits_{s=1}^{m+n} B_{16}^{s}(\delta_s - \delta_{s-1})}, \qquad K_{16} = 0; \tag{15.5}$$

$$\Delta = \Delta_{26} = \frac{\sum\limits_{s=1}^{m+n} B_{26}^{s}(\delta_s^2 - \delta_{s-1}^2)}{2\sum\limits_{s=1}^{m+n} B_{26}^{s}(\delta_s - \delta_{s-1})}, \qquad K_{26} = 0. \tag{15.6}$$

Examining (15.1)-(15.6), we note that in the general case of anisotropy of the layers of a shell all values of $\Delta = \Delta_{jk}$ differ. Hence we conclude that in the general case of an anisotropic laminar shell, when there are no limits placed on the elastic characteristics of the materials of the layers, there is no single location of the coordinate surface of the shell for which all interaction stiffnesses K_{jk} become zero.

In the presence of a single coordinate surface for the shell all interaction stiffnesses K_{jk} become zero if we assume the following condition:

$$\Delta = \Delta_{11} = \Delta_{22} = \Delta_{66} = \Delta_{12} = \Delta_{16} = \Delta_{26}; \tag{15.7}$$

but this condition positively imposes substantial limitations on the elastic characteristics of the materials of the layers. As an example let us discuss the two-layer shell (Figure 16).

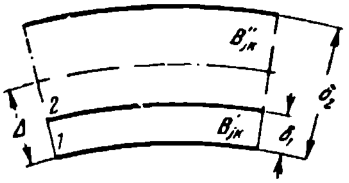

Figure 16

Assuming

$$m+n=2, \quad B^1_{jk}=B'_{jk}, \quad B^2_{jk}=B''_{jk}, \tag{15.8}$$

from (15.7), by virtue of (15.1)-(15.6), we obtain

$$\begin{aligned}\Delta &= \frac{B'_{11}\delta_1^2+B''_{11}(\delta_2^2-\delta_1^2)}{B'_{11}\delta_1+B''_{11}(\delta_2-\delta_1)} = \frac{B'_{22}\delta_1^2+B''_{22}(\delta_2^2-\delta_1^2)}{B'_{22}\delta_1+B''_{22}(\delta_2-\delta_1)} = \\ &= \frac{B'_{66}\delta_1^2+B''_{66}(\delta_2^2-\delta_1^2)}{B'_{66}\delta_1+B''_{66}(\delta_2-\delta_1)} = \frac{B'_{12}\delta_1^2+B''_{12}(\delta_2^2-\delta_1^2)}{B'_{12}\delta_1+B''_{22}(\delta_2-\delta_1)} = \\ &= \frac{B'_{16}\delta_1^2+B''_{16}(\delta_2^2-\delta_1^2)}{B'_{16}\delta_1+B''_{16}(\delta_2-\delta_1)} = \frac{B'_{26}\delta_1^2+B''_{26}(\delta_2^2-\delta_1^2)}{B'_{26}\delta_1+B''_{26}(\delta_2-\delta_1)}\end{aligned} \tag{15.9}$$

Hence, it is easily seen that the equalities in (15.9) will apply if

$$\frac{B'_{11}}{B''_{11}}=\frac{B'_{22}}{B''_{22}}=\frac{B'_{12}}{B''_{12}}=\frac{B'_{66}}{B''_{66}}=\frac{B'_{16}}{B''_{16}}=\frac{B'_{26}}{B''_{26}}, \tag{15.10}$$

which is a substantial limitation for the presented problem.

In the special case of an isotropic shell we have from (12.9) for the elastic characteristics of the layers:

$$\left.\begin{aligned} B'_{11}&=B'_{22}=\frac{E_1}{1-\nu_1^2}, \quad B'_{12}=\nu B'_{11}, \quad B'_{66}=\frac{E_1}{2(1+\nu_1)}, \\ B''_{11}&=B''_{22}=\frac{E_2}{1-\nu_2^2}, \quad B''_{12}=\nu_2 B''_{11}, \quad B''_{66}=\frac{E_2}{2(1+\nu_2)}, \end{aligned}\right\} \tag{15.11}$$

where E_1 and E_2 are the elastic moduli of the first and second layers, respectively; ν_1 and ν_2 are the Poisson's ratios for the first and second layers respectively.

Inserting the values of B_{jk} from (15.11) into (15.10), we obtain:

$$\frac{E_1(1-\nu_2^2)}{E_2(1-\nu_1^2)}=\frac{\nu_1E_1(1-\nu_2^2)}{\nu_2E_2(1-\nu_1^2)}=\frac{2E_1(1+\nu_2)}{2E_2(1+\nu_1)}. \qquad (15\ 12)$$

Here it is easily seen that these equalities apply only in the case of equality of the Poisson's ratios of the shell layers, that is, when $\nu_1=\nu_2=\nu$. In this case, from (15.9) it is easy to obtain the value of Δ defining the position of the coordinate surface of the shell for which the stiffnesses K_{jk} become zero:

$$\Delta=\frac{E_1\delta_1^2+E_2(\delta_2^2-\delta_1^2)}{E_1\delta_1+E_2(\delta_2-\delta_1)}. \qquad (15.13)$$

Thus, we conclude that in the general case of a multi-layer shell in the absence of limitation (15.7) there does not exist such a position of the coordinate surface of the shell at which we also obtain elasticity relationships of type (13.5) and (13.6) for the multi-layer shell. It must be noted that in a few special cases there may be chosen such a position of the coordinate surface of the shell that certain of the interaction stiffnesses K_{jk} become zero; herein only certain elasticity relationships will be of type (13.5) and (13.6).

The question of selection of the initial coordinate surface in the case of isotropic plates and shells is discussed in the literature*,**,***.

SECTION 16. CALCULATION OF STIFFNESSES FOR ARBITRARY DIRECTIONS

In Section 4 of Chapter I it was shown that the elastic constants a_{jk} of an anisotropic body change with a transformation from one system of coordinates to another. Using (4.5), we easily see that this also occurs for coefficients B_{jk}.

---

*I.I. Yenikeyev, Some problems in the bending of nonuniform plates with transverse structural asymmetry. Author's abstract, 1954.

**V.I. Korolev, Thin two-layer plates and shells, Inzhenernyy sbornik, Vol. XXII, 1955.

***E.I. Grigolyuk, Selection of the initial surface in the theory of nonuniform shells, Izvestiya OTN AN SSSR, No. 8, 1956.

As before, let us discuss the generalized plane stress of an anisotropic plate, the material of which at each point has only one plane of elastic symmetry parallel to the median plane of the plate, which coincides with the coincident coordinate plates $\alpha\beta$ and $\alpha'\beta'$.

Let the coordinate systems (α, β, γ) and $(\alpha', \beta', \gamma')$ be obtained from one another by rotation through a certain angle about the γ axis. Moreover, in system (α, β, γ) let the coefficients B_{jk} be equal to B_{11}, B_{22}, B_{12}, B_{66}, B_{16}, B_{26} and correspondingly in the system $(\alpha', \beta', \gamma')$ be equal to B'_{11}, B'_{22}, B'_{12}, B'_{66}, B'_{16}, B'_{26}. Then, following the known procedure,* we obtain the following transformation formulas:

$$\left.\begin{aligned}
B'_{11} &= B_{11}\cos^4\varphi + 2(B_{12} + 2B_{66})\sin^2\varphi\cos^2\varphi + \\
&\qquad + B_{22}\sin^4\varphi + 2(B_{16}\cos^2\varphi + B_{26}\sin^2\varphi)\sin 2\varphi; \\
B'_{22} &= B_{11}\sin^4\varphi + 2(B_{12} + 2B_{66})\sin^2\varphi\cos^2\varphi + \\
&\qquad + B_{22}\cos^4\varphi - 2(B_{16}\sin^2\varphi + B_{26}\cos^2\varphi)\sin 2\varphi; \\
B'_{12} &= B_{12} + [B_{11} + B_{22} - 2(B_{12} + 2B_{66})]\sin^2\varphi\cos^2\varphi + \\
&\qquad + (B_{26} - B_{16})\cos 2\varphi\sin 2\varphi; \\
B'_{66} &= B_{66} + [B_{11} + B_{22} - 2(B_{12} + 2B_{66})]\sin^2\varphi\cos^2\varphi + \\
&\qquad + (B_{26} - B_{16})\cos 2\varphi\sin 2\varphi; \\
B'_{16} &= \frac{1}{2}[B_{22}\sin^2\varphi - B_{11}\cos^2\varphi + (B_{12} + 2B_{66})\cos 2\varphi]\sin 2\varphi + \\
&\qquad + B_{16}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) + B_{26}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi); \\
B'_{26} &= \frac{1}{2}[B_{22}\cos^2\varphi - B_{11}\sin^2\varphi - (B_{12} + 2B_{66})\cos 2\varphi]\sin 2\varphi + \\
&\qquad + B_{16}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi) + B_{26}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi).
\end{aligned}\right\} \quad (16.1)$$

Thus, having the values of B_{jk} for each layer, by means of formulas (16.1) we may find the values of B'_{jk} in the new system of coordinates for each discussed layer. Inserting the values of $[B'_{jk}]^s$ into formulas (8.3)-(8.5) or (8.6)-(8.8), it is not difficult to find the values of stiffnesses C'_{jk}, K'_{jk} and D'_{jk} in the coordinated system of $(\alpha', \beta', \gamma')$. If stiffnesses C_{jk}, K_{jk} and D_{jk}

---

*S.G. Lekhnitskiy, see pp. 274-276 of work cited in footnote * on p. 27.

are known in coordinate system (α, β, γ), then by means of similar transformation formulas (16.1) it is easy to find the values of stiffnesses C'_{jk} K'_{jk} and D'_{jk} in coordinate system $(\alpha', \beta', \gamma')$ ');*

In the special case of an orthotropic plate, when the directions of coordinate lines α, β, γ coincide with the principal directions of elasticity of the plate, transformation formulas (16.1) are substantially simplified and take the following form:

$$\left.\begin{aligned} B'_{11} &= B_{11}\cos^4\varphi + 2(B_{12} + 2B_{66})\sin^2\varphi\cos^2\varphi + B_{22}\sin^4\varphi; \\ B'_{22} &= B_{11}\sin^4\varphi + 2(B_{12} + 2B_{66})\sin^2\varphi\cos^2\varphi + B_{22}\cos^4\varphi; \\ B'_{12} &= B_{12} + [B_{11} + B_{22} - 2(B_{12} + 2B_{66})]\sin^2\varphi\cos^2\varphi; \\ B'_{66} &= B_{66} + [B_{11} + B_{22} - 2(B_{12} + 2B_{66})]\sin^2\varphi\cos^2\varphi. \end{aligned}\right\} \tag{16.2}$$

Here the secondary coefficients B'_{16} and B'_{26}, which in the system of coordinates (α, β, γ) are equal to zero, in the system of coordinates $(\alpha', \beta', \gamma')$ have the form

$$\left.\begin{aligned} B'_{16} &= \frac{1}{2}[B_{22}\sin^2\varphi - B_{11}\cos^2\varphi + (B_{12} + 2B_{66})\cos 2\varphi]\sin 2\varphi; \\ B'_{26} &= \frac{1}{2}[B_{22}\cos^2\varphi - B_{11}\sin^2\varphi - (B_{12} + 2B_{66})\cos 2\varphi]\sin 2\varphi. \end{aligned}\right\} \tag{16.3}$$

Let us also list the following invariants:

$$\left.\begin{aligned} B'_{11} + B'_{22} + 2B'_{12} &= B_{11} + B_{22} + 2B_{12}; \\ B'_{66} - B'_{12} &= B_{66} - B_{12}. \end{aligned}\right\} \tag{16.4}$$

---

*See the last cited work by S.G. Lekhnitskiy γ 804

CHAPTER III

MEMBRANE THEORY OF ANISOTROPIC SHELLS

SECTION 1. GENERAL PREMISES AND INITIAL RELATIONSHIPS IN THE MEMBRANE THEORY OF SINGLE-LAYER ISOTROPIC SHELLS

Here, as in the case of isotropic shells,* by "membrane theory" we shall understand that approximate method of calculation in which in the first three equilibrium equations (2.7.4) we disregard the transverse forces (N_1, N_2) and in the sixth equation we disregard the torsional moments (H_{12}, H_{21}). In view of this, in order to construct a membrane theory of anisotropic shells from the general equations of the theory of anisotropic shells we have the following initial equations and relationships.

Equilibrium equations (Chapter II, Section 7):

$$\left.\begin{aligned} &\frac{\partial BT_1}{\partial \alpha} - \frac{\partial B}{\partial \alpha} T_2 + \frac{\partial AS_{21}}{\partial \beta} + \frac{\partial A}{\partial \beta} S_{12} = -ABX; \\ &\frac{\partial AT_2}{\partial \beta} - \frac{\partial A}{\partial \beta} T_1 + \frac{\partial BS_{12}}{\partial \alpha} + \frac{\partial B}{\partial \alpha} S_{21} = -ABY; \\ &\qquad\qquad\qquad\qquad k_1T_1 + k_2T_2 = Z; \end{aligned}\right\} \qquad (1.1)$$

$$\left.\begin{aligned} &\frac{\partial BM_1}{\partial \alpha} + \frac{\partial AH_{21}}{\partial \beta} + \frac{\partial A}{\partial \beta} H_{12} - \frac{\partial B}{\partial \alpha} M_2 = ABN_1; \\ &\frac{\partial AM_2}{\partial \beta} + \frac{\partial BH_{12}}{\partial \alpha} + \frac{\partial B}{\partial \alpha} H_{21} - \frac{\partial A}{\partial \beta} M_1 = ABN_2; \\ &\qquad\qquad\qquad\qquad S_{12} - S_{21} = 0. \end{aligned}\right\} \qquad (1.2)$$

---

*A.A. Gol'denveyzer. Teoriya uprugikh tonkikh obolochek. Gostekhiz-

Elasticity relationships (Chapter II, Section 14):

$$\left.\begin{aligned} T_1 = h\,(B_{11}\varepsilon_1 + B_{12}\varepsilon_2 + B_{16}\omega);\quad T_2 = h\,(B_{22}\varepsilon_2 + B_{12}\varepsilon_1 + B_{26}\omega);\\ S = S_{12} = S_{21} = h\,(B_{66}\omega + B_{16}\varepsilon_1 + B_{26}\varepsilon_2)\end{aligned}\right\} \qquad (1.3)$$

or by virtue of (2.4.1)-(2.4.6),

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{h}(a_{11}T_1 + a_{12}T_2 + a_{16}S);\\ \varepsilon_2 &= \frac{1}{h}(a_{12}T_1 + a_{22}T_2 + a_{26}S);\\ \omega &= \frac{1}{h}(a_{16}T_1 + a_{26}T_2 + a_{66}S);\end{aligned}\right\} \qquad (1.3')$$

$$\left.\begin{aligned} M_1 &= \frac{h^3}{12}(B_{11}\varkappa_1 + B_{12}\varkappa_2 + B_{16}\tau),\\ M_2 &= \frac{h^3}{12}(B_{22}\varkappa_2 + B_{12}\varkappa_1 + B_{26}\tau);\\ H = H_{12} = H_{21} &= \frac{h^3}{12}(B_{66}\tau + B_{16}\varkappa_1 + B_{26}\varkappa_2).\end{aligned}\right\} \qquad (1.4)$$

Formulas associating the components of deformation and change in curvature with the components of displacement are (Chapter II, Section 2):

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + k_1 w;\\ \varepsilon_2 &= \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + k_2 w;\\ \omega &= \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right),\end{aligned}\right\} \qquad (1.5)$$

$$\left.\begin{aligned} \varkappa_1 &= -\frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} + \frac{\partial k_1}{\partial \alpha}\frac{u}{A} + \frac{\partial k_1}{\partial \beta}\frac{v}{B} - k_1^2 w;\\ \varkappa_2 &= -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{\partial k_2}{\partial \alpha}\frac{u}{A} + \frac{\partial k_2}{\partial \beta}\frac{v}{B} - k_2^2 w;\\ \tau &= -\frac{2}{AB}\left(\frac{\partial^2 w}{\partial \alpha\,\partial \beta} - \frac{1}{A}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \alpha} - \frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \beta}\right) +\\ &\qquad + (k_1 - k_2)\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) - \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right)\right].\end{aligned}\right\} \qquad (1.6)$$

In examining the presented equations, relationships and formulas we note that they fall essentially into two groups. In the first group we have: equilibrium equations (1.1), elasticity relationships (1.3) and formulas associating the relative deformations of elongation and shear of the median surface with the components of displacement (1.5). In the second group we have equilibrium equations (1.2), elasticity relationships (1.4) and formulas associating the relative changes in curvature and torsion with the components of displacement (1.6). These groups are fundamentally different.

The first group consists of ten equations, dealing with the ten unknown functions T_1, T_2, S, ε_1, ε_2, ω, u, v, w. It constitutes a system of differential equations and in the presence of given boundary conditions gives the solution of the problem of a shell according to the membrane theory, that is, it permits determining the tangential forces (T_1, T_2, S), the relative deformations of the median surface (ε_1, ε_2, ω) and displacements (u, v, w) without dependence on the bending and torsional moments (M_1, M_2, H), the lateral forces (N_1, N_2) and the bending and torsional deformations ($\varkappa_1$, $\varkappa_2$, τ).

The second group of relationships which has no value in calculation may be used for a rough estimate of the error in the membrane theory. Using the values of displacements u, v, w, obtained from solution of the equations of the first group, by means of the relationships in the second group it is easy to determine the values of the moments and transverse forces. In order to apply the membrane theory in the solution of various problems of anisotropic shells it is necessary that the stresses from the moments and transverse forces be practically negligible in comparison with stresses from tangential forces T_1, T_2, S.

Thus, if the membrane theory is applicable for solution of a given problem, the solution according to this theory is considered complete if we have determined the internal tangential forces T_1, T_2, S (and thereby also stresses σ_α, σ_β, $\tau_{\alpha\beta}$) and the corresponding displacements u, v, w.* Herein the equations of the second group are replaced by the approximate relationships

$$N_1 = N_2 = M_1 = M_2 = H_{12} = H_{21} = 0, \tag{1.7}$$

which is equivalent to assumption of the well-known hypothesis concerning uniform distribution of stresses over the thickness of the shell,**,*** in accordance with which we obtain the following well-known formulas for determining the stresses:

$$\sigma_\alpha = \frac{T_1}{h}, \quad \sigma_\beta = \frac{T_2}{h}, \quad \tau_{\alpha\beta} = \frac{S}{h}. \tag{1.8}$$

---

*S.A. Ambartsumyan, Membrane theory of anisotropic shells, Izvestiya AN Arm SSR (FMYeiT nauki), Vol. I, No. 6, 1948.

**V.V. Novozhilov, Teoriya tonkikh obolochek, Sudpromgiz, 1951, pp. 85-89.

***A. L. Gol'denveyzer, see pp. 421-422 of work cited in footnote * on p. 61.

SECTION 2. BOUNDARY CONDITIONS

In Section 10 of Chapter II it was stated that the boundary conditions of the general (moment) theory of anisotropic shells do not differ from the corresponding boundary conditions for a homogeneous isotropic shell. Naturally the same may be said in the case of the membrane theory of anisotropic shells. In this connection, without going into detail concerning the general aspects of boundary conditions in the membrane theory of shells,*,**, let us present a few examples of the boundary conditions which naturally follow from the boundary conditions in the general theory of shells. For the sake of brevity we shall list the boundary conditions only for the edge which is defined by the coordinate line $\alpha = \alpha_0 = \text{const}$.

I. Uniform Boundary Conditions:

a) Free edge (2.10.1)

$$T_1 = 0, \quad S_{12} = 0; \tag{2.1}$$

b) Hinged edge free in the tangential direction (2.10.3), (2.10.4)

or

$$\left.\begin{aligned} T_1 = 0, \quad v = 0 \\ S_{12} = 0, \quad u = 0; \end{aligned}\right\} \tag{2.2}$$

c) Absolutely fixed edge (2.10.2), (2.10.4), (2.10.5)

$$u = 0, \quad v = 0. \tag{2.3}$$

II. Nonuniform Boundary Conditions:

a) Loaded edge (2.10.7)

$$T_1 = T_1^*, \quad S_{12} = S_{12}^*; \tag{2.4}$$

b) Hinged edge free in the tangential direction and loaded by a tangential force (2.10.9), (2.10.10)

or

$$\left.\begin{aligned} T_1 = T_1^*, \quad v = 0 \\ S_{12} = S_{12}^*, \quad u = 0; \end{aligned}\right\} \tag{2.5}$$

---

*A. L. Gol'denveyzer, see pp. 109-110 of work cited in footnote *, p. 61.

**V. V. Novozhilov, see pp. 89-91 of work cited in footnote **, p. 63.

c) Displaced edge (2.10.12)

$$u = u^*, \quad v = v^*. \tag{2.6}$$

In similar manner we may also describe the boundary conditions for the edge defined by coordinate line $\beta = \beta_0 = \text{const}$.

Of course, other boundary conditions are possible, but as a rule they may be represented only by means of tangential forces T_i and S_{ik} and tangential displacements u and v.

SECTION 3. AREA OF APPLICABILITY OF THE MEMBRANE THEORY

The membrane theory, being an approximate theory for calculation of shells, provides an approximately correct picture of the stress and displacement of a shell only upon the fulfillment of certain conditions. Five sufficient conditions for applicability of the membrane theory to isotropic shells were first formulated by A. L. Gol'denveyzer.* Without any changes the conditions described by A. L. Gol'denveyzer may be extended to anisotropic shells. These five conditions for applicability of the membrane theory of shells are given here without detailed explanation (detailed discussion will be found in the work by A. L. Gol'denveyzer).

First Condition. The lines of distortion at the median surface of the shell must not form an excessively dense network.

Second Condition. None of the lines of distortion may touch asymptotic lines of the median surface, that is, those lines at this surface along which the normal curvature of the surface vanishes.

Third Condition. External loads and reactions must change smoothly, that is, they must not have an extremely large variability index.

Fourth Condition. The median surface of the shell must not possess certain properties:

a) A cylindrical shell must not be excessively long:

b) A conical shell must not include a conical vertex;

c) The median surface of the shell must not touch the plane along the closed curve.

---

*A. L. Gol'denveyzer, see pp. 423–432 of work cited in footnote on p. 61.

Fifth Condition. The so-called condition of V. V. Novodvorskiy.* The median surface of the shell must be rigid, that is, it must not be deformed without elongation (compression) and shear.

Let us also point out that these conditions, being sufficient, are not necessary and, as A. L. Gol'denveyzer has shown, when they are not fulfilled, calculation according to the membrane theory may, under known circumstances remain valid. Without dwelling further on the general premises of the membrane theory of anisotropic shells, let us turn to discussion of specific problems and examples.

SECTION 4. FUNDAMENTAL EQUATIONS OF THE MEMBRANE THEORY OF SYMMETRICALLY LOADED SHELLS OF REVOLUTION

Let us discuss a shell whose median surface is a surface of revolution with axis of revolution z. The location of any point M of the median surface of the shell will be defined by Gaussian coordinates: the angle $\varphi = \frac{\beta}{r}$, which is the azimuth of the plane extended through point M and the axis of revolution z, and the meridian arc $s = \alpha$, taken along the meridian from a certain initial point M_0.

In such a system of coordinates lines $\varphi = \text{const}$ are meridians of the meridian surface, $s = \text{const}$ are parallel circles (Figure 17). For the discussed surface let us introduce two more geometric quantities: r, representing the distance MM_2 from point M to the axis of revolution z, and ϑ, representing the angle between the tangent to the meridian and the axis of revolution z (Figure 18).

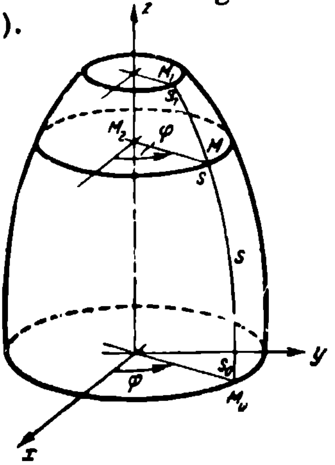

Figure 17

---

*V. V. Novodvorskiy, Concerning the "momentless" theory of shells, Stroitel'naya promyshlennost', No. 1, 1933.

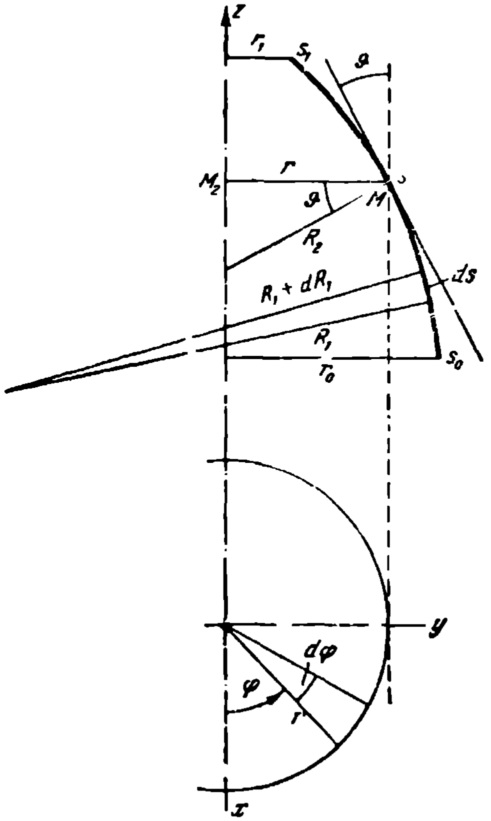

Figure 18

In the chosen coordinate system we have for the principal curvatures of the surface of revolution

$$k_1=\frac{1}{R_1}=\frac{d\vartheta}{ds},\quad k_2=\frac{1}{R_2}=\frac{\cos\vartheta}{r}, \tag{4.1}$$

where R_1 is the radius of curvature of the meridian; R_2 is the second radius of curvature of the surface, representing the length of the segment of the norma to the surface to the axis of revolution.

For the square of a linear element of the discussed surface, in virtue of (1.1.4), (1.1.5) and (2.1.1), we have

$$dl^2=ds^2+r^2\,d\varphi^2. \tag{4.2}$$

Hence for the coefficients of the first quadratic form we obtain

$$A=1,\quad B=R_2\cos\vartheta=r. \tag{4.3}$$

Let us list a few relationships which will be used in subsequent discussion:

$$\frac{dr}{ds} = -\sin\vartheta, \quad \frac{d}{ds}\left(\frac{1}{R_2}\right) = -\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\sin\vartheta}{r}. \tag{4.4}$$

The system of coordinates presented here may be examined in detail in the work by A.I. Lur'ye.*

In the adopted coordinate system for an arbitrarily loaded shell of revolution we obtain: from (1.1), the following equilibrium equations:

$$\frac{\partial T_1}{\partial s} + (T_2 - T_1)\frac{\sin\vartheta}{r} + \frac{1}{r}\frac{\partial S}{\partial \varphi} = -X; \tag{4.5}$$

$$\frac{\partial S}{\partial s} - 2S\frac{\sin\vartheta}{r} + \frac{1}{r}\frac{\partial T_2}{\partial \varphi} = -Y; \tag{4.6}$$

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} = Z; \tag{4.7}$$

from (1.5), the following deformation relationships:

$$\varepsilon_1 = \frac{\partial u}{\partial s} + \frac{w}{R_1};$$

$$\varepsilon_2 = \frac{1}{r}\frac{\partial v}{\partial \varphi} - \frac{\sin\vartheta}{r}u + \frac{w}{R_2};$$

$$\omega = \frac{1}{r}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial s} + \frac{\sin\vartheta}{r}v;$$

from (1.3) the following elasticity relationships:

$$\varepsilon_1 = \frac{1}{h}(a_{11}T_1 + a_{12}T_2 + a_{16}S);$$

$$\varepsilon_2 = \frac{1}{h}(a_{12}T_1 + a_{22}T_2 + a_{26}S);$$

$$\omega = \frac{1}{h}(a_{16}T_1 + a_{26}T_2 + a_{66}S).$$

Finally, by means of the last two groups of relationships we obtain the following inhomogeneous geometric equations:

$$\frac{\partial u}{\partial s} + \frac{w}{R_1} = \frac{1}{h}(a_{11}T_1 + a_{12}T_2 + a_{16}S); \tag{4.8}$$

$$\frac{1}{r}\frac{\partial v}{\partial \varphi} - \frac{\sin\vartheta}{r}u + \frac{w}{R_2} = \frac{1}{h}(a_{22}T_2 + a_{12}T_1 + a_{26}S); \tag{4.9}$$

$$\frac{1}{r}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial s} + \frac{\sin\vartheta}{r}v = \frac{1}{h}(a_{66}S + a_{16}T_1 + a_{26}T_2). \tag{4.10}$$

---

*A.I. Lur'ye, Statika tonkostennykh uprugykh obolochek, Gostekhizdat, 1947, pp. 7-12.

where, as usual, a_{ik} are the elastic constants of the shell material (Chapter I, Section 3).

For symmetrically loaded shells we must assume that the external loading may only be a function of arc s, that is,

$$X=X(s),\quad Y=Y(s),\quad Z=Z(s), \tag{4.11}$$

while the boundary conditions are independent of angular coordinate φ. Considering this as well as the fact that in the case of shells of revolution

$$R_1=R_1(s),\quad R_2=R_2(s),\quad r=r(s),\quad \vartheta=\vartheta(s), \tag{4.12}$$

on the basis of equations (4.5)-(4.6) it is easy to establish that in the case of a symmetrically loaded shell of revolution the internal forces (T_1, T_2, S) and displacements (u, v, w) cannot be functions of angular coordinate φ. By virtue of this, from (4.5)-(4.7) we obtain the following equilibrium equations:

$$\frac{dT_1}{ds}+(T_2-T_1)\frac{\sin\vartheta}{r}=-X; \tag{4.13}$$

$$\frac{dS}{ds}-2S\frac{\sin\vartheta}{r}=-Y; \tag{4.14}$$

$$T_2=R_2Z-\frac{R_2}{R_1}T_1. \tag{4.15}$$

Following V.V. Novozhilov,* let us introduce the new unknowns

$$U=T_1r\cos\vartheta,\quad V=Sr^2. \tag{4.16}$$

From the first two equilibrium equations (4.13) and (4.14), by virtue of (4.15) and (4.16), we obtain

$$\frac{dU}{ds}=-r(X\cos\vartheta+Z\sin\vartheta); \tag{4.17}$$

$$\frac{dV}{ds}=-r^2Y. \tag{4.18}$$

Integrating the latter for the new sought functions, we obtain

$$U=-\int_{s_0}^{s} r(X\cos\vartheta+Z\sin\vartheta)\,ds+U_0; \tag{4.19}$$

$$V=-\int_{s_0}^{s} r^2Y\,ds+V_0. \tag{4.20}$$

---

*V.V. Novozhilov, see p. 95 of work cited in footnote **, p. 63.

where U_0 and V_0 are integration constants which are determined from the boundary conditions of the shell while the lower limit of integration of s_0 may be arbitrarily chosen. Inserting the values of U and V from (4.19) and (4.20) into (4.16) and then into (4.15), we obtain for the internal forces T_1, T_2 and S

$$T_1 = -\frac{1}{R_2 \cos^2 \vartheta}\left[\int_{s_0}^{s} r(X\cos\vartheta + Z\sin\vartheta)\,ds - U_0\right]; \qquad (4.21)$$

$$T_2 = \frac{1}{R_1 \cos^2 \vartheta}\left[\int_{s_0}^{s} r(X\cos\vartheta + Z\sin\vartheta)\,ds - U_0\right] + R_2 Z; \qquad (4.22)$$

$$S = -\frac{1}{R_2^2 \cos^2 \vartheta}\left(\int_{s_0}^{s} r^2 Y\,ds - V_0\right). \qquad (4.23)$$

which, as was to be expected, do not differ from the corresponding formulas in the theory of a symmetrically loaded isotropic shell of revolution.

Let us turn to definition of the displacements. As in the case of the statical problem, from (4.8)-(4.10) we have

$$\frac{du}{ds} + \frac{w}{R_1} = \frac{1}{h}(a_{11}T_1 + a_{12}T_2 + a_{16}S); \qquad (4.24)$$

$$\frac{\sin\vartheta}{r}u + \frac{w}{R_2} = \frac{1}{h}(a_{22}T_2 + a_{12}T_1 + a_{26}S); \qquad (4.25)$$

$$\frac{dv}{ds} + \frac{\sin\vartheta}{r}v = \frac{1}{h}(a_{16}T_1 + a_{26}T_2 + a_{66}S). \qquad (4.26)$$

From (4.24), considering (4.15), for the normal displacement we obtain

$$w = \frac{R_1}{h}\left[\left(a_{11} - a_{12}\frac{R_1}{R_1}\right)T_1 + a_{16}S + a_{12}R_2 Z\right] - R_1\frac{du}{ds}; \qquad (4.27)$$

inserting the value of w into (4.25) and (4.26) and with consideration of (4.15), we obtain

$$\frac{du}{ds} + \frac{R_2}{R_1}\frac{\sin\vartheta}{r}u = \left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{T_1}{h} +$$
$$+\left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{S}{h} + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2}{h}Z; \qquad (4.28)$$

$$\frac{dv}{ds} + \frac{\sin\vartheta}{r}v = \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{T_1}{h} + a_{66}\frac{S}{h} + a_{26}\frac{R_2}{h}Z. \qquad (4.29)$$

As before, following V.V. Novozhilov,* let us introduce the new unknowns

---

*V.V. Novozhilov, see p. 96 of work cited in footnote **, p. 63.

$$\varphi = \frac{u}{\cos\vartheta}, \quad \psi = \frac{v}{r}; \tag{4.30}$$

then instead of (4.28) and (4.29) we obtain

$$\frac{d\varphi}{ds} = \left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{R_2}{r}\frac{T_1}{h} +$$

$$+\left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{R_2}{r}\frac{S}{h} + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2^2}{rh}Z; \tag{4.31}$$

$$\frac{d\psi}{ds} = \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{T_1}{rh} + a_{66}\frac{S}{rh} + a_{26}\frac{R_2}{rh}Z. \tag{4.32}$$

Integrating the latter, for the new sought functions we obtain

$$\varphi = \frac{1}{h}\left\{\int\limits_{s_0}^{s}\left[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{R_2}{r}T_1 +\right.\right.$$

$$\left.\left. + \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{R_2}{r}S + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2^2}{r}Z\right]ds\right\} + \varphi_0; \tag{4.33}$$

$$\psi = \frac{1}{h}\left\{\int\limits_{s_0}^{s}\left[\left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{T_1}{r} + a_{66}\frac{S}{r} + a_{26}\frac{R_2}{r}Z\right]ds\right\} + \psi_0. \tag{4.34}$$

where φ_0 and ψ_0 are integration constants which are determined from the boundary conditions of the shell, while the lower limit of integration s_0 is chosen as in the case of the statical problem. Inserting the values of φ and ψ from (4.33) and (4.34) into (4.30) and then into (4.27), we obtain for displacements u, v and w

$$u = \frac{\cos\vartheta}{h}\left\{\int\limits_{s_0}^{s}\left[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{R_2}{r}T_1 +\right.\right.$$

$$\left.\left. + \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{R_2}{r}S + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2^2}{r}Z\right]ds\right\} + \varphi_0\cos\vartheta; \tag{4.35}$$

$$v = \frac{r}{h}\left\{\int\limits_{s_0}^{s}\left[\left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{T_1}{r} + a_{66}\frac{S}{r} + a_{26}\frac{R_2}{r}Z\right]ds\right\} + \psi_0 r; \tag{4.36}$$

$$w = \frac{\sin\vartheta}{h}\left\{\int\limits_{s_0}^{s}\left[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{R_2}{r}T_1 +\right.\right.$$

$$\left.\left. + \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{R_2}{r}S + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2^2}{r}Z\right]ds\right\} +$$

$$+ \frac{R_2}{h}\left[\left(a_{12} - a_{22}\frac{R_2}{R_1}\right)T_1 + a_{26}S + a_{22}R_2Z\right] + \varphi_0\sin\vartheta. \tag{4.37}$$

The formulas derived here for the displacements are fundamentally different from the corresponding formulas obtained in the case of a symmetrically loaded, isotropic shell of revolution. Here, in distinction from the problem of an isotropic shell, each displacement (u,v,w) depends on all three components (X,Y,Z) of the external surface load. By virtue of this it is easily seen that when a symmetrically loaded anisotropic shell of revolution is statically indeterminate (that is, when the boundary conditions are such that the integration constants U_0 and V_0 may not be determined without use of relationships (4.35)-(4.37)), each internal force (T_1, T_2, S) depends on all three components of the external surface load. In the case where the shell is statically determinate (that is, the boundary conditions are such that integration constants U_0 and V_0 are determined directly from the boundary conditions and relationships (4.21)-(4.23) without use of (4.35)-(4.37)), then, as in isotropic shells, the components of the external load X and Z in the shell cause only internal forces T_1 and T_2, while component Y causes only internal force S. The same remarks apply to the displacements, each of them depending on all three components of the external surface load.

In the special case of an orthotropic shell (Chapter II, Section 14), when at each point of the shell one of the planes of elastic symmetry is parallel to the median surface of the shell and the remaining two planes are perpendicular to coordinate lines $s = \text{const}$, and $\varphi = \text{const}$, the formulas for internal forces (4.21)-(4.23) remain unchanged and for the displacements we obtain

$$
\begin{aligned}
u = \frac{\cos\vartheta}{h}\Bigg\{\int_{s_0}^{s}\Bigg[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{R_2}{r}T_1 &+ \\
+ \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2^2}{r}Z\Bigg]\,ds\Bigg\} &+ \varphi_0\cos\vartheta; \qquad (4.38)
\end{aligned}
$$

$$
v = a_{66}\frac{r}{h}\int_{s_0}^{s}\frac{S}{r}\,ds + \psi_0 r; \qquad (4.39)
$$

$$
\begin{aligned}
w = \frac{\sin\vartheta}{h}\Bigg\{\int_{s_0}^{s}\Bigg[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{R_2}{r}T_1 &+ \\
+ \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R_2^2}{r}Z\Bigg]\,ds\Bigg\} &+ \\
+ \frac{R_2}{h}\left[\left(a_{12} - a_{22}\frac{R_2}{R_1}\right)T_1 + a_{22}R_2 Z\right] &+ \varphi_0\sin\vartheta. \qquad (4.40)
\end{aligned}
$$

Here, as in the case of a symmetrically loaded, isotropic shell of revolution, displacements u and w are caused by only two components (X and Z) of the external load, while displacement v is caused by only one component (Y) of the external load. Generally, in the case of a symmetrically loaded, orthotropic shell* of revolution components of the external load X and Z in the shell cause only the internal forces T_1 and T_2, while the Y component causes only the internal force S.

We will note that the first group of final relationships (4.21)-(4.23) for the statical problem of a symmetrically loaded shell of revolution contains two integration constants (U_0, V_0), while the second group of final relationships (4.35)-(4.37) for the geometric problem of a symmetrically loaded, anisotropic shell of revolution contains four integration constants U_0, V_0, φ_0, ψ_0. Hence, as in the case of an isotropic shell, for definiteness of solution of the presented problem, at each of the two parallel circles delimiting the shell two boundary conditions must be given, of which conditions at least two must be given in displacements, otherwise the shell may undergo pure shear deformation or will be displaced as a solid body. In the special case in which the shell of revolution has one edge (that is, it is closed at the vertex), the number of boundary conditions is decreased by half.**

SECTION 5. EXAMPLES OF CALCULATION OF SYMMETRICALLY LOADED SHELLS OF REVOLUTION

A. Cylindrical Shells.*** For the median (coordinate) surface of a circular cylindrical shell, by virtue of (4.1)-(4.3), we have (Figure 19):

$$A=1, \quad B=r=R_2=R, \quad R_1=\infty, \quad \vartheta=0. \tag{5.1}$$

Then, from (4.21)-(4.23), for the internal forces we obtain

$$T_1=-\int_{s_0}^{s} X\,ds+\frac{U_0}{K}; \tag{5.2}$$

$$T_2=RZ; \tag{5.3}$$

---

*Here and hereafter, unless otherwise indicated, the term "orthotropic shell" will signify a shell of orthotropic material for which the principal directions of elasticity at each point of the shell coincide with the directions of the coordinate lines.

**V.V. Novozhilov, see p. 99 of work cited in footnote ** on p. 63.

***L.A. Movsisyan, Some specific characteristics of anisotropic shells, Izvestiya AN ArmSSR (fiz.-mat. nauki), Vol. XI, No. 4, 1958.

$$S = -\int_{s_0}^{s} Y\,ds + \frac{V_0}{R^2}. \tag{5.4}$$

From (4.35)-(4.37) we obtain for the displacements

$$u = \frac{1}{h}\int_{s_0}^{s} (a_{11}T_1 + a_{16}S + a_{12}RZ)\,ds + \varphi_0; \tag{5.5}$$

$$v = \frac{1}{h}\int_{s_0}^{s} (a_{16}T_1 + a_{66}S + a_{26}RZ)\,ds + \psi_0 R; \tag{5.6}$$

$$w = \frac{R}{h}(a_{12}T_1 + a_{26}S + a_{22}RZ). \tag{5.7}$$

Let us solve a few examples of cylindrical shells.

1. A closed circular shell (radius of curvature R, length L), supports a uniformly distributed, normally applied surface load of intensity q. The shell boundaries are defined by two transverse sections perpendicular to the axis of revolution z. One of the ends $(s_0 = 0)$ of the shell is completely fixed and the other end $(s_1 = L)$ is completely free.

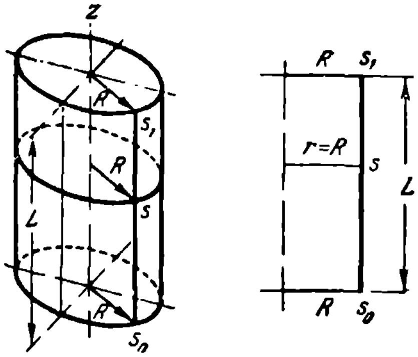

Figure 19

We have the following boundary conditions:

$$s = 0:\ u = 0,\ v = 0; \tag{5.8}$$

$$s = L:\ T_1 = 0,\ S = 0 \tag{5.9}$$

and the components of the surface load

$$X=0, \quad Y=0, \quad Z=q. \tag{5.10}$$

From (5.2)-(5.4), by virtue of (5.9) and (5.10), for the internal forces we obtain

$$T_1=0, \quad S=0, \quad T_2=Rq. \tag{5.11}$$

Inserting the values of T_1, T_2 and S from (5.11) into (5.5)-(5.7) and performing appropriate transformations (here and in the following examples it is assumed that the lower limit of integration $s_0=0$). for the displacements we obtain

$$\begin{aligned} u &= a_{12}\frac{R}{h}sq+\varphi_0; \\ v &= a_{26}\frac{R}{h}sq+\psi_0 R; \\ w &= a_{22}\frac{R^2}{h}q. \end{aligned}$$

Hence, by virtue of (5.8) for the integration constants we obtain $\varphi_0=0$, and $\psi_0=0$, and, finally, for the displacements we obtain

$$u=a_{12}\frac{R}{h}qs, \quad v=a_{26}\frac{R}{h}qs, \quad w=a_{22}\frac{R^2}{h}q. \tag{5.12}$$

In the special case of an orthotropic shell and the internal forces (5.11) remain unchanged and for the displacements we obtain

$$u=a_{12}\frac{R}{h}qs, \quad v=0, \quad w=a_{22}\frac{R^2}{h}q. \tag{5.13}$$

2. A closed circular shell has one end $(s_0=0)$ completely fixed and the other $(s_1 = L)$ sustains a uniformly distributed shear force of intensity S^*. We have the following boundary conditions:

for $s=0$: $u=0, \quad v=0;$ (5.14)
for $s=L$: $T_1=0, \quad S=S^*,$ (5.15)

and the components of the surface load

$$X=0, \quad Y=0, \quad Z=0. \tag{5.16}$$

From (5.2)-(5.4), by virtue of (5.15) and (5.16), for the internal forces we obtain

$$T_1 = 0, \quad T_2 = 0, \quad S = S^*. \tag{5.17}$$

Inserting the values of T_1, T_2 and S from (5.17) into (5.5)-(5.7) and performing transformations, for the displacements we obtain

$$\begin{aligned} u &= a_{16}\frac{s}{h}S^* + \varphi_0; \\ v &= a_{66}\frac{s}{h}S^* + \psi_0 R; \\ w &= a_{26}\frac{R}{h}S^*. \end{aligned}$$

Hence, on the basis of (5.14), for the integration constants we obtain $\varphi_0 = 0$, and $\psi_0 = 0$, and, finally, for the displacements we have

$$u = a_{16}\frac{s}{h}S^*, \quad v = a_{66}\frac{s}{h}S^*, \quad w = a_{26}\frac{R}{h}S^*. \tag{5.18}$$

In the special case of an orthotropic shell the internal forces (5.17) remain unchanged and we obtain for the displacements

$$u = 0, \quad v = a_{66}\frac{s}{h}S^*, \quad w = 0. \tag{5.19}$$

3. A closed circular shell sustains a uniformly distributed, normally applied surface load of intensity q. The ends of the shell ($s_0 = 0$ and $s_1 = L$) are completely fixed. Here we have the following boundary conditions:

$$s = 0: \quad u = 0, \quad v = 0; \tag{5.20}$$

$$s = L: \quad u = 0, \quad v = 0, \tag{5.21}$$

and the components of the surface load

$$X = 0, \quad Y = 0, \quad Z = q. \tag{5.22}$$

From (5.2)-(5.4), by virtue of (5.22), for the internal forces we have

$$T_1 = \frac{U_0}{R}, \quad T_2 = Rq, \quad S = \frac{V_0}{R^2}, \tag{5.23}$$

where, in distribution from the two previous examples, the integration constants U_0 and V_0 cannot be determined by means of (5.23) and boundary conditions (5.20) and (5.21). Thus we have an external statically indeterminate problem and the integration constants U_0 and V_0 may be determined only with the use of formulas (5.5) and (5.6) for the tangential displacements u and v.

From (5.5) and (5.6) by virtue of (5.22) and (5.23), for the tangential displacements we obtain

$$\left.\begin{aligned} u &= \frac{s}{h}\left(a_{11}\frac{U_0}{R}+a_{16}\frac{V_0}{R^2}+a_{12}Rq\right)+\varphi_0;\\ v &= \frac{s}{h}\left(a_{16}\frac{U_0}{R}+a_{66}\frac{V_0}{R^2}+a_{26}Rq\right)+\psi_0 R. \end{aligned}\right\} \qquad (5.24)$$

Hence, from the boundary conditions in (5.20) and (5.21), for the integration constants φ_0, and ψ_0, U_0 and V_0 we obtain

$$\varphi_0=0, \quad \psi_0=0;$$

$$U_0=\frac{a_{16}a_{26}-a_{12}a_{66}}{a_{11}a_{66}-a_{16}^2}R^2q;$$

$$V_0=\frac{a_{12}a_{16}-a_{11}a_{26}}{a_{11}a_{66}-a_{16}^2}R^3q.$$

Then from (5.23) and (5.24), for the internal forces and displacements we finally obtain

$$T_1=\frac{a_{16}a_{26}-a_{12}a_{66}}{a_{11}a_{66}-a_{16}^2}Rq; \qquad (5.25)$$

$$T_2=Rq; \qquad (5.26)$$

$$S=\frac{a_{12}a_{16}-a_{11}a_{26}}{a_{11}a_{66}-a_{16}^2}Rq; \qquad (5.27)$$

$$u=0; \quad v=0; \qquad (5.28)$$

$$w=\frac{(a_{11}a_{22}-a_{12}^2)a_{66}-a_{22}a_{16}^2+2a_{12}a_{16}a_{26}-a_{11}a_{26}^2}{a_{11}a_{66}-a_{16}^2}\,\frac{R^2q}{h}. \qquad (5.29)$$

In the special case of an orthotropic shell, for the internal forces and displacements we obtain

$$T_1=-\frac{a_{12}}{a_{11}}Rq; \quad T_2=Rq; \quad S=0; \qquad (5.30)$$

$$u=0; \quad v=0; \quad w=\frac{a_{11}a_{22}-a_{12}^2}{a_{11}}\,\frac{R^2q}{h}. \qquad (5.31)$$

B. Conical Shells. On the basis of the fourth condition of applicability of the membrane theory of shells it is here assumed that the investigated conical shells do not contain the vertex of a cone.

For the median surface of a circular conical shell, by virtue of (4.1) and (4.3), we have (Figure 20)

$$\begin{aligned} A &= 1, \\ r = B &= (s' - s)\sin\alpha, \\ R_1 &= \infty, \\ R_2 &= (s' - s)\operatorname{tg}\alpha, \\ \vartheta &= \alpha, \end{aligned} \tag{5.32}$$

where s' is the length of the generatrix of the complete cone.

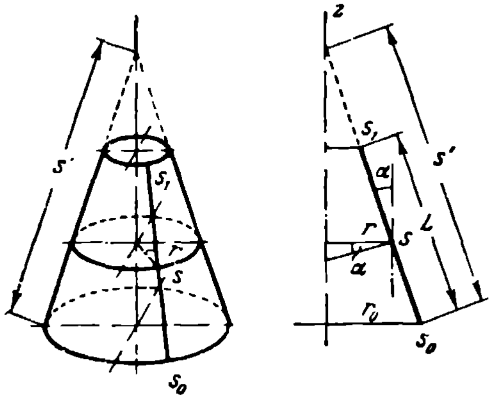

Figure 20

On the basis of (5.32), from (4.21)-(4.23) for the internal forces we obtain

$$T_1 = \frac{1}{(s' - s)\cos\alpha\sin\alpha}\left[\int_{s_0}^{s} (X\cos\alpha + Z\sin\alpha)(s' - s)\sin\alpha\, ds - U_0\right]; \tag{5.33}$$

$$T_2 = Z(s' - s)\operatorname{tg}\alpha; \tag{5.34}$$

$$S = -\frac{1}{(s' - s)^2\sin^2\alpha}\left[\int_{s_0}^{s} Y(s' - s)^2\sin^2\alpha\, ds - V_0\right]. \tag{5.35}$$

From (4.35)-(4.37) we have for the displacements

$$u = \frac{1}{h}\int_{s_0}^{s} [a_{11}T_1 + a_{16}S + a_{12}(s' - s)Z\operatorname{tg}\alpha]\, ds + \varphi_0\cos\alpha; \tag{5.36}$$

$$v = \frac{(s' - s)}{h}\int_{s_0}^{s} [a_{16}T_1 + a_{66}S + a_{26}(s' - s)Z\operatorname{tg}\alpha]\frac{ds}{(s' - s)} + \\ + \psi_0(s' - s)\sin\alpha; \tag{5.37}$$

$$w = \frac{\operatorname{tg} \alpha}{h} \int_{s_0}^{s} [a_{11} T_1 + a_{16} S + a_{12} (s' - s) Z \operatorname{tg} \alpha]\, ds +$$
$$+ \frac{(s' - s) \operatorname{tg} \alpha}{h} [a_{12} T_1 + a_{26} S + a_{22} (s' - s) Z \operatorname{tg} \alpha] + \varphi_0 \sin \alpha. \tag{5.38}$$

Let us discuss a few examples of calculation of conical shells.

1. A closed truncated cone sustains a uniformly distributed, normally applied surface load of intensity q. The boundaries of the shell are defined by two transverse sections perpendicular to the axis of revolution z. One of the ends of the shell $(s_0 = 0)$ is completely fixed and the other end $(s_1 = L)$ is completely free. We have the following boundary conditions:

$$\text{for} \qquad s = 0: \quad u = 0, \quad v = 0; \tag{5.39}$$
$$\text{for} \qquad s = L: \quad T_1 = 0, \quad S = 0, \tag{5.40}$$

and the components of the surface load

$$X = 0, \quad Y = 0, \quad Z = q. \tag{5.41}$$

From (5.33)-(5.35), by virtue of (5.41) we have for the internal forces

$$T_1 = -\frac{1}{(s' - s) \sin \alpha \cos \alpha} \left[q \left(s's - \frac{s^2}{2} \right) \sin^2 \alpha - U_0 \right]$$
$$T_2 = q (s' - s) \operatorname{tg} \alpha, \quad S = \frac{V_0}{(s' - s)^2 \sin^2 \alpha}.$$

Hence, from the boundary conditions in (5.40), for the integration constants U_0 and V_0 we obtain

$$U_0 = q \left(s'L - \frac{L^2}{2} \right) \sin^2 \alpha, \quad V_0 = 0.$$

Then, finally, for the internal forces we write

$$\left. \begin{aligned} T_1 &= q \frac{s^2 - L^2 + 2s' (L - s)}{2 (s' - s)} \operatorname{tg} \alpha; \\ T_2 &= q (s' - s) \operatorname{tg} \alpha; \quad S = 0. \end{aligned} \right\} \tag{5.42}$$

Inserting the values of T_1, S and Z from (5.42) and (5.41) into (5.36) and (5.37) and performing transformations, for the tangential displacements we obtain

$$u = q\left\{a_{11}\left[s's - \frac{s^2}{2} + (s' - L)^2 \ln (s' - s)\right] + \right.$$
$$\left. + 2a_{12}\left(s's - \frac{s^2}{2}\right)\right\} \frac{\operatorname{tg} \alpha}{2h} + \varphi_0 \cos \alpha;$$
$$v = q\left\{a_{16} \frac{\left[s - (s' - L)^2 \frac{1}{s' - s}\right]}{2} + a_{26}s\right\} \frac{s' - s}{h} \operatorname{tg} \alpha + \psi_0 (s' - s) \sin \alpha.$$

Hence, from the boundary conditions in (5.39), for the integration constants we obtain

$$\varphi_0 = -\frac{q}{2h} \frac{\operatorname{tg} \alpha}{\cos \alpha} [a_{11} (s' - L)^2 \ln s'];$$
$$\psi_0 = \frac{q}{2h} \frac{1}{\cos \alpha} a_{16} \frac{(s' - L)^2}{s'}.$$

Then, finally, for the displacements we obtain

$$u = \left\{a_{11}\left[s's - \frac{s^2}{2} + (s' - L)^2 \ln\left(1 - \frac{s}{s'}\right)\right] + \right.$$
$$\left. + 2a_{12}\left(s's - \frac{s^2}{2}\right)\right\} \frac{\operatorname{tg} \alpha}{2h} q, \tag{5.43}$$
$$v = \left\{\frac{a_{16}}{2}\left[s - \frac{(s' - L)^2 s}{s' (s' - s)}\right] + a_{26}s\right\} \frac{(s' - s) \operatorname{tg} \alpha}{h} q, \tag{5.44}$$
$$w = \left\{\left[a_{12} \frac{s^2 - L^2 + 2 (L - s) s'}{(s' - s)} + 2a_{22} (s' - s)\right] (s' - s) + \right.$$
$$+ a_{11}\left[s's - \frac{s^2}{2} + (s' - L)^2 \ln\left(1 - \frac{s}{s'}\right)\right] +$$
$$\left. + 2a_{12}\left(s's - \frac{s^2}{2}\right)\right\} \frac{\operatorname{tg}^2 \alpha}{2h} q. \tag{5.45}$$

In the special case of an orthotropic shell all the internal forces and displacements except tangential displacement v remain unchanged. Displacement v becomes zero.

2. A closed truncated conical shell is completely fixed at one end ($s_0 = 0$) and the other end section ($s_1 = L$) without undergoing any longitudinal displacement, sustains a uniformly distributed shear force of intensity S*. Here we have the following boundary conditions:

$$\text{for} \quad s = 0: \quad u = 0, \quad v = 0; \tag{5.46}$$
$$\text{for} \quad s = L: \quad u = 0, \quad S = S^*, \tag{5.47}$$

and the components of the surface load

$$X = 0, \quad Y = 0, \quad Z = 0. \tag{5.48}$$

From (5.33)-(5.35), by virtue of (5.48), for the internal forces we have

$$T_1=\frac{U_0}{(s'-s)\sin\alpha\cos\alpha},\quad T_2=0,\quad S=\frac{V_0}{(s'-s)^2\sin^2\alpha}. \tag{5.49}$$

From the second boundary condition (5.47) and the last relationship for S we obtain the following value for integration constant V_0:

$$V_0=S^*(s'-L)^2\sin^2\alpha;$$

finally, for s we find

$$S=\frac{(s'-L)^2}{(s'-s)^2}S^* \tag{5.50}$$

Integration constant U_0 cannot be determined by means of the boundary conditions and relationship (5.49) derived for T_1. Here, as in the case of the third example of a cylindrical shell, we have an external statically indeterminate problem and integration constant U_0 may be determined only with use of the formulas for tangential displacement u.

From (5.36), by virtue of (5.48)-(5.50), for displacement u we have

$$u=-a_{11}\frac{U_0}{h\sin\alpha\cos\alpha}\ln(s'-s)+a_{16}\frac{(s'-L)^2}{h}\frac{1}{(s'-s)}S^*+\varphi_0\cos\alpha.$$

Hence, by means of the first of the boundary conditions (5.46) and (5.47) for the integration constants U_0 and φ_0 we obtain

$$U_0=S^*\frac{a_{16}}{a_{11}}\frac{(s'-L)L}{s'\ln\left(1-\frac{L}{s'}\right)}\sin\alpha\cos\alpha;$$

$$\varphi_0=S^*\frac{a_{16}}{h}\left[\frac{(s'-L)L\ln(s'-L)}{s'\ln\left(1-\frac{L}{s'}\right)}-(s'-L)\right]\frac{1}{\cos\alpha}.$$

Inserting the resulting values of the integration constants into the formulas for internal force T_1 and tangential displacement u, we finally obtain

$$T_1=\frac{a_{16}}{a_{11}}\frac{(s'-L)L}{(s'-s)s'\ln\left(1-\frac{L}{s'}\right)}S^*; \tag{5.51}$$

$$u=a_{16}\left[\frac{s'-L}{s'-s}+\frac{L\ln\left(\frac{s'-L}{s'-s}\right)}{s'\ln\left(1-\frac{L}{s'}\right)}-1\right]\frac{s'-L}{h}S^*. \tag{5.52}$$

Inserting the values of T, Z and S into (5.37) and considering the second of the boundary conditions in (5.46), after conversions we obtain for the tangential displacements v

$$v=\frac{a_{16}}{a_{11}}\frac{(s'-L)Ls}{h(s')^2\ln\left(1-\frac{L}{s'}\right)}S^*+\frac{a_{66}}{2h}(s'-L)^2\left[\frac{1}{s'-s}-\frac{s'-s}{(s')^2}\right]S^*. \tag{5.53}$$

Finally, from (5.38), for the normal displacement ω we find:

$$w=\left[\frac{a_{12}a_{16}}{a_{11}}\frac{(s'-L)L}{s'\ln\left(1-\frac{L}{s'}\right)}+a_{26}\frac{(s'-L)^2}{(s'-s)}\right]\frac{\operatorname{tg}\alpha}{h}S^*+$$
$$+a_{16}(s'-L)\left[\frac{s'-L}{s'-s}+\frac{L\ln\left(\frac{s'-L}{s'-s}\right)}{s'\ln\left(1-\frac{L}{s'}\right)}-1\right]\frac{\operatorname{tg}\alpha}{h}S^*. \tag{5.54}$$

In the special case of an orthotropic shell, for the internal forces and displacements we have:

$$T_1=0;\ T_2=0;\ S=\frac{(s'-L)^2}{(s'-s)^2}S^*; \tag{5.55}$$

$$u=0;\ w=0;\ v=\frac{a_{66}}{2h}(s'-L)^2\left[\frac{1}{s'-s}-\frac{s'-s}{(s')^2}\right]S^*. \tag{5.56}$$

C. Spherical Shells. In the case of a spherical shell, if it is assumed that arc s is reckoned from the equator of the sphere, for the median surface of the spherical shell, on the basis of (4.1) and (4.3), we have (Figure 21)

$$\left.\begin{aligned}&A=1;\ B=r=R\cos\frac{s}{R};\\&R_1=R_2=R;\ \vartheta=\frac{s}{R}.\end{aligned}\right\} \tag{5.57}$$

where R is the radius of the sphere.

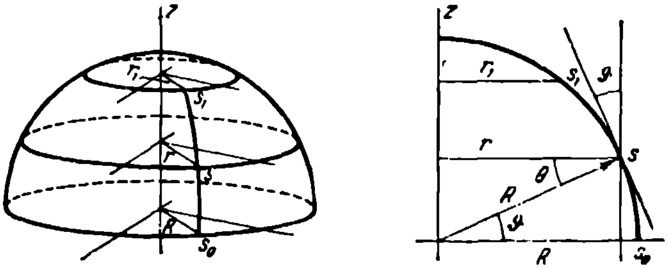

Figure 21

Using (5.57), from (4.21)-(4.23) for the internal forces we obtain

$$T_1 = -\frac{1}{R\cos^2\frac{s}{R}}\left[\int_{s}^{s}\left(X\cos\frac{s}{R} + Z\sin\frac{s}{R}\right)R\cos\frac{s}{R}\,ds - U_0\right]; \tag{5.58}$$

$$T_2 = \frac{1}{R\cos^4\frac{s}{R}}\left[\int_{s_0}^{s}\left(X\cos\frac{s}{R} + Z\sin\frac{s}{R}\right)R\cos\frac{s}{R}\,ds - U_0\right] + RZ; \tag{5.59}$$

$$S = -\frac{1}{R^2\cos^2\frac{s}{R}}\left(\int_{s_0}^{s} YR^2\cos^2\frac{s}{R}\,ds - V_0\right). \tag{5.60}$$

From (4.35)-(4.37) for the displacements we have

$$u = \frac{\cos\frac{s}{R}}{h}\int_{s_0}^{s}[(a_{11} - 2a_{12} + a_{22})T_1 + (a_{16} - a_{26})S + $$

$$+ (a_{12} - a_{22})RZ]\frac{ds}{\cos\frac{s}{R}} + \varphi_0\cos\frac{s}{R}; \tag{5.61}$$

$$v = \frac{\cos\frac{s}{R}}{h}\int_{s_0}^{s}[(a_{16} - a_{26})T_1 + a_{66}S + a_{26}RZ]\frac{ds}{\cos\frac{s}{R}} + \psi_0 R\cos\frac{s}{R}; \tag{5.62}$$

$$w = \frac{\sin\frac{s}{R}}{h}\int_{s_0}^{s}[(a_{11} - 2a_{12} + a_{22})T_1 + (a_{16} - a_{26})S + (a_{12} - a_{22})RZ]\times$$

$$\times\frac{ds}{\cos\frac{s}{R}} + \frac{R}{h}[(a_{12} - a_{22})T_1 + a_{26}S + a_{22}RZ] + \varphi_0\sin\frac{s}{R}. \tag{5.63}$$

Let us discuss a few examples of calculation of spherical shells.

1. One of the ends ($s_0 = 0$) of a closed spherical section is completely fixed and the other end ($s_1 = L$) sustains a uniformly distributed tangential force of intensity T*. We have the following boundary conditions:

$$s = 0:\quad u = 0,\quad v = 0; \tag{5.64}$$

$$s = L:\quad T_1 = T^*,\quad S = 0, \tag{5.65}$$

and the components of the surface load

$$X = 0,\quad Y = 0,\quad Z = 0. \tag{5.66}$$

From (5.58)-(5.60), by virtue of (5.66), for the internal forces we have

$$T_1 = \frac{U_0}{R\cos^2\frac{s}{R}}, \quad T_2 = -\frac{U_0}{R\cos^2\frac{s}{R}}, \quad S = \frac{V_0}{R^2\cos^2\frac{s}{R}}.$$

hence, by virtue of the boundary conditions in (5.65), for the integration constants U_0 and V_0 we obtain

$$U_0 = T^* R\cos^2\frac{L}{R}, \quad V_0 = 0.$$

Then, finally, for the internal forces we write

$$T_1 = \frac{\cos^2\frac{L}{R}}{\cos^2\frac{s}{R}} T^*, \quad T_2 = -\frac{\cos^2\frac{L}{R}}{\cos^2\frac{s}{R}}, \quad S = 0. \tag{5.67}$$

Inserting the values of T_1, S and Z into (5.61) and (5.62), for the tangential displacements we obtain

$$u = \frac{R\cos^2\frac{L}{R}}{2h}(a_{11} - 2a_{12} + a_{22})\left[\ln \operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right)\cos\frac{s}{R} + \operatorname{tg}\frac{s}{R}\right]T^* + \\ + \varphi_0 \cos\frac{s}{R};$$

$$v = \frac{R\cos^2\frac{L}{R}}{2h}(a_{16} - a_{26})\left[\ln \operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right)\cos\frac{s}{R} + \operatorname{tg}\frac{s}{R}\right]T^* + \\ + \psi_0 R\cos\frac{s}{R}.$$

Hence, from the boundary conditions in (5.64), for the integration constants we obtain $\varphi_0 = 0$, and $\psi_0 = 0$. Then, finally, for the displacements we find

$$u = \frac{RT^*}{2h}(a_{11} - 2a_{12} + a_{22})\left[\operatorname{tg}\frac{s}{R} + \right. \\ \left. + \cos\left(\frac{s}{R}\right)\ln \operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right)\right]\cos^2\frac{L}{R}, \tag{5.68}$$

$$v = \frac{RT^*}{2h}(a_{16} - a_{26})\left[\cos\left(\frac{s}{R}\right)\ln \operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right) + \operatorname{tg}\frac{s}{R}\right]\cos^2\frac{L}{R}, \tag{5.69}$$

$$w = \frac{RT^*}{h}(a_{12} - a_{22})\frac{\cos^2\frac{L}{R}}{\cos^2\frac{s}{R}} + \frac{RT^*}{2h}(a_{11} - 2a_{12} + a_{22}) \times \\ \times \left[\sin\left(\frac{s}{R}\right)\ln \operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right) + \operatorname{tg}^2\frac{s}{R}\right]\cos^2\frac{L}{R}. \tag{5.70}$$

In the special case of an orthotropic shell all internal forces and displacements (except tangential displacement v) remain unchanged. Displacement v becomes zero.

2. A closed hemispherical shell (shell with a pole) sustains a uniformly distributed, normally applied surface load of intensity q. The one edge of the shell $(s_0 = 0)$, which coincides with the equatorial line is completely fixed.

We have the following boundary conditions:

$$\text{for} \qquad s = 0: \quad u = 0, \quad v = 0, \tag{5.71}$$

and the components of the surface load are

$$X = 0, \quad Y = 0, \quad Z = q. \tag{5.72}$$

From (5.58)-(5.60), on the basis of (5.72) we obtain for the internal forces

$$T_1 = \frac{Rq}{2} + \frac{U_0}{R\cos^2\frac{s}{R}}; \quad T_2 = \frac{Rq}{2} - \frac{U_0}{R\cos^2\frac{s}{R}}; \quad S = \frac{V_0}{R^2\cos^2\frac{s}{R}}.$$

From the condition that the internal forces at the pole of the sphere $\left(s = \frac{\pi R}{2}\right)$ are finite, we conclude that $U_0 = 0$, and $V_0 = 0$; finally, for the internal forces we obtain

$$T_1 = \frac{Rq}{2}, \quad T_2 = \frac{Rq}{2}, \quad S = 0. \tag{5.73}$$

Inserting the values of T_1, S and Z into (5.61) and (5.62), for the tangential displacements we obtain

$$u = \frac{R^2q}{2h}(a_{11} - a_{22})\cos\left(\frac{s}{R}\right)\ln\operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right) + \varphi_0\cos\frac{s}{R},$$
$$v = \frac{R^2q}{2h}(a_{16} - a_{26})\cos\left(\frac{s}{R}\right)\ln\operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right) + \psi_0 R\cos\frac{s}{R}.$$

Hence, by virtue of the boundary conditions in (5.71), for the integration constants we obtain $\varphi_0 = 0$, and $\psi_0 = 0$ and, finally, for the displacements we obtain

$$u = \frac{R^2q}{2h}(a_{11} - a_{22})\cos\left(\frac{s}{R}\right)\ln\operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right); \tag{5.74}$$
$$v = \frac{R^2q}{2h}(a_{16} - a_{26})\cos\left(\frac{s}{R}\right)\ln\operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right); \tag{5.75}$$

$$w = \frac{R^2 q}{2h}\left[(a_{11} - a_{22})\sin\left(\frac{s}{R}\right)\ln \operatorname{tg}\left(\frac{s}{2R} + \frac{\pi}{4}\right) + (a_{12} + a_{22})\right]. \qquad (5.76)$$

In the special case of an orthotropic shell all the internal forces and displacements except tangential displacement v remain unchanged; v becomes zero.

SECTION 6. EVALUATION OF RESULTS OBTAINED IN THE PRECEDING SECTION

The examples discussed in the preceding section permit us to draw certain conclusions concerning the specific characteristics of the stress and displacement of anisotropic shells in which moments are absent.

Analysis of the first examples of cylindrical and conical shells as well as of both examples of a spherical shell shows that, in distinction from isotropic shells, in anisotropic symmetrical shells under the action of uniformly distributed, normally applied load q or of a load T* uniformly applied to the edge there appear tangential displacements v (that is, there occurs twisting relative to the axis of revolution z); after deformation initially plane meridian lines of the shell become helical. It is also characteristic that when a normally applied surface load q or an external edgewise force T* change sign, the discussed shells undergo twisting in the opposite direction (that is, the sign of tangential displacement v also changes).

In examining the second examples of cylindrical and conical shells it is easily seen that, in distinction from the similar problems of isotropic shells, the longitudinal internal forces T_1 as well as the normal and tangential displacements ω and u are different from zero. Despite the absence of normal Z component and tangential X, Y components of external surface load as well as the presence of complete axial symmetry, in the discussed shells, under the action of uniformly distributed edgewise shear forces S* (or, which is the same thing, under the action of the corresponding torsional moments), there appear longitudinal forces T_1, normal ω and longitudinal u displacements which, depending on the direction of torsion, change sign (that is, if, for example, in the one case there appear in the shell tensile forces T_1 or there is a decrease in the radius of curvature of the shell, while in the other case, when the external force (or moment) changes sign, there appear in the shell compressive forces T_1 or there is an increase in the radius of curvature of the shell).

Finally, from the third example of a cylindrical shell it follows that, despite the complete axial symmetry of the discussed shell, under the action of a normally applied uniformly distributed surface load q there appear in it internal shear forces S.

In the case of orthotropic shells, when the principal directions of elasticity at each point of the shell coincide with the directions of the coordinate lines, we encounter problems in which the derived results, differing quantitatively from similar results for the corresponding isotropic shells, do not contain any fundamental differences. However, problems of orthotropic shells often contain qualitatively new results which differ fundamentally from the similar results for the corresponding isotropic shells. For example, in the second example of a spherical shell the tangential u displacement and normal w displacements contain terms with the factor $(a_{11} - a_{22})$, which, depending on the ratio $\frac{a_{11}}{a_{22}}$ may lead to results which are fundamentally different from the corresponding results for isotropic shells.

SECTION 7. CONTINUATION OF SECTION 5

The conclusions of the previous section may also be extended to other types of shells of revolution. In this connection the present section discusses additional examples of shells of revolution.

A. Shells of Revolution Generated by Revolution of an Arc of a Circle. Let a shell of positive curvature be generated by reduction of a certain part of the arc of a circle of radius R about the axis of revolution of the shell z, removed from the axis 0-0 of the corresponding sphere (of radius R) by a distance c (Figure 22). If we assume that the arc s is reckoned from the equator of the corresponding sphere, then for the median surface of the discussed shell of revolution, by virtue of (4.1) and (4.3), we obtain

$$A = 1, \quad B = r = R\cos\frac{s}{R} - c,$$
$$R_1 = R, \quad R_2 = \frac{R\cos\frac{s}{R} - c}{\cos\frac{s}{R}} = \frac{r}{\cos\frac{s}{R}}. \tag{7.1}$$

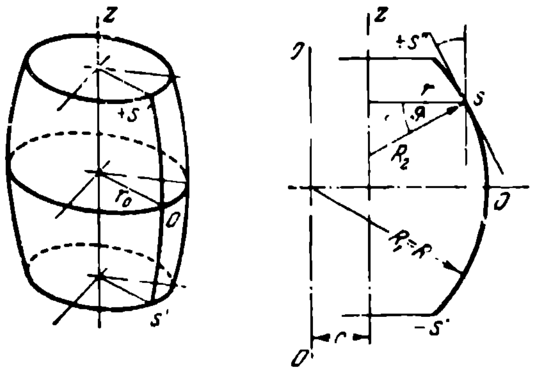

Figure 22

By virtue of (7.1), from (4.21)-(4.23) we obtain for the internal forces

$$T_1 = -\frac{1}{\left(R\cos\frac{s}{R} - c\right)\cos\frac{s}{R}} \times$$

$$\times\left[\int_0^s \left(R\cos\frac{s}{R} - c\right)\left(X\cos\frac{s}{R} + Z\sin\frac{s}{R}\right)ds - U_0\right]; \qquad (7.2)$$

$$T_2 = \frac{1}{R\cos^2\frac{s}{R}}\left[\left(R\cos\frac{s}{R} - c\right)\left(X\cos\frac{s}{R} + Z\sin\frac{s}{R}\right)ds - U_0\right] +$$

$$+\frac{R\cos\frac{s}{R} - c}{\cos\frac{s}{R}}Z; \qquad (7.3)$$

$$S = -\frac{1}{\left(R\cos\frac{s}{R} - c\right)^2}\left[\int_0^s \left(R\cos\frac{s}{R} - c\right)^2 Y\,ds - V_0\right]. \qquad (7.4)$$

while for the displacements, from (4.35)-(4.37), we have

$$u = \frac{\cos\frac{s}{R}}{h}\left\{\int_0^s \left[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{T_1}{\cos\frac{s}{R}} +\right.\right.$$

$$\left.\left. + \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{S}{\cos\frac{s}{R}} + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R\cos\frac{s}{R} - c}{\cos^2\frac{s}{R}}Z\right]ds\right\} +$$

$$+\varphi_0\cos\frac{s}{R}; \qquad (7.5)$$

$$v = \frac{R\cos\frac{s}{R} - c}{h}\left\{\int_0^s \left[\left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{T_1}{R\cos\frac{s}{R} - c} +\right.\right.$$

$$\left.\left. + a_{26}\frac{Z}{\cos\frac{s}{R}} + a_{66}\frac{S}{R\cos\frac{s}{R} - c}\right]ds\right\} + \psi_0\left(R\cos\frac{s}{R} - c\right); \qquad (7.6)$$

$$w = \frac{\sin\frac{s}{R}}{h}\left\{\int_0^s \left[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)\frac{T_1}{\cos\frac{s}{R}} +\right.\right.$$

$$\left.\left. + \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{S}{\cos\frac{s}{R}} + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)\frac{R\cos\frac{s}{R} - c}{\cos^2\frac{s}{R}}Z\right]ds\right\} +$$

$$+\frac{R\cos\frac{s}{R} - c}{h\cos\frac{s}{R}}\left[\left(a_{12} - a_{22}\frac{R_2}{R_1}\right)T_1 +\right.$$

$$\left. + a_{22}\frac{R\cos\frac{s}{R} - c}{\cos\frac{s}{R}}Z + a_{26}S\right] + \varphi_0\sin\frac{s}{R}. \qquad (7.7)$$

On the basis of the general formulas for internal forces (7.2)-(7.4) and displacements (7.5)-(7.7), let us examine an example. Let a barrel-shaped shell sustain a uniformly distributed, normal surface load of intensity q. Let the boundaries of the shell be defined by two transverse sections perpendicular to the axis of revolution z. One of the ends of the shell $(s = -s_0)$ is completely fixed and the other end $(s = s_0)$ is completely free. We have the following boundary conditions:

$$\left.\begin{array}{llll} \text{for} & s = -s_0: & u = 0, & v = 0; \\ \text{for} & s = s_0: & T_1 = 0, & S = 0, \end{array}\right\} \tag{7.8}$$

and the components of the surface load

$$X = 0, \quad Y = 0, \quad Z = q. \tag{7.9}$$

In the same manner as in the case of the first example of a cylindrical shell (Chapter III, Section 5), for the internal forces and displacements of the discussed shell we obtain

$$T_1 = \frac{qR^2\left(\cos^2\frac{s}{R} - \cos^2\frac{s_0}{R}\right) - 2qRc\left(\cos\frac{s}{R} - \cos\frac{s_0}{R}\right)}{2\cos\frac{s}{R}\left(R\cos\frac{s}{R} - c\right)}; \tag{7.10}$$

$$T_2 = \frac{qR}{2} + q\,\frac{A^0}{2\cos^2\frac{s}{R}}, \quad S = 0; \tag{7.11}$$

$$u = q\,\frac{\cos\frac{s}{R}}{h}\left\{\left[\left(a_{11} - \frac{a_{22}}{2}\right)R^2 + \frac{A^0R}{2}\left(a_{11}\frac{R^2}{c^2} + a_{12} - \frac{a_{22}}{2}\right)\right]\times\right.$$

$$\times \ln\frac{\operatorname{tg}\left(\frac{\pi}{4} + \frac{s}{2R}\right)}{\operatorname{tg}\left(\frac{\pi}{4} - \frac{s_0}{2R}\right)} + \left[\left(a_{12} + \frac{a_{22}}{2}\right)Rc + \right.$$

$$\left. + \frac{A^0R}{2}\left(a_{11}\frac{R}{c} + u_{22}\frac{2}{3}\frac{c}{R}\right)\right]\left(\operatorname{tg}\frac{s}{R} + \operatorname{tg}\frac{s_0}{R}\right) +$$

$$+ \frac{A^0R}{2}\left(a_{11} - \frac{a_{22}}{2}\right)\left(\frac{\sin\frac{s}{R}}{\cos^2\frac{s}{R}} + \frac{\sin\frac{s_0}{R}}{\cos^2\frac{s_0}{R}}\right) -$$

$$- a_{11}\frac{R^3}{2\sqrt{R^2 - c^2}}\left(\frac{A^0R}{c^2} + 1\right)\left[\ln\frac{(R + c)\operatorname{tg}\frac{s}{2R} + \sqrt{R^2 - c^2}}{(R + c)\operatorname{tg}\frac{s}{2R} - \sqrt{R^2 - c^2}} - \right.$$

$$\left. - \ln\frac{(R + c)\operatorname{tg}\frac{s_0}{2R} - \sqrt{R^2 - c^2}}{(R + c)\operatorname{tg}\frac{s_0}{2R} + \sqrt{R^2 - c^2}}\right] +$$

$$+a_{22}\frac{A^0c}{6}\left(\frac{\sin\frac{s}{R}}{\cos^3\frac{s}{R}}+\frac{\sin\frac{s_0}{R}}{\cos^3\frac{s_0}{R}}\right)\Bigg\}; \tag{7.12}$$

$$v=q\frac{R\cos\frac{s}{R}-c}{h}\Bigg[\Bigg\{a_{16}\left[\frac{R^3}{2\sqrt{R^2-c^2}}-\frac{R^3c^2}{2(R^2-c^2)^{3/2}}+\right.$$

$$\left.+\frac{A^0R}{2}\left(\frac{R^2}{c^2\sqrt{R^2-c^2}}-\frac{R^2}{(R^2-c^2)^{3/2}}\right)\right]+$$

$$+a_{26}\frac{R^2c^2+R^3A^0}{2c^2\sqrt{R^2-c^2}}\Bigg\}\left[\ln\frac{(R+c)\operatorname{tg}\frac{s}{2R}+\sqrt{R^2-c^2}}{(R+c)\operatorname{tg}\frac{s}{2R}-\sqrt{R^2-c^2}}-\right.$$

$$\left.-\ln\frac{(R+c)\operatorname{tg}\frac{s_0}{2R}-\sqrt{R^2-c^2}}{(R+c)\operatorname{tg}\frac{s_0}{2R}+\sqrt{R^2-c^2}}\right]-$$

$$-a_{16}\left[\frac{R^3c}{2(R^2-c^2)}+\frac{R^4A^0}{2c(R^2-c^2)}\right]\left(\frac{\sin\frac{s}{R}}{R\cos\frac{s}{R}-c}+\frac{\sin\frac{s_0}{R}}{R\cos\frac{s_0}{R}-c}\right)-$$

$$\left.-(a_{26}+a_{16})\frac{R^2A^0}{2c^2}\ln\frac{\operatorname{tg}\left(\frac{\pi}{4}+\frac{s}{2R}\right)}{\operatorname{tg}\left(\frac{\pi}{4}-\frac{s_0}{2R}\right)}-a_{26}\frac{RA^0}{2c}\left(\operatorname{tg}\frac{s}{R}+\operatorname{tg}\frac{s_0}{R}\right)\right], \tag{7.13}$$

where

$$A^0=\left(R\cos\frac{s_0}{R}-2c\right)\cos\frac{s_0}{R}. \tag{7.14}$$

In view of its extreme unwieldiness we shall not derive the expression for normal displacement ω.

In examining the derived formulas it is easily seen that in the sufficiently general case of a symmetrical loaded anisotropic shell of revolution (even if the shell is externally statically determinate) the derived design formulas are complex. In this connection it is of particular interest to examine a shell of revolution for which the meridian arc is shallow, the ratio of the rise f of the meridian arc to the length $2s_0$ of the arc itself (or to the length of the chord of the arc $\overline{00_1}$) being small (Figure 23).

B. Shells of Revolution With Extremely Shallow Meridian Arc. Let a shell be generated by the revolution of an extremely shallow arc of a certain curve around the axis of revolution z; herein the chord of the meridian arc $\overline{00_1}$ is parallel to the axis of revolution z or, if it is not parallel, the angle formed by the z axis and chord $\overline{00_1}$ must be so small that angle ϑ (Figure 23) at all points of the shell remains small).

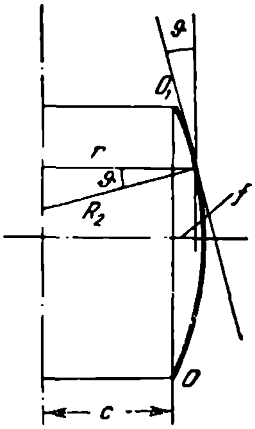

Figure 23

For shells of the described type we assume that the coefficients of the first quadratic form A and B and the principal radii of curvature R_1 and R_2 behave as constants*,**,***,****,***** upon differentiation. Of course, this assumption refers to shells in which the Gaussian curvature at all points retains one and the same sign.****** Considering this, on the basis of (4.1)-(4.4), for the coefficients of the first quadratic form we obtain

$$A=1, \quad B=r=R_2. \tag{7.15}$$

Then, from (4.21)-(4.23), for the internal forces we have

$$T_1=-\frac{1}{R_2}\left(R_2\int_{s_0}^{s} X\,ds-U_0\right); \tag{7.16}$$

$$T_2=\frac{1}{R_1}\left(R_2\int_{s_0}^{s} X\,ds-U_0\right)+R_2Z; \tag{7.17}$$

---

*S. M. Feynberg, on the approximate moment theory of thin-walled shells of arbitrary configuration, Symposium — Issledovaniya po teorii sooruzheniy (Investigations in the theory of structures), Gosstroyizdat, 1939.

**Yu. N. Rabotnov, Boundary zone equations in the theory of shells, DAN SSSR, Vol. XVII, No. 5, 1945.

***S. A. Ambartsumyan, Approximate method for calculation of shallow thin shells, DAN ArmSSR, VI, No. 3, 1947.

****S. A. Ambartsumyan, Calculation of shallow shells, PMM, Vol. XI, No. 5, 1947.

*****A. L. Gol'denveyzer, see pp. 458-460 of work cited on p. 61.

******V. Z. Vlasov, Obshchaya teoriya obolochek, Gostekhizdat, 1949, pp. 436-438.

$$S = -\frac{1}{R_2^2}\left(R_2^2 \int_{s_0}^{s} Y\,ds - V_0\right), \tag{7.18}$$

and for the displacements, from (4.35)-(4.37), we obtain

$$u = \frac{1}{h}\int_{s_0}^{s}\left[\left(a_{11} - 2a_{12}\frac{R_2}{R_1} + a_{22}\frac{R_2^2}{R_1^2}\right)T_1 + \right.$$
$$\left. + \left(a_{16} - a_{26}\frac{R_2}{R_1}\right)S + \left(a_{12} - a_{22}\frac{R_2}{R_1}\right)R_2 Z\right]ds + \varphi_0; \tag{7.19}$$

$$v = \frac{R_2}{h}\int_{s_0}^{s}\left[\left(a_{16} - a_{26}\frac{R_2}{R_1}\right)\frac{T_1}{R_2} + a_{66}\frac{S}{R_2} + a_{26}Z\right]ds + \varphi_0 R_2; \tag{7.20}$$

$$w = \frac{R_2}{h}\left[\left(a_{12} - a_{22}\frac{R_2}{R_1}\right)T_1 + a_{26}S + a_{22}R_2 Z\right]. \tag{7.21}$$

Here, as is easily seen, we have the approximate assumption that the internal geometry of the discussed shell does not differ from the internal geometry of a circular cylindrical shell of radius R_2 (5.1) and (7.15). However, notwithstanding this, the design formulas for these two types of shells (5.2)-(5.7) and (7.16)-(7.21) are fundamentally different. The formulas for shallow shells, in distinction from the corresponding formulas for cylindrical shells, contain terms with the factor $1/R_1$.

Let us examine few examples.

1. A barrel-shaped shell sustains a uniformly distributed, normally applied surface load of intensity q. The boundaries of the shall are defined by two lateral cross-sections perpendicular to the axis of revolution z. One of the ends of the shell $(s = 0)$ is completely fixed and the other end $(s = L)$ is completely free. We have the following boundary conditions:

$$\left.\begin{array}{llll} \text{for} & s = 0: & u = 0, & v = 0; \\ \text{for} & s = L: & T_1 = 0, & S = 0, \end{array}\right\} \tag{7.22}$$

and the components of the surface load are

$$X = 0, \quad Y = 0, \quad Z = q. \tag{7.23}$$

Proceeding as in the previous analogous cases (Chapter III, Section 5), for the internal forces and displacements we obtain

$$T_1 = 0, \quad T_2 = R_2 q, \quad S = 0; \tag{7.24}$$

$$\left.\begin{array}{l} u = R_2 q\,\frac{s}{h}\left(a_{12} - a_{22}\frac{R_2}{R_1}\right); \\ v = a_{26}\frac{s}{h}R_2 q, \quad w = a_{22}\frac{R_2^2}{h}q. \end{array}\right\} \tag{7.25}$$

In the special case where the shell is generated by the revolution of a certain part of a circular arc (the first example in this section) by virtue of (7.1) we obtain the following formulas.

$$T_1 = 0, \quad T_2 = \frac{R\cos\frac{s}{R} - c}{\cos\frac{s}{R}}\,q, \quad S = 0; \tag{7.26}$$

$$u = q\,\frac{R\cos\frac{s}{R} - c}{\cos\frac{s}{R}}\,\frac{s}{n}\left(a_{12} - a_{22}\,\frac{R\cos\frac{s}{R} - c}{R\cos\frac{s}{R}}\right); \tag{7.27}$$

$$v = q a_{26}\frac{s}{h}\,\frac{R\cos\frac{s}{R} - c}{\cos\frac{s}{R}}; \tag{7.28}$$

$$w = q a_{22}\,\frac{1}{h}\,\frac{\left(R\cos\frac{s}{R} - c\right)^2}{\cos^2\frac{s}{R}}, \tag{7.29}$$

or, in view of the small value of angle θ, assuming $\cos\frac{s}{R} = 1$, we obtain,

$$\left.\begin{aligned} &T_1 = 0, \quad T_2 = (R - c)q, \quad S = 0; \\ &u = q(R - c)\,\frac{s}{h}\left(a_{12} - a_{22}\frac{R - c}{R}\right); \\ &v = q a_{26}\frac{s}{h}(R - c), \quad w = q a_{22}\frac{1}{h}(R - c)^2. \end{aligned}\right\} \tag{7.30}$$

Comparing the resulting approximate formulas (7.26)-(7.30) with the corresponding exact formulas (7.10)-(7.13), we note that with sufficient flatness of the meridian arc $\left(\frac{s_0}{R} \leqslant \frac{\pi}{6}, \ \frac{f}{OO_1} < \frac{1}{7}\right)$ the approximate formulas, from the standpoint of the membrane theory, provide wholly acceptable results. In this case the approximate calculations differ from the exact calculations by not more than 15 percent; with a decrease in the relative rise of the meridian arc there is a decrease in this difference. For example, when $\frac{f}{OO_1} < \frac{1}{10}$ that is, when $\frac{s_0}{R} < \frac{\pi}{9}$ this difference does not exceed 5-6 percent.

The investigations and numerical calculations performed by the author show that the limit of applicability of the approximate theory of calculation established here is not so exclusive that it may not be extended to other types of symmetrically loaded anisotropic shells of revolution with shallow meridian arc.

2. Let the shell of the previous example be filled with liquid. For simplicity of calculation it is assumed (with sufficient accuracy) that the hydrostatic pressure is replaced by a distributed load which varies linearly over the arc of meridian. Assuming that the pressure at the bottom point of the shell (s = 0) is q, for the components of surface load we have

$$X = 0, \quad Y = 0, \quad Z = q\left(1 - \frac{s}{L}\right). \tag{7.31}$$

From (7.16)-(7.21), by virtue of the boundary conditions (7.22) and by proceeding in the usual manner, we obtain for the internal forces and displacements

$$T_1 = 0, \quad T_2 = qR_2\left(1 - \frac{s}{L}\right), \quad S = 0; \tag{7.32}$$

$$u = q\frac{R_2}{h}\left(a_{12} - \frac{R_2}{R_1}a_{22}\right)s\left(1 - \frac{s}{2L}\right); \tag{7.33}$$

$$v = q\frac{R_2}{h}a_{26}s\left(1 - \frac{s}{2L}\right); \tag{7.34}$$

$$w = q\frac{R_2^2}{h}a_{22}\left(1 - \frac{s}{L}\right). \tag{7.35}$$

In the special case where the shell is formed by revolution of the halfwave of a sinusoid about the axis of revolution z at a distance c from the axis of the sinusoid $\overline{OO_1}$ (Figure 23), for the principal radii of curvature of the shell

we have, with sufficient accuracy

$$R_1 = \frac{L^2}{f\pi^2 \sin\frac{\pi s}{L}}, \quad R_2 = f\sin\frac{\pi s}{L} + c. \tag{7.36}$$

(In determining the radii of curvature it is here assumed that $\frac{f^2}{L^2} \ll 1$.)

Inserting the values of R_1 and R_2 from (7.36) into formulas (7.32)-(7.35), we obtain for the internal forces and displacements

$$T_1 = 0, \quad T_2 = q\left(f\sin\frac{\pi s}{L} + c\right)\left(1 - \frac{s}{L}\right), \quad S = 0;$$

$$u = \frac{q\left(f\sin\frac{\pi s}{L} + c\right)}{h}\left(a_{12} - a_{22}\frac{\left(f\sin\frac{\pi s}{L} + c\right)f\pi^2\sin\frac{\pi s}{L}}{L^2}\right)s\left(1 - \frac{s}{2L}\right);$$

$$v = q\frac{f\sin\frac{\pi s}{L} + c}{h}a_{26}s\left(1 - \frac{s}{2L}\right);$$

$$w = q\frac{\left(f\sin\frac{\pi s}{L} + c\right)^2}{h}a_{22}\left(1 - \frac{s}{L}\right).$$

3. A barrel-shaped shell sustains a uniformly distributed, normally applied surface load of intensity q. The ends of the shell ($s = 0$, $s = L$) are completely fixed. Here we have the following boundary conditions:

$$\left.\begin{aligned} &\text{for} \quad s = 0: \quad u = 0; \quad v = 0; \\ &\text{for} \quad s = L: \quad u = 0, \quad v = 0, \end{aligned}\right\} \tag{7.37}$$

and the components of the external surface load are

$$X=0, \quad Y=0, \quad Z=q. \tag{7.38}$$

The discussed external statically indeterminate problem is solved just as simply as the corresponding problem of a cylindrical shell. Proceeding in the same manner as in the case of the third example of a cylindrical shell (Chapter III, Section 5) we obtain for the internal forces and displacements

$$T_1=C_1R_2q, \quad T_2=\left(1-C_1\frac{R_2}{R_1}\right)R_2q, \quad S=C_2R_2q; \tag{7.39}$$

$$u=\frac{s}{h}\left[\left(a_{11}-2a_{12}\frac{R_2}{R_1}+a_{22}\frac{R_2^2}{R_1^2}\right)C_1+\right.$$
$$\left.+\left(a_{16}-a_{26}\frac{R_2}{R_1}\right)C_2+\left(a_{12}-a_{22}\frac{R_2}{R_1}\right)\right]R_2q; \tag{7.40}$$

$$v=\frac{s}{h}\left[\left(a_{16}-a_{26}\frac{R_2}{R_1}\right)C_1+a_{66}C_2+a_{26}\right]R_2q; \tag{7.41}$$

$$w=\frac{R_2}{h}\left[\left(a_{12}-a_{22}\frac{R_2}{R_1}\right)C_1+a_{26}C_2+a_{22}\right]R_2q, \tag{7.42}$$

where

$$C_1=\frac{a_{26}\left(a_{16}-a_{26}\frac{R_2}{R_1}\right)-a_{66}\left(a_{12}-a_{22}\frac{R_2}{R_1}\right)}{a_{66}\left(a_{11}-2a_{12}\frac{R_2}{R_1}+a_{22}\frac{R_2^2}{R_1^2}\right)-\left(a_{16}-a_{26}\frac{R_2}{R_1}\right)^2}, \tag{7.43}$$

$$C_2=\frac{a_{16}\left(a_{12}-a_{22}\frac{R_2}{R_1}\right)-a_{26}\left(a_{11}-a_{12}\frac{R_2}{R_1}\right)}{a_{66}\left(a_{11}-2a_{12}\frac{R_2}{R_1}+a_{22}\frac{R_2^2}{R_1^2}\right)-\left(a_{16}-a_{26}\frac{R_2}{R_1}\right)^2}. \tag{7.44}$$

In the special case, assuming in the derived formulas $a_{16}=a_{26}=0$, we find the formulas for the corresponding orthotropic shells.

In the case where the meridian arc of the shell is shallow all problems of symmetrically loaded anisotropic shells of revolution are simply solved without unwieldy calculations and design formulas.

SECTION 8. AN ARBITRARILY LOADED CYLINDRICAL SHELL OF ARBITRARY SHAPE

Let us discuss a cylindrical shell of arbitrary shape. The location of any point M on the median surface of the shell is defined by orthogonal coordinates α and β, where β is the distance from any initial cross-section to the cross-section containing point M (α,β), and β is the length of arc of the directing curve from a certain initial generatix to point M (α, β). The coordinate lines of the given system are the generating and directing lines of the discussed cylinder (Figure 24).

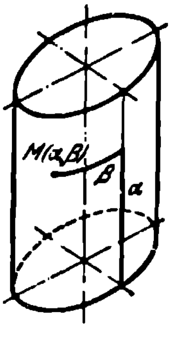

Figure 24

The coefficients of the first quadratic form A and B and the principal radii of curvature of the median surface of the discussed cylindrical shell are defined as follows:

$$A=1,\quad B=1,\quad R_1=\frac{1}{k_1}=\infty,\quad R_2=\frac{1}{k_2}=R(\beta). \tag{8.1}$$

By virtue of (8.1), from (1.1) we obtain the following equilibrium equations

$$\frac{\partial T_1}{\partial \alpha}+\frac{\partial S}{\partial \beta}=-X; \tag{8.2}$$

$$\frac{\partial S}{\partial \alpha}+\frac{\partial T_2}{\partial \beta}=-Y; \tag{8.3}$$

$$T_2=RZ, \tag{8.4}$$

and from (1.5) the following values of deformations:

$$\varepsilon_1=\frac{\partial u}{\partial \alpha};$$

$$\varepsilon_2=\frac{\partial v}{\partial \beta}+\frac{w}{R};$$

$$\omega=\frac{\partial u}{\partial \beta}+\frac{\partial v}{\partial \alpha}.$$

For the elasticity relationships we have from (1.3)

$$\varepsilon_1=\frac{1}{h}(a_{11}T_1+a_{12}T_2+a_{16}S);$$

$$\varepsilon_2=\frac{1}{h}(a_{12}T_1+a_{22}T_2+a_{26}S);$$

$$\omega=\frac{1}{h}(a_{16}T_1+a_{26}T_2+a_{66}S).$$

From the last two groups of relationships we obtain the following inhomogeneous geometric equations:

$$\frac{\partial u}{\partial \alpha} = \frac{1}{h}(a_{11}T_1 + a_{12}T_2 + a_{16}S); \tag{8.5}$$

$$\frac{\partial v}{\partial \beta} + \frac{w}{K} = \frac{1}{h}(a_{12}T_1 + a_{22}T_2 + a_{26}S); \tag{8.6}$$

$$\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} = \frac{1}{h}(a_{16}T_1 + a_{26}T_2 + a_{66}S). \tag{8.7}$$

Integration of the systems of differential equations in (8.2)-(8.4) and (8.5)-(8.7) is not difficult and may be completed in general form.* However, for greater clarity the discussed examples will be solved by successive integration of the indicated equations.**

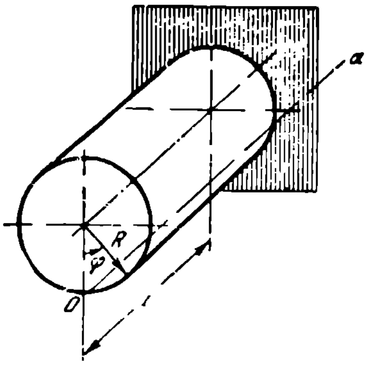

Figure 25

1. A horizontal circular tube (of radius R and length L), one end of which ($\alpha = l$) is completely fixed and the other ($\alpha = 0$) is completely free, is entirely filled with a liquid of specific gravity γ. The weight of the tube itself is not considered. Angle $\varphi = \beta/R$ is reckoned from the low point of the tube (Figure 25).

We have the following boundary conditions:

for $\alpha = 0$: $T_1 = 0$, $S = 0$; (8.8)

for $\alpha = L$: $u = 0$, $v = 0$. (8.9) and the components of the surface load are

---

*A. L. Gol'denveyzer, see pp. 125-129 of work cited in footnote on p. 61.

**V. V. Novozhilov, see pp. 154-159 of work cited in footnote ** on p. 63.

$$X=0, \quad Y=0, \quad Z=R\gamma\left(1+\cos\frac{\beta}{R}\right). \tag{8.10}$$

From (8.4), by virtue of (8.10), for the internal force T_2 we have

$$T_2=R^2\gamma\left(1+\cos\frac{\beta}{R}\right). \tag{8.11}$$

Inserting the value of T_2 from (8.11) into (8.3) and considering (8.10) and the second boundary condition in (8.8), we obtain for internal force S

$$S=R\gamma\alpha\sin\frac{\beta}{R}. \tag{8.12}$$

Inserting the value S from (8.12) into (8.2) and considering (8.10) and (8.8), we obtain for internal force T_1

$$T_1=-\gamma\frac{\alpha^2}{2}\cos\frac{\beta}{R}. \tag{8.13}$$

Inserting the values of T_1, T_2 and S from (8.11)-(8.13) into (8.5), integrating with respect to α and considering (8.9), we obtain for tangential displacement u

$$u=\frac{\gamma}{h}\left[a_{11}\frac{L^3-\alpha^3}{6}\cos\frac{\beta}{R}+a_{12}(\alpha-L)R^2\left(1+\cos\frac{\beta}{R}\right)+\right.$$
$$\left.+a_{16}\frac{\alpha^2-L^2}{2}R\sin\frac{\beta}{R}\right]. \tag{8.14}$$

Inserting the values of T_1, T_2, S and u from (8.11)-(8.14) into (8.7), integrating with respect to α and considering (8.9), we obtain for tangential displacement v

$$v=\frac{\gamma}{h}\left[\frac{a_{16}}{6}(3\alpha L^2-2\alpha^3-L^3)\cos\frac{\beta}{R}+\right.$$
$$+a_{26}(\alpha-L)R^2\left(1+\cos\frac{\beta}{R}\right)+a_{66}\frac{\alpha^2-L^2}{2}R\sin\frac{\beta}{R}+$$
$$\left.+\frac{a_{11}}{6R}\left(\alpha L^3-\frac{\alpha^4}{4}-\frac{3}{4}L^4\right)\sin\frac{\beta}{R}+a_{12}\frac{(\alpha-L)^2}{2}R\sin\frac{\beta}{R}\right]. \tag{8.15}$$

From (8.6), by virtue of (8.15), we obtain for the normal displacement

$$\begin{aligned} w = & -\frac{\gamma}{h}\left[\frac{a_{11}}{6R}\left(\alpha L^3 - \frac{\alpha^4}{4} - \frac{3}{4}L^4\right)\cos\frac{\beta}{R} - a_{22}R^3\left(1+\cos\frac{\beta}{R}\right) + \right. \\ & + a_{12}R\left(\alpha^2 - \alpha L + \frac{L^2}{2}\right)\cos\frac{\beta}{R} + a_{66}\frac{\alpha^2 - L^2}{2}R\cos\frac{\beta}{R} + \\ & \left. + \frac{a_{16}}{6}(2\alpha^3 + L^3 - 3\alpha L^2)\sin\frac{\beta}{R} - a_{26}(2\alpha - L)R^2\sin\frac{\beta}{R}\right]. \end{aligned} \qquad (8.16)$$

2. A closed cylindrical shell of arbitrary cross-section (radius of curvature $R_2 = R(\beta)$, length L) sustains a uniformly distributed, normally applied surface load of intensity q. One of the ends of the shell ($\alpha = 0$) is completely fixed and the other end ($\alpha = L$) is completely free. For the discussed shell we have the following boundary conditions:

$$\left.\begin{aligned} &\text{for} \quad \alpha = 0: \quad u = 0, \quad v = 0; \\ &\text{for} \quad \alpha = L: \quad T_1 = 0, \quad S = 0, \end{aligned}\right\} \qquad (8.17)$$

and the components of the surface load are

$$X = 0, \quad Y = 0, \quad Z = q. \qquad (8.18)$$

Proceeding as in the previous example, we obtain for the internal forces and displacements of the discussed shell

$$\left.\begin{aligned} &T_2 = Rq, \quad S = q(L-\alpha)\frac{dR}{d\beta}, \\ &T_1 = q\frac{(L-\alpha)^2}{2}\frac{d^2R}{d\beta^2}; \end{aligned}\right\} \qquad (8.19)$$

$$\left.\begin{aligned} u = & q\frac{\alpha}{h}\left[a_{12}R + a_{16}\left(L - \frac{\alpha}{2}\right)\frac{dR}{d\beta} + \right. \\ & \left. + a_{11}\left(\frac{L^2}{2} - \frac{\alpha L}{2} + \frac{\alpha^2}{6}\right)\frac{d^2R}{d\beta^2}\right]; \\ v = & q\frac{\alpha}{h}\left\{a_{26}R - \left[a_{12}\frac{\alpha}{2} - a_{66}\left(L - \frac{\alpha}{2}\right)\right]\frac{dR}{d\beta} + \right. \\ & + a_{16}\left(\frac{L^2}{2} - \alpha L + \frac{\alpha^2}{3}\right)\frac{d^2R}{d\beta^2} - \\ & \left. - a_{11}\left(\frac{\alpha L^2}{4} - \frac{\alpha^2 L}{6} + \frac{\alpha^3}{24}\right)\frac{d^3R}{d\beta^3}\right\}; \\ w = & q\frac{R}{h}\left\{a_{22}R + a_{26}(L - 2\alpha)\frac{dR}{d\beta} + \left[a_{12}\left(\frac{L^2}{2} - \alpha L + \alpha^2\right) - \right.\right. \\ & \left. - a_{66}\left(\alpha L - \frac{\alpha^2}{2}\right)\right]\frac{d^2R}{d\beta^2} - a_{16}\left(\frac{\alpha L^2}{2} - \alpha^2 L + \frac{\alpha^3}{3}\right)\frac{d^3R}{d\beta^3} + \\ & \left. + a_{11}\left(\frac{\alpha^2L^2}{4} - \frac{\alpha^3 L}{6} + \frac{\alpha^4}{24}\right)\frac{d^4R}{d\beta^4}\right\}. \end{aligned}\right\} \qquad (8.20)$$

Let us discuss a numerical example. Let the discussed shell have an elliptical cross-section for which* where a and b are the major and minor semi

---

*V.V. Novozhilov, see pp. 157-158 of work cited in footnote **, on p. 1C

$$R = \frac{b^2}{a}\,\frac{1}{(1-\varepsilon^2\cos^2\psi)^{3/2}},$$

axes of the ellipse, ϵ is the eccentricity of the ellipse $\left(\varepsilon^2 = 1 - \frac{b^2}{a^2}\right)$ and ψ is the angle formed by the normal to the ellipse with its minor axis (Figure 26).

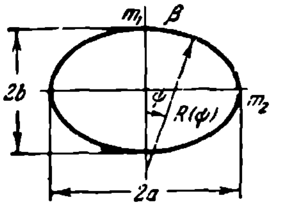

Figure 26

In addition to the value of R, we shall also be interested in

$$\frac{dR}{d\psi} = -\frac{3}{2}\,\frac{\varepsilon^2\sin 2\psi}{1-\varepsilon^2\cos^2\psi};$$

$$\frac{d^2R}{d\psi^2} = -\frac{3a}{b^2}\,\varepsilon^2\,\frac{\cos 2\psi - \varepsilon^2\cos^2\psi}{\sqrt{1-\varepsilon^2\cos^2\psi}};$$

$$\frac{d^3R}{d\psi^3} = \frac{6a^2}{b^4}\,\varepsilon^2\left[1 - \frac{3}{4}\,\varepsilon^2 - \frac{1}{2}\,\varepsilon^2\left(1 - \frac{1}{2}\,\varepsilon^2\right)\cos^2\psi\right]\sin 2\psi;$$

$$\frac{d^4R}{d\psi^4} = \frac{12a^3}{b^6}\,\varepsilon^2(1-\varepsilon^2\cos^2\psi)^{1/2}\left[\left(1 - \frac{3}{4}\,\varepsilon^2\right)\cos 2\psi - \right.$$
$$\left. - \frac{1}{2}\,\varepsilon^2\left(1 - \frac{1}{2}\,\varepsilon^2\right)(\cos^2\psi - 3\sin^2\psi)\cos^2\psi\right]$$

Let the shell be made on an orthotropic material with the following elastic characteristics:**

$$E_1 = 2\cdot 10^5\,\text{kg/cm}^2,\quad E_2 = 0.5\cdot 10^5\,\text{kg/cm}^2;$$
$$\nu_1 = 0.2;\quad \nu_2 = 0.05;\quad G = 0.455\cdot 10^5\,\text{kg/cm}^2.$$

The material of the shell is so arranged that at each point of the shell one principal direction of elasticity of the material coincides with coordinate γ and the other two do not coincide with coordinate lines α and b. At each point of the shell let the principal direction of elasticity of a material with a subscript 1 form with direction α an angle φ = 30°; then, by means of formulas (1.4.8)-(1.4.13), we find the elastic constants a_{jk} for the coordinate system

---

*V.V. Novozhilov, see pp. 157-158 of work cited in footnote **, p. 63.

**S.G. Lekhnitskiy, Anizotropnyye plastinki, Gostekhizdat, 1947, p. 48.

α and β:

$$a_{11} = 0{,}781 \cdot 10^{-5}\ \text{см}^2/\text{kg};\quad a_{22} = 1{,}531 \cdot 10^{-5}\ \text{см}^2/\text{kg};$$
$$a_{12} = -0{,}006 \cdot 10^{-5}\ \text{см}^2/\text{kg};\ a_{66} = 2{,}575 \cdot 10^{-5}\ \text{см}^2/\text{kg};$$
$$a_{16} = -0{,}541 \cdot 10^{-5}\ \text{см}^2/\text{kg};\ a_{26} = -0{,}758 \cdot 10^{-5}\ \text{см}^2/\text{kg};$$

The internal forces and displacements within a tube of elliptical cross-section are easily obtained by use of the above data. In view of the fact that the discussed shell is externally statically determinate, the internal forces of the anisotropic shell do not differ from the internal forces of the corresponding isotropic shell.

Let us determine the displacements in cross-section $\alpha = L$. We shall limit ourselves to the investigation of points $m_1 (\psi = 0)$ and m_2 ($\psi = \pi/2$). For the sake of clarity and by way of example we shall also assume that $b/a = 0.5$ and assign three different values (2,3,4) for ratio L/b. Representing the displacements as follows:

$$u = q\frac{b^2}{h}K_u \cdot 10^{-5}\ \text{см},\ v = q\frac{b^2}{h}K_v \cdot 10^{-5}\,\text{см},\ w = q\frac{b^2}{h}K_w \cdot 10^{-5}\ \text{см}$$

(the values of b and h must be expressed in centimeters and the value of q in kg/cm^2), for coefficients K_u, K_v, K_w we have the quantities listed in the following table:

Table of Values of K_u, K_v, K_w

	K_u		K_v		K_w	
$\frac{L}{b}$ \ ψ	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$
2,0	−2,39	4,68	−7,68	2,49	82,40	−35,88
3,0	−7,98	15,81	−14,57	9,81	186,96	−150,60
4,0	−18,85	37,49	−25,12	24,46	392,80	−440,56

In the special case of an orthotropic shell, as before, at points m_1 and

m_2 the tangential displacements v become zero. Here, in the sufficiently general case of anisotropy, points m_1 and m_2 have tangential displacements v which are of the same order of magnitude as tangential displacements u.

3. Let us derive the general integral of the equations of the membrane theory of arbitrarily loaded anisotropic cylindrical shells of arbitrary shape. The general integral is constructed in the same manner as in the case of an isotropic shell.* Assuming

$$T_1 = T_1^* + T_1', \quad T_2 = T_2^* + T_2', \quad S = S^* + S', \tag{8.21}$$

where (T_1^*, T_2^*, S^*) are the partial integrals of the inhomogeneous system of equations in (8.2)-(8.4) while (T_1', T_2', S') is the general integral of the corresponding homogeneous system, we obtain for T_j^* and T_j'

$$\left.\begin{aligned} T_1^* &= -\int_{\alpha_1}^{\alpha} X\,d\alpha + \int_{\alpha_1}^{\alpha} \frac{\partial}{\partial\beta}\left\{\int_{\alpha_1}^{\alpha}\left[\frac{\partial}{\partial\beta}(RZ) + Y\right]d\alpha\right\}d\alpha; \\ T_2^* &= RZ, \quad S^* = -\int_{\alpha_1}^{\alpha}\left[\frac{\partial}{\partial\beta}(RZ) + Y\right]d\alpha; \end{aligned}\right\} \tag{8.22}$$

$$\left.\begin{aligned} T_1' &= -\int_{\alpha_1}^{\alpha} \frac{\partial f_1}{\partial\beta}\,d\alpha + f_2; \\ S' &= f_1, \quad T_2' = 0. \end{aligned}\right\} \tag{8.23}$$

where $f_1 = f_1(\beta)$ and $f_2 = f_2(\beta)$ are arbitrary functions of integration and the lower limit of integration α_1 is arbitrarily chosen.

With known T_1, T_2 and S, the system of equations in (8.5)-(8.7) is also integrated in an elementary manner.

Assuming

$$u = u^* + u' + u'', \quad v = v^* + v' + v'', \quad w = w^* + w' + w'', \tag{8.24}$$

where (u^*, v^*, w^*) and (u', v', w') are partial integrals of the inhomogeneous system in (8.5)-(8.7) which correspond to the group of internal forces

---

*A.L. Gol'denveyzer, see footnote *, p. 97.

(T_1^*, T_2^*, S^*) and (T_1', T_2', S') respectively; (u'', v'', w'') is the general integral of the corresponding homogeneous system; for u_j^*, u_j', u_j'' we obtain

$$\left.\begin{aligned}
u^* &= \frac{1}{h}\int_{\alpha_1}^{\alpha}\left(a_{11}T_1^* + a_{12}T_2^* + a_{16}S^*\right)d\alpha;\\
v^* &= \frac{1}{h}\left\{\int_{\alpha_1}^{\alpha}\left(a_{16}T_1^* + a_{26}T_2^* + a_{66}S^*\right)d\alpha -\right.\\
&\qquad\left. - \int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\beta}\left(a_{11}T_1^* + a_{12}T_2^* + a_{16}S^*\right)d\alpha\right\};\\
w^* &= \frac{R}{h}\left\{\left(a_{12}T_1^* + a_{22}T_2^* + a_{26}S^*\right) -\right.\\
&\qquad - \int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\beta}\left(a_{16}T_1^* + a_{26}T_2^* + a_{66}S^*\right)d\alpha +\\
&\qquad\left. + \int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial^2}{\partial\beta^2}\left(a_{11}T_1^* + a_{12}T_2^* + a_{16}S^*\right)d\alpha\right\};
\end{aligned}\right\}\qquad(8.25)$$

$$\left.\begin{aligned}
u' &= -\frac{1}{h}\left[a_{11}\left(\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial f_1}{\partial\beta}d\alpha - \int_{\alpha_1}^{\alpha}f_2\,d\alpha\right) - a_{16}\int_{\alpha_1}^{\alpha}f_1\,d\alpha\right];\\
v' &= -\frac{1}{h}\left[a_{16}\left(\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial f_1}{\partial\beta}d\alpha - \int_{\alpha_1}^{\alpha}f_2\,d\alpha\right) - a_{66}\int_{\alpha_1}^{\alpha}f_1\,d\alpha +\right.\\
&\qquad + a_{11}\left(\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial^2 f_1}{\partial\beta^2}d\alpha - \int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial f_2}{\partial\beta}d\alpha\right) -\\
&\qquad\left. - a_{16}\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial f_1}{\partial\beta}d\alpha\right];
\end{aligned}\right\}\qquad(8.26)$$

$$\begin{aligned}
w' &= \frac{R}{h}\left[a_{12}\left(f_2 - \int_{\alpha_1}^{\alpha}\frac{\partial f_1}{\partial\beta}d\alpha\right) + a_{26}f_1 +\right.\\
&\qquad + a_{16}\left(\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial^2 f_1}{\partial\beta^2}d\alpha - \int_{\alpha_1}^{\alpha}\frac{\partial f_2}{\partial\beta}d\alpha\right) - a_{66}\int_{\alpha_1}^{\alpha}\frac{\partial f_1}{\partial\beta}d\alpha -\\
&\qquad - a_{11}\left(\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial^3 f_1}{\partial\beta^3}d\alpha - \int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial^2 f_2}{\partial\beta^2}d\alpha\right) +\\
&\qquad\left. + a_{16}\int_{\alpha_1}^{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\frac{\partial^2 f_1}{\partial\beta^2}d\alpha\right];
\end{aligned}$$

$$\left.\begin{aligned} &u'' = f_3; \quad v'' = -\int_{\alpha_1}^{\alpha} \frac{\partial f_3}{\partial \beta}\, d\alpha + f_4; \\ &w'' = R \int_{\alpha_2}^{\alpha} \frac{\partial^2 f_3}{\partial \beta^2}\, d\alpha - R\frac{\partial f_4}{\partial \beta}, \end{aligned}\right\} \qquad (8.27)$$

where $f_3 = f_3(\beta)$, $f_4 = f_4(\beta)$ are arbitrary functions of integration and the lower limit of integration α_2 (as in the case of α_1) is arbitrarily chosen. The formulas derived here by elementary means may be generalized for the case of a conical shell.

SECTION 9. SOME REMARKS CONCERNING THE MEMBRANE THEORY OF ANISOTROPIC LAMINAR SHELLS

The results obtained in the previous sections may, without significant changes, be extended to the case of a shell consisting of an odd number of layers symmetrically arranged relative to the coordinate surface of the shell* (Chapter II, Section 13).

As is known, in the case of an anisotropic laminar shell the equilibrium equations (1.1), (1.2) and the formulas associating the deformations with displacements (1.5), (1.6) remain unchanged; only the elasticity relationships change, which in this case have the form of (2.13.15)

$$\left.\begin{aligned} T_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{16}\omega; \\ T_2 &= C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + C_{26}\omega; \\ S &= C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2. \end{aligned}\right\} \qquad (9.1)$$

where

$$C_{ik} = 2\left[B_{ik}^{m+1} h_{m+1} + \sum_{s=1}^{m} B_{ik}^{s}(h_s - h_{s+1})\right] \qquad (9.2)$$

Solving system (9.1) for deformations ε_1, ε_2, and ω, we obtain

$$\left.\begin{aligned} \varepsilon_1 &= A_{11}T_1 + A_{12}T_2 + A_{16}S; \\ \varepsilon_2 &= A_{12}T_1 + A_{22}T_2 + A_{26}S; \\ \omega &= A_{16}T_1 + A_{26}T_2 + A_{66}S. \end{aligned}\right\} \qquad (9.3)$$

---

*S.A. Ambartsumyan, see footnote *, p. 63.

where

$$\left.\begin{aligned} A_{11} &= \frac{C_{22}C_{66}-C_{26}^2}{\Delta}; & A_{22} &= \frac{C_{11}C_{66}-C_{16}^2}{\Delta}; \\ A_{12} &= \frac{C_{16}C_{26}-C_{12}C_{66}}{\Delta}; & A_{66} &= \frac{C_{11}C_{22}-C_{12}^2}{\Delta}; \\ A_{16} &= \frac{C_{12}C_{26}-C_{22}C_{16}}{\Delta}; & A_{26} &= \frac{C_{12}C_{16}-C_{11}C_{26}}{\Delta}; \end{aligned}\right\} \tag{9.4}$$

$$\Delta = (C_{11}C_{22}-C_{12}^2)C_{66} + 2C_{12}C_{16}C_{26} - C_{11}C_{26}^2 - C_{22}C_{16}^2. \tag{9.5}$$

Comparing (9.9) with (1.3), it is easily seen that they differ only in the coefficients of T_1, T_2 and S. Here, instead of $\frac{a_{jk}}{h}$ in the presence of internal forces T_1, T_2 and S, we have the new coefficients A_{jk}.

Thus, in solution of problems of membrane theory of symmetrically arranged multi-layer shells wide use may be made of the results obtained in the solution of the corresponding problems in the membrane theory of single-layer shells. For this purpose, in all design formulas for single-layer shells, instead of coefficients a_{jk}, it is necessary to insert coefficients hA_{jk}. As for the determination of stresses, here, in distinction from single-layer shells (where the stresses are obtained by direct division of the values of internal forces by the thickness of the shell h), it is necessary to use the well-known formulas (2.4.2)-(2.4.4), which for the i-th layer of a momentless shell have the form*

$$\left.\begin{aligned} \sigma_\alpha^i &= B_{11}^i\varepsilon_1 + B_{12}^i\varepsilon_2 + B_{16}^i\omega; \\ \sigma_\beta^i &= B_{12}^i\varepsilon_1 + B_{22}^i\varepsilon_2 + B_{26}^i\omega; \\ \tau_{\alpha\beta} &= B_{16}^i\varepsilon_1 + B_{26}^i\varepsilon_2 + B_{66}^i\omega. \end{aligned}\right\} \tag{9.6}$$

Inserting the values of deformations ε_1, ε_2, ω from (9.3) into (9.6), we obtain for the stresses

---

*By direct substitution it may be seen that in the case of a single-layer shell the formulas in (9.6) are transformed to (1.8).

$$\left.\begin{aligned}
\sigma_{\alpha}^{i} &= \left(A_{11}B_{11}^{i} + A_{12}B_{12}^{i} + A_{16}B_{16}^{i}\right)T_1 + \left(A_{12}B_{11}^{i} + A_{22}B_{12}^{i} + \right.\\
&\qquad \left. + A_{26}B_{16}^{i}\right)T_2 + \left(A_{16}B_{11}^{i} + A_{26}B_{12}^{i} + A_{66}B_{16}^{i}\right)S;\\
\sigma_{\beta}^{i} &= \left(A_{11}B_{12}^{i} + A_{12}B_{22}^{i} + A_{16}B_{26}^{i}\right)T_1 + \left(A_{12}B_{12}^{i} + A_{22}B_{22}^{i} + \right.\\
&\qquad \left. + A_{26}B_{26}^{i}\right)T_2 + \left(A_{16}B_{12}^{i} + A_{26}B_{22}^{i} + A_{66}B_{26}^{i}\right)S;\\
\tau_{\alpha\beta}^{i} &= \left(A_{11}B_{16}^{i} + A_{12}B_{26}^{i} + A_{16}B_{66}^{i}\right)T_1 + \left(A_{12}B_{16}^{i} + A_{22}B_{26}^{i} + \right.\\
&\qquad \left. + A_{26}B_{66}^{i}\right)T_2 + \left(A_{16}B_{16}^{i} + A_{26}B_{26}^{i} + A_{66}B_{66}^{i}\right)S.
\end{aligned}\right\} \qquad (9.7)$$

Thus, with the exception of the above-mentioned slight differences, the entire problem of the membrane theory of an anisotropic laminar shell consisting of an odd number of layers symmetrically arranged relative to the coordinate surface of the shell coincides completely with the problem of the membrane theory of a single-layer shell.

Let us examine an example. A three-layer circular cylindrical shell (radius of curvature R, length L) sustains a uniformly distributed, normally applied surface load of intensity q. The ends of the shell (s = 0, s = L) are completely fixed (Chapter III, Section 5, the third problem of a cylindrical shell). For the layers of the shell we are given:

Layers I and III are steel. The layer thicknesses are $\delta_1 = \delta_3 = 0{,}2$ *см*,

the moduli of elasticity and the Poisson ratios of the material of the layers are

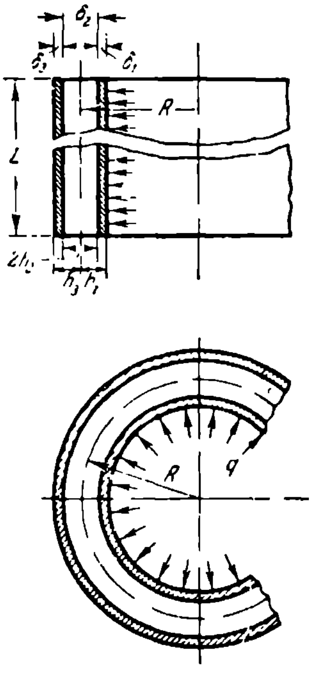

Figure 27

$E_1^{(1)} = E_1^{(3)} = E_2^{(1)} = E_2^{(3)} = 2 \cdot 10^6$ kg/см^2 $G_{12}^{(1)} = G_{12}^{(3)} = 0{,}77 \times 10^6$ kg/см^2, $\nu_1^{(1)} = \nu_2^{(1)} =$

$= \nu_1^{(3)} = \nu_2^{(3)} = 0{,}3$*).

Layer II is made of plastic wood. The layer thickness is $\delta = 6{,}0$ см; the moduli of elasticity and the Poisson ratio of the material are

$E_1^{(2)} = 0{,}5 \cdot 10^5$ kg/см^2, $E_2^{(2)} = 2 \cdot 10^5$ kg/см^2, $G_{12}^{(2)} = 0{,}5 \cdot 10^5$ kg/см^2 $\nu_1^{(2)} = 0{,}05$, $\nu_2^{(2)} = 0{,}2$.

The outer layers of the shell (I and III) are made of isotropic material and the inner layer (II) is made of an orthotropic material. The principal directions of elasticity of the material of the inner layer coincide with the coordinate directions of the shell (Figure 27).

From (2.12.2), for the coefficients B_{jk} we obtain

Layers I and III $B_{11}^{(1)} = B_{11}^{(3)} = B_{22}^{(1)} = B_{22}^{(3)} = 2{,}2 \cdot 10^6$ kg/см^2, $B_{12}^{(1)} =$

$= B_{12}^{(3)} = 0{,}66 \cdot 10^6$ kg/см^2, $B_{66}^{(1)} = B_{66}^{(3)} = 0{,}77 \cdot 10^6$ kg/см^2.

Layer II $B_{11}^{(2)} = 0{,}505 \cdot 10^5$ kg/см^2, $B_{22}^{(2)} = 2{,}02 \cdot 10^5$ kg/см^2,

$B_{12}^{(2)} = 0{,}101 \cdot 10^5$ kg/см^2, $B_{66}^{(2)} = 0{,}5 \cdot 10^5$ kg/см^2.

For the design thicknesses of the shell we have (Figure 27)

$$h_1 = h_3 = 4{,}2 \text{ см}, \quad h = 4{,}0 \text{ см}.$$

From (9.2), for the stiffnesses C_{jk} we obtain

$$C_{11} = 1{,}28 \cdot 10^6 \text{ kg/см}, \quad C_{22} = 2{,}50 \cdot 10^6 \text{ kg/см},$$
$$C_{12} = 0{,}345 \cdot 10^6 \text{ kg/см}, \quad C_{66} = 0{,}71 \cdot 10^6 \text{ kg/см}.$$

In calculating the stiffnesses C_{jk} it is considered that for a three-layer shell $m = 1$ (Chapter II, Section 12, Figure 13).

From (9.4) and (9.5), for coefficients A_{jk} we have

$$A_{11} = 0{,}812 \cdot 10^{-6} \text{ см/kg}, \quad A_{22} = 0{,}416 \cdot 10^{-6} \text{ см/kg};$$
$$A_{12} = -0{,}112 \cdot 10^{-6} \text{ см/kg}, \quad A_{66} = 1{,}41 \cdot 10^{-6} \text{ см/kg}.$$

---

*The parenthesized superscripts indicate the layer numbers.

Having all the design data for the shell, we may proceed to the calculation of the internal forces, displacements and stresses. In design formulas (5.30)-(5.31), by replacing a_{jk} with hA_{11}, we obtain

$$T_1 = -\frac{A_{12}}{A_{11}} Rq, \quad T_2 = Rq, \quad S = 0;$$
$$u = 0, \quad v = 0, \quad w = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}} R^2 q.$$

Inserting the values of A_{jk}, we finally obtain for the design quantities with values other than zero,

$$T_1 = 0{,}138 \ Rq \ \text{kg/cм}, \quad T_2 = Rq \ \text{kg/cм},$$
$$w = 0.401 \cdot 10^{-6} R^2 q \ \text{cм}.$$

(here and henceforth the value of R must be expressed in centimeters and the value of q in kg/cm2).

From (9.9), considering that for an orthotropic shell $A_{16} = A_{26} = 0$ (since $B_{16}^{(I,\ II,\ III)} = B_{26}^{(I,\ II,\ III)} = 0$), from the values of the internal forces we obtain for the deformations

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 0{,}401 \cdot 10^{-6} Rq, \quad \omega = 0.$$

From (9.6), considering the values of coefficients $B_{jk}^{(i)}$ as well as of the deformations $\cong \varepsilon_1, \varepsilon_2, \omega$, for the stresses within the layers we obtain

$$\sigma_s^{(1)} = \sigma_s^{(3)} = 0{,}264 Rq \ \text{kg/cм}^2, \quad \sigma_s^{(2)} = 0{,}004 Rq \ \text{kg/cм}^2,$$
$$\sigma_\varphi^{(1)} = \sigma_\varphi^{(3)} = 0{,}881 Rq \ \text{kg/cм}^2, \quad \sigma_\varphi^{(2)} = 0{,}081 Rq \ \text{kg/cм}^2.$$
$$\tau_{s\varphi}^{(1)} = \tau_{s\varphi}^{(2)} = \tau_{s\varphi}^{(3)} = 0.$$

The stresses in the layers may also be derived directly by means of formulas (9.7).

In similar manner we may solve other, more complex problems for multi-layer anisotropic shells.

22805

CHAPTER IV

SYMMETRICALLY LOADED ANISOTROPIC SHELLS OF REVOLUTION

SECTION 1. BASIC PREMISES. INITIAL RELATIONSHIPS AND EQUATIONS.

It is assumed that the coordinate surface ($\gamma = 0$) is a surface of revolution with axis of symmetry z and that it coincides with the internal surface of the shell. The location of any point M of the coordinate surface of the shell is defined by angle φ (the azimuth of the plane extended through point M and the axis of revolution z) and by the meridian arc s reckoned along the meridian from a certain initial point M_0 (Figures 17 and 28). For the discussed surface we introduce two more geometric quantities: r, which is the distance $\overline{MM_2}$ from point M to the axis of revolution z, and ϑ, which is the angle between the tangent to the meridian and the axis of revolution z (Figures 18 and 28).

In the chosen system of coordinates, for the principal curvatures and coefficients of the first quadratic form we have (Chapter III, Section 4)

$$\left.\begin{aligned} k_1 = \frac{1}{R_1} = \frac{d\vartheta}{ds}, \quad k_2 = \frac{1}{R_2} = \frac{\cos\vartheta}{r}, \\ A = 1, \quad B = r = R_2 \cos\vartheta. \end{aligned}\right\} \tag{1.1}$$

It is considered that the discussed shell is symmetrically loaded relative to the axis of revolution (X = 0, Y = 0, Z = 0) and that it has corresponding symmetrical boundary conditions. It consists of layers of orthotropic material so arranged that at each point of each layer one of the planes of elastic symmetry is parallel to the coordinate surface and the other two planes are perpendicular to the corresponding meridians and parallels*,**,***. The shell as a whole represents an orthotropic body of revolution possessing anisotropy of revolution.

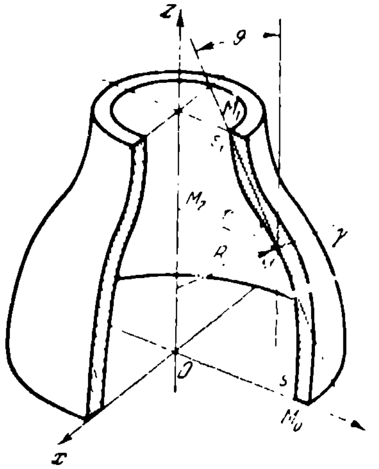

Figure 28

Since at the same time the shell will be deformed while remaining a body of revolution, its internal forces and displacements will not be functions of angular coordinate φ and within it there will arise only internal stresses T_1, T_2, $N_1 = N$ and bending moments M_1 and M_2, while only u and w of the displacements will differ from zero.

From the general equations and relationships in the theory of anisotropic laminar shells (Chapter II), for a symmetrically loaded anisotropic laminar s shell, considering (1.1), we obtain:

the equilibrium equations (Chapter II, Section 7)

$$\frac{d}{ds}(rT_1) + T_2 \sin\vartheta + \frac{r}{R_1} N = -rX; \tag{1.2}$$

$$\frac{d}{ds}(rN) - r\left(\frac{T_1}{R_1} + \frac{T_2}{R_2}\right) = -rZ; \tag{1.3}$$

$$\frac{d}{ds}(rM_1) + M_2 \sin\vartheta - rN_1 = 0; \tag{1.4}$$

---

*S.A. Ambartsumyan, Symmetrically loaded anisotropic shells of revolution, DAN ArmSSR, Vol. IX, No. 5, 1948.

**S.A. Ambartsumyan, Calculation of laminar shells of revolution, DAN ArmSSR, Vol. XI, No. 2, 1949.

***S.A. Ambartsumyan, On Calculation of laminar anisotropic shells, Izvestiya AN ArmSSR (FMYeiT nauki), Vol. VI, No. 3, 1953.

formulas associating the components of deformations, changes in curvature with the components of displacement (Chapter II, Section 2) where W is the

$$\varepsilon_1 = \frac{du}{ds} + \frac{w}{R_1}; \quad \varepsilon_2 = \frac{1}{r}(w\cos\vartheta - u\sin\vartheta); \tag{1.5}$$

$$\varkappa_1 = -\frac{dW}{ds}; \quad \varkappa_2 = W\frac{\sin\vartheta}{r}, \tag{1.6}$$

angle of rotation of a normal element of the shell in the plane of a meridian and has the following form:

$$W = \frac{dw}{ds} - \frac{u}{R_1}; \tag{1.7}$$

The equation of compatibility of deformations (Chapter II, Section 3)

$$r\frac{d\varepsilon_2}{ds} - (\varepsilon_2 - \varepsilon_1)\sin\vartheta - W\cos\vartheta = 0. \tag{1.8}$$

Equation (1.8) may be derived both from (1.3.3) and from relationships (1.5)-(1.7) by elimination of displacements u and w.

The equations and relationships obtained so far in this chapter do not differ from the corresponding equations and relationships in the theory of symmetrically loaded isotropic shells of revolution*;

the elasticity relationships (Chapter II, Sections 8 and 12)

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + K_{11}\varkappa_1 + K_{12}\varkappa_2; \tag{1.9}$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + K_{22}\varkappa_2 + K_{12}\varkappa_1; \tag{1.10}$$

$$M_1 = D_{11}\varkappa_1 + D_{12}\varkappa_2 + K_{11}\varepsilon_1 + K_{12}\varepsilon_2; \tag{1.11}$$

$$M_2 = D_{22}\varkappa_2 + D_{12}\varkappa_1 + K_{22}\varepsilon_2 + K_{12}\varepsilon_1. \tag{1.12}$$

where, in view of the fact that the coordinate surface coincides with the internal surface of the shell, that is, Δ = 0 (Figures 8 and 11), we have for the stiffnesses

$$C_{jk} = \sum_{s=1}^{p} B_{jk}^{s}(\delta_s - \delta_{s-1}); \tag{1.13}$$

$$K_{jk} = \frac{1}{2}\sum_{s=1}^{p} B_{jk}^{s}(\delta_s^2 - \delta_{s-1}^2); \tag{1.14}$$

$$D_{jk} = \frac{1}{3}\sum_{s=1}^{p} B_{jk}^{s}(\delta_s^3 - \delta_{s-1}^3). \tag{1.15}$$

---

*A. L. Lur'ye, Statika tonkostennykh obolochek, Gostekhizdat, 1947, pp. 7-34.

where p = m + n is the total number of layers, B^s_{jk} is the same as in (2.12.2); the stresses within the layers are (Chapter II, Section 12) (Figure 29)

$$\sigma^i_s = \frac{E^i_1}{1 - \nu^i_1\nu^i_2}\left[\varepsilon_1 + \nu^i_2\varepsilon_2 + \gamma\left(\varkappa_1 + \nu^i_2\varkappa_2\right)\right]; \quad (1.16)$$

$$\sigma^i_\varphi = \frac{E^i_2}{1 - \nu^i_1\nu^i_2}\left[\varepsilon_2 + \nu^i_1\varepsilon_1 + \gamma\left(\varkappa_2 + \nu^i_1\varkappa_1\right)\right]. \quad (1.17)$$

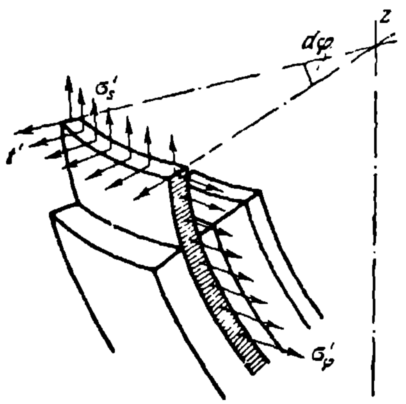

Figure 29

Inserting the values σ^i_s, and σ^i_φ into (2.4.7), after certain transformations we obtain with an accuracy of $k_j h$, the following value for the shearing stress $\tau^i_{s\gamma} = t^i$:

$$t^i = -(\gamma - \delta_{i-1})\,\Delta\left(B^i_{jk}\varepsilon\right) - \frac{\gamma^2 - \delta^2_{i-1}}{2}\,\Delta\left(B^i_{jk}\varkappa\right) + \varphi_i(s), \quad (1.18)$$

where for the operators $\Delta_i(B_{jk}q)$ we have

$$\Delta\left(B^i_{jk}q\right) = \frac{1}{r}\left\{\frac{d}{ds}\left[r\left(B^i_{11}q_1 + B^i_{12}q_2\right) + \left(B^i_{22}q_2 + B^i_{12}q_1\right)\sin\vartheta\right]\right\}. \quad (1.19)$$

From the conditions of contact of adjacent layers (2.5.2) and at the outer surfaces of the shell (2.5.3) and (1.5.4), we obtain for the integration function

$$\varphi_i(s) = -\sum_{s=1}^{i-1}\left[(\delta_s - \delta_{s-1})\,\Delta\left(B^s_{jk}\varepsilon\right) + \frac{\delta^2_s - \delta^2_{s-1}}{2}\,\Delta\left(B^s_{jk}\varkappa\right)\right] - X^-. \quad (1.20)$$

Then, finally, for t^i we have*

$$t^i = -(\gamma - \delta_{i-1})\,\Delta(B^i_{jk}\varepsilon) - \frac{\gamma^2 - \delta^2_{i-1}}{2}\,\Delta(B^i_{jk}\varkappa) - $$
$$- \sum_{s=1}^{i-1}\left[(\delta_s - \delta_{s-1})\,\Delta(B^s_{jk}\varepsilon) + \frac{\delta_s^2 - \delta^2_{s-1}}{2}\,\Delta(B^s_{jk}\varkappa)\right] - X^i. \tag{1.21}$$

SECTION 2. EQUATIONS OF SOLUTION AND DESIGN FORMULAS

After E. Meissner**, ***, let us introduce the auxiliary function $V = V(s)$, by means of which the internal forces are represented as follows:

$$T_2 = \frac{dV}{ds}, \quad T_1 = -\frac{\sin\vartheta}{r}V + \frac{1}{r}F(s); \tag{2.1}$$

$$N = \frac{\cos\vartheta}{r}V + \frac{1}{r}F_2(s). \tag{2.2}$$

Here $F_i(s)$ are functions of the outer surface load and are defined by the following formulas:

$$F_1 = \sin\vartheta\int_{s_0}^{s} rE_r\,ds + \cos\vartheta\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right); \tag{2.3}$$

$$F_2 = -\cos\vartheta\int_{s_0}^{s} rE_r\,ds + \sin\vartheta\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right), \tag{2.4}$$

where E_r and E_z are the components of the external surface load in the r and z directions, respectively, and P_z^0 is the value of the principal vector of the external forces applied to the parallel circle $s = s_0$ with radius r_0. It is evident that E_r, E_z, and P_z^0 have the following form (Figure 30):

---

*Stresses t^i (determined by means of (1.1)) are satisfied by all conditions of contact of the layers and by conditions at the outer surfaces. However, from (1.21) this is not always evident. In checking this it is necessary to consider equation (1.2) and formula (1.7.1) as well as the fact that $\delta_0 = 0$.

**E. Meissner, Das Elastizitätsproblem für dünne Schalen von Ringflächen, Kugel-oder Kegelform, Phys. Zs., XIV, No. 8, 1914.

***A. I. Lur'ye, see pp. 24-25 of work cited in footnote *, p. 111.

$$E_r = Z\cos\vartheta - X\sin\vartheta, \tag{2.5}$$

$$E_z = Z\sin\vartheta + X\cos\vartheta, \tag{2.6}$$

$$P_z^0 = (T_1^0\cos\vartheta_0 + N^0\sin\vartheta^0)\,2\pi r_0. \tag{2.7}$$

The lower limit of integration s_0 may be arbitrarily chosen on the basis of convenience of calculation.

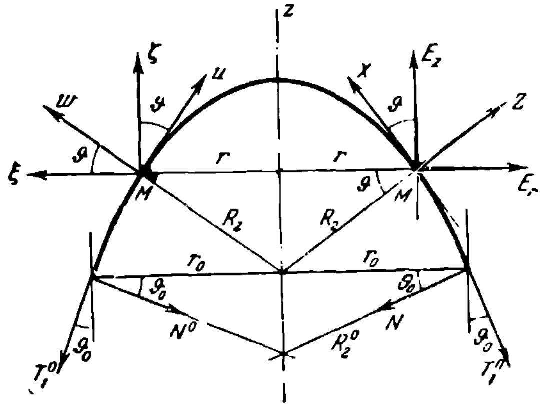

Figure 30

Inserting the values of T_1, T_2, and N from (2.1) and (2.2) into (1.2) and (1.3), we identically satisfy the first two equilibrium equations. The third equilibrium equation (1.4), by virtue of (2.2), takes the following form:

$$\frac{d}{ds}(rM_1) + M_2\sin\vartheta - V\cos\vartheta = F_2. \tag{2.8}$$

Simultaneous solution of relationships (1.9) and (1.10) for deformations ε_1 and ε_2, taking into account (2.1) and (1.6), yields

$$\varepsilon_1 = -\frac{1}{\Omega}\left[C_{22}\frac{\sin\vartheta}{r}V + C_{12}\frac{dV}{ds} + (K_{12}C_{22} - K_{22}C_{12})\frac{\sin\vartheta}{r}W - \right.$$
$$\left. - (K_{11}C_{22} - K_{12}C_{12})\frac{dW}{ds} - C_{22}\frac{1}{r}F_1(s)\right], \tag{2.9}$$

$$\varepsilon_2 = \frac{1}{\Omega}\left[C_{12}\frac{\sin\vartheta}{r}V + C_{11}\frac{dV}{ds} - (K_{22}C_{11} - K_{12}C_{12})\frac{\sin\vartheta}{r}W + \right.$$
$$\left. + (K_{12}C_{11} - K_{11}C_{12})\frac{dW}{ds} - C_{12}\frac{1}{r}F_1(s)\right]; \tag{2.10}$$

$$\Omega = C_{11}C_{22} - C_{12}^2. \tag{2.11}$$

Inserting the values of $\varkappa_1$, and $\varkappa_2$ from (1.6) and of ε_1, and ε_2 from (2.9) and (2.10) into (2.11) and (1.12), we obtain for the bending moments

$$\begin{aligned}M_1 = -\Big[&(D_{11} - D_{11}^0)\frac{dW}{ds} - (D_{12} - D_{12}^0)\frac{\sin\vartheta}{r}W + \\ &+ \frac{K_{11}C_{12} - K_{12}C_{11}}{\Omega}\frac{dV}{ds} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\frac{\sin\vartheta}{r}V - \\ &- \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\frac{1}{r}F_1(s)\Big],\end{aligned} \tag{2.12}$$

$$\begin{aligned}M_2 = -\Big[&(D_{12} - D_{12}^0)\frac{dW}{ds} - (D_{22} - D_{22}^0)\frac{\sin\vartheta}{r}W - \\ &- \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}\frac{dV}{ds} - \frac{K_{22}C_{12} - K_{12}C_{22}}{\Omega}\frac{\sin\vartheta}{r}V + \\ &+ \frac{K_{22}C_{12} - K_{12}C_{22}}{\Omega}\frac{1}{r}F_1(s)\Big],\end{aligned} \tag{2.13}$$

where, along with the known stiffnesses (1.13)-(1.15), we have

$$D_{11}^0 = \frac{(K_{11})^2 C_{22} - 2K_{11}K_{12}C_{12} + (K_{12})^2 C_{11}}{\Omega}; \tag{2.14}$$

$$D_{22}^0 = \frac{(K_{22})^2 C_{11} - 2K_{22}K_{12}C_{12} + (K_{12})^2 C_{22}}{\Omega}; \tag{2.15}$$

$$D_{12}^0 = \frac{K_{11}K_{12}C_{22} - [K_{11}K_{22} + (K_{12})^2]\, C_{12} + K_{22}K_{12}C_{11}}{\Omega}. \tag{2.16}$$

Inserting the values of ε_1 and ε_2 from (2.9) and (2.10) into (1.8) and the values of M_1 and M_2 from (2.12) and (2.13) into (2.8), after transformation we obtain

$$\begin{aligned}&\frac{d^2V}{ds^2} - \frac{\sin\vartheta}{r}\frac{dV}{ds} + \left(\frac{C_{12}}{C_{11}}\frac{1}{R_1R_2} - \frac{C_{22}}{C_{11}}\frac{\sin^2\vartheta}{r^2}\right)V = \\ &\quad = \frac{P_1}{C_{11}}\frac{d^2W}{ds^2} + \frac{P_2 - P_1}{C_{11}}\frac{\sin\vartheta}{r}\frac{dW}{ds} + \Big[\frac{\Omega}{C_{11}}\frac{1}{R_2} + \\ &\quad + \frac{K_{22}C_{11} - K_{12}C_{12}}{C_{11}}\frac{1}{R_1R_2} - \frac{P_3}{C_{11}}\frac{\sin^2\vartheta}{r^2}\Big]W + \Phi_1(s);\end{aligned} \tag{2.17}$$

$$\begin{aligned}&\frac{d^2W}{ds^2} - \frac{\sin\vartheta}{r}\frac{dW}{ds} - \left(\frac{D_{12} - D_{12}^0}{D_{11} - D_{11}^0}\frac{1}{R_1R_2} + \frac{D_{22} - D_{22}^0}{D_{11} - D_{11}^0}\frac{\sin^2\vartheta}{r^2}\right)W = \\ &\quad = -\frac{P_1}{\Omega(D_{11} - D_{11}^0)}\frac{d^2V}{ds^2} + \frac{P_2 + P_1}{\Omega(D_{11} - D_{11}^0)}\frac{\sin\vartheta}{r}\frac{dV}{ds} - \\ &\quad - \Big[\frac{1}{(D_{11} - D_{11}^0)}\frac{1}{R_2} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega(D_{11} - D_{11}^0)}\frac{1}{R_1R_2} - \\ &\quad - \frac{P_3}{\Omega(D_{11} - D_{11}^0)}\frac{\sin^2 3}{r^2}\Big]V + \Phi_2(s).\end{aligned} \tag{2.18}$$

where we have used the following notation:

$$\left.\begin{aligned} P_1 &= K_{11}C_{12} - K_{12}C_{11}; \\ P_2 &= K_{22}C_{11} - K_{11}C_{22}; \\ P_3 &= K_{22}C_{12} - K_{12}C_{22}; \end{aligned}\right\} \tag{2.19}$$

$$\Phi_1 = \frac{C_{12}}{C_{11}} \frac{1}{r} \frac{d}{ds} F_1(s) - \frac{C_{22}}{C_{11}} \frac{\sin\vartheta}{r^2} F_1(s); \tag{2.20}$$

$$\begin{aligned} \Phi_2 = {} & \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega\left(D_{11} - D_{11}^0\right)} \frac{1}{r} \frac{d}{ds} F_1(s) - \frac{1}{r\left(D_{11} - D_{11}^0\right)} F_2(s) - \\ & - \frac{K_{22}C_{12} - K_{12}C_{22}}{\Omega\left(D_{11} - D_{11}^0\right)} \frac{\sin\vartheta}{r^2} F_1(s). \end{aligned} \tag{2.21}$$

Equations (2.17) and (2.18) comprise a complete system of differential equations in terms of the two required auxiliary functions of Meissner, and W, by means of which (using formulas (2.1), (2.2), (2.12), and (2.13)) we determine all the design forces of the problem*.

From (1.16) and (1.17), by virtue of (1.6), (2.9) and (2.10), we obtain for the stresses in each layer of the shell

$$\begin{aligned} \sigma_s^i = {} & -\frac{1}{\Omega}\left(\Delta_{12}^i \frac{\sin\vartheta}{r} V + \Delta_{11}^i \frac{dV}{ds}\right) + \\ & + \left[\frac{1}{\Omega}\left(K_{11}\Delta_{12}^i - K_{12}\Delta_{11}^i\right) - \gamma B_{11}^i\right] \frac{dW}{ds} - \left[\frac{1}{\Omega}\left(K_{12}\Delta_{12}^i - K_{22}\Delta_{11}^i\right) - \right. \\ & \left. - \gamma B_{12}^i\right] \frac{\sin\vartheta}{r} W + \frac{\Delta_{12}^i}{\Omega} \frac{1}{r} F_1(s); \end{aligned} \tag{2.22}$$

$$\begin{aligned} \sigma_\varphi^i = {} & \frac{1}{\Omega}\left(\Delta_{22}^i \frac{\sin\vartheta}{r} V + \Delta_{21}^i \frac{dV}{ds}\right) - \\ & - \left[\frac{1}{\Omega}\left(K_{11}\Delta_{22}^i - K_{12}\Delta_{21}^i\right) + \gamma B_{12}^i\right] \frac{dW}{ds} + \left[\frac{1}{\Omega}\left(K_{12}\Delta_{22}^i - K_{22}\Delta_{21}^i\right) + \right. \\ & \left. + \gamma B_{22}^i\right] \frac{\sin\vartheta}{r} W - \frac{\Delta_{22}^i}{r} \frac{1}{r} F_1(s); \end{aligned} \tag{2.23}$$

$$\left.\begin{aligned} \Delta_{11}^i &= B_{11}^i C_{12} - B_{12}^i C_{11}, & \Delta_{22}^i &= B_{22}^i C_{12} - B_{12}^i C_{22}; \\ \Delta_{12}^i &= B_{11}^i C_{22} - B_{12}^i C_{12}, & \Delta_{21}^i &= B_{22}^i C_{11} - B_{12}^i C_{12}. \end{aligned}\right\} \tag{2.24}$$

The only shearing stress t^i differing from zero is determined from (2.18).

The final expression for the shearing stress t^i for the general case is not derived here due to its unwieldiness.

---

*S. A. Ambartsumyan, see footnote *** on p. 110.

The total displacement vector, as is known, may be represented not only by means of the usual displacement components u and w but also by means of two other components e_z and e_r, where e_z is the displacement in the direction of the z axis and e_r is the displacement in the direction of the radius r (Figure 30). It is evident that these displacement components are associated by the relationships*

$$\left.\begin{aligned} e_z &= u\cos\vartheta + w\sin\vartheta; \\ e_r &= w\cos\vartheta + u\sin\vartheta. \end{aligned}\right\} \tag{2.25}$$

Hence, by virtue of (1.5) and (1.7), the displacement components e_z and e_r will be represented in terms of deformations $\varepsilon_1, \varepsilon_2$ by means of the following formulas:

$$e_r = r\varepsilon_2, \quad e_z = e_z^0 + \int_{s_0}^{s} (\varepsilon_1\cos\vartheta + W\sin\vartheta)\,ds. \tag{2.26}$$

From (2.25), on the basis of (2.26), for the other pair of displacement components we obtain

$$\left.\begin{aligned} u &= -\varepsilon_2 r\sin\vartheta + \left[e_z^0 + \int_{s_0}^{s} (\varepsilon_1\cos\vartheta + W\sin\vartheta)\,ds\right]\cos\vartheta; \\ w &= \varepsilon_2 r\cos\vartheta + \left[e_z^0 + \int_{s_0}^{s} (\varepsilon_1\cos\vartheta + W\sin\vartheta)\,ds\right]\sin\vartheta. \end{aligned}\right\} \tag{2.27}$$

In these formulas e_z^0 is a constant defining the translation of the shell along the z axis.

SECTION 3. SHELLS OF REVOLUTION CONSISTING OF AN ODD NUMBER OF LAYERS SYMMETRICALLY ARRANGED RELATIVE TO THE MEDIAN SURFACE OF THE SHELL

Let us examine a symmetrically loaded orthotropic shell of revolution consisting of an odd number (2m + 1) of homogeneous orthotropic layers (Figure 15) (Chapter II, Section 13).

---

*A. I. Lur'ye, see pp. 12-15 of work cited in footnote on p. 111.

In this case, for the stiffnesses we have (1.13.2)-(1.13.4)

$$C_{jk}=2\left[B_{jk}^{m+1}h_{m+1}+\sum_{s=1}^{m}B_{jk}^{s}(h_s-h_{s+1})\right]; \tag{3.1}$$

$$D_{jk}=\frac{2}{3}\left[B_{jk}^{m+1}h_{m+1}^{3}+\sum_{s=1}^{m}B_{jk}^{s}(h_s^{3}-h_{s+1}^{3})\right]; \tag{3.2}$$

$$K_{jk}=0. \tag{3.3}$$

Then, from (1.9)-(1.12), for the internal forces and moments we obtain

$$T_1=C_{11}\varepsilon_1+C_{12}\varepsilon_2;\quad T_2=C_{22}\varepsilon_2+C_{12}\varepsilon_1; \tag{3.4}$$

$$M_1=D_{11}\varkappa_1+D_{12}\varkappa_2;\quad M_2=D_{22}\varkappa_2+D_{12}\varkappa_1. \tag{3.5}$$

For changes in curvature $\varkappa_1$, $\varkappa_2$ formulas (1.6) remain valid while for the deformations, from (2.9) and (2.10), we obtain

$$\varepsilon_1=-\frac{1}{\Omega}\left(C_{22}\frac{\sin\vartheta}{r}V+C_{12}\frac{dV}{ds}-C_{22}\frac{1}{r}F_1\right); \tag{3.6}$$

$$\varepsilon_2=\frac{1}{\Omega}\left(C_{12}\frac{\sin\vartheta}{r}V+C_{11}\frac{dV}{ds}-C_{12}\frac{1}{r}F_1\right). \tag{3.7}$$

For normal stresses in the shell layers we obtain from (2.22) and (2.23)

$$\sigma_s^i=-\frac{1}{\Omega}\left(\Delta_{12}^i\frac{\sin\vartheta}{r}V+\Delta_{11}^i\frac{dV}{ds}\right)-$$
$$-\gamma\left(B_{11}^i\frac{dW}{ds}-B_{12}^i\frac{\sin\vartheta}{r}W\right)+\frac{\Delta_{12}^i}{\Omega}\frac{1}{r}F_1; \tag{3.8}$$

$$\sigma_\varphi^i=\frac{1}{\Omega}\left(\Delta_{22}^i\frac{\sin\vartheta}{r}V+\Delta_{21}^i\frac{dV}{ds}\right)-$$
$$-\gamma\left(B_{12}^i\frac{dW}{ds}-B_{22}^i\frac{\sin\vartheta}{r}W\right)-\frac{\Delta_{22}^i}{\Omega}\frac{1}{r}F_1. \tag{3.9}$$

For shearing stresses t^i, after transformations, we obtain from (1.21):

for the outer layers

$$t^i=X^+-\frac{\gamma^2}{2}\Delta\left(B_{jk}^{2m+2-i}\varkappa\right)-\gamma\Delta\left(B_{jk}^{2m+2-i}\varepsilon\right)+$$
$$+\frac{h_0^2}{2}\Delta\left(P_{jk}^{2m+2-i}\varkappa\right)+h_0\Delta\left(R_{jk}^{2m+2-i}\varepsilon\right); \tag{3.10}$$

for the inner layers $(i=1,\ 2,\ \ldots,\ m+1)$

$$t^i = X^- - \frac{\gamma^2}{2}\Delta\left(B^i_{jk}\varkappa\right) - \gamma\Delta\left(B^i_{jk}\varepsilon\right) + \frac{h_0^2}{2}\Delta\left(P^i_{jk}\varkappa\right) - h_0\Delta\left(R^i_{jk}\varepsilon\right). \tag{3.11}$$

where for the given elasticity coefficients we have

$$P^i_{jk} = \frac{1}{h_0^2}\left[\delta_i^2 B^i_{jk} + \sum_{s=1}^{i-1} B^i_{jk}\left(\delta_s^2 - \delta_{s+1}^2\right)\right]; \tag{3.12}$$

$$R^i_{jk} = \frac{1}{h_0}\left[\delta_i B^i_{jk} + \sum_{s=1}^{i-1} B^i_{jk}\left(\delta_s - \delta_{s+1}\right)\right], \tag{3.13}$$

where $2h_0 = h$ (See Figure 15).

The internal forces T_1, T_2, N are expressed through fundamental function V by formulas (2.1) and (2.2), while for the moments, from (2.12) and (2.13), on the basis of (3.3) we obtain

$$M_1 = -D_{11}\frac{dW}{ds} + D_{12}\frac{\sin\vartheta}{r}W; \tag{3.14}$$

$$M_2 = -D_{12}\frac{dW}{ds} + D_{22}\frac{\sin\vartheta}{r}W. \qquad (3.15$$

The formulas for displacements (2.26) and (2.27) remain unchanged.

Finally, from (2.17) and (2.18) we obtain as a solution the system of differential equations

$$\frac{d^2V}{ds^2} - \frac{\sin\vartheta}{r}\frac{dV}{ds} + \left(\frac{C_{12}}{C_{11}}\frac{1}{R_1R_2} - \frac{C_{22}}{C_{11}}\frac{\sin^2\vartheta}{r^2}\right)V = \frac{\Omega}{C_{11}}\frac{1}{R_2}W + \Phi_1; \tag{3.16}$$

$$\frac{d^2W}{ds^2} - \frac{\sin\vartheta}{r}\frac{dW}{ds} - \left(\frac{D_{12}}{D_{11}}\frac{1}{R_1R_2} + \frac{D_{22}}{D_{11}}\frac{\sin^2\vartheta}{r^2}\right)W = -\frac{1}{D_{11}}\frac{1}{R_2}V + \Phi_2, \tag{3.17}$$

where for Φ_1 we have (2.20) and for Φ_2, from (2.21), we obtain

$$\Phi_2 = -\frac{1}{D_{11}}\frac{1}{r}F_2. \tag{3.18}$$

SECTION 4. SINGLE-LAYER SHELLS OF REVOLUTION

Let us discuss a symmetrically loaded orthotropic shell of revolution consisting of one homogeneous layer*. Assuming that the coordinate surface $(\gamma = 0)$ coincides with the median surface of the shell, for stiffnesses C_{jk} and D_{jk} we obtain (Chapter II, Section 14)

$$C_{jk} = hB_{jk}, \quad D_{jk} = \frac{h^3}{12} B_{jk}. \tag{4.1}$$

Then for the internal forces and moments we find

$$T_1 = h(B_{11}\varepsilon_1 + B_{12}\varepsilon_2); \quad T_2 = h(B_{22}\varepsilon_2 + B_{12}\varepsilon_1); \tag{4.2}$$

$$M_1 = \frac{h^3}{12}(B_{11}\varkappa_1 + B_{12}\varkappa_2); \quad M_2 = \frac{h^3}{12}(B_{22}\varkappa_2 + B_{12}\varkappa_1). \tag{4.3}$$

For changes in curvature formulas (1.6) remain unchanged and for the deformations, from (3.6) and (3.7), we obtain

$$\varepsilon_1 = -\frac{1}{h\left(B_{11}B_{22} - B_{12}^2\right)}\left[B_{22}\frac{\sin\vartheta}{r}V + B_{12}\frac{dV}{ds} - B_{22}\frac{1}{r}F_1\right]; \tag{4.4}$$

$$\varepsilon_2 = \frac{1}{h\left(B_{11}B_{22} - B_{12}^2\right)}\left[B_{12}\frac{\sin\vartheta}{r}V + B_{11}\frac{dV}{ds} - B_{12}\frac{1}{r}F_1\right]. \tag{4.5}$$

For the normal stresses, from (3.8) and (3.9), we have

$$\sigma_s = -\frac{\sin\vartheta}{r}\frac{V}{h} - \gamma\left(B_{11}\frac{dW}{ds} - B_{12}\frac{\sin\vartheta}{r}W\right) + \frac{1}{hr}F_1; \tag{4.6}$$

$$\sigma_\varphi = \frac{1}{h}\frac{dV}{ds} - \gamma\left(B_{12}\frac{dW}{ds} - B_{22}\frac{\sin\vartheta}{r}W\right). \tag{4.7}$$

The internal forces T_1, T_2, N are expressed in terms of the fundamental function V by formulas (2.1) and (2.2) and for the moments, from (3.14) and (3.15), we obtain

$$M_1 = -\frac{h^3}{12}\left(B_{11}\frac{dW}{ds} - B_{12}\frac{\sin\vartheta}{r}W\right); \tag{4.8}$$

$$M_2 = -\frac{h^3}{12}\left(B_{12}\frac{dW}{ds} - B_{22}\frac{\sin\vartheta}{r}W\right). \tag{4.9}$$

---

*S. A. Ambartsumyan, see footnote *, on p. 110.

As before, the formulas for displacements (2.26) and (2.27) remain unchanged.

The system of equations of solution (3.16) and (3.17) take the following form:

$$\frac{d^2V}{ds^2} - \frac{\sin\vartheta}{r}\frac{dV}{ds} + \left(\frac{B_{12}}{B_{11}}\frac{1}{R_1R_2} - \frac{B_{22}}{B_{11}}\frac{\sin^2\vartheta}{r^2}\right)V = \\ = \frac{h\left(B_{11}B_{22} - B_{12}^2\right)}{B_{11}}\frac{1}{R_2}W + \Phi_1. \tag{4.10}$$

$$\frac{d^2W}{ds^2} - \frac{\sin\vartheta}{r}\frac{dW}{ds} - \left(\frac{B_{12}}{B_{11}}\frac{1}{R_1R_2} + \frac{B_{22}}{B_{11}}\frac{\sin^2\vartheta}{r^2}\right)W = \\ = -\frac{12}{B_{11}h^3}\frac{1}{R_2}V + \Phi_2. \tag{4.11}$$

where for Φ_1 and Φ_2, from (2.20) and (3.18), we have

$$\Phi_1 = -\frac{B_{12}}{B_{11}}\frac{1}{r}\frac{d}{ds}F_1 - \frac{B_{22}}{B_{11}}\frac{\sin\vartheta}{r^2}F_1. \tag{4.12}$$

$$\Phi_2 = -\frac{12}{B_{11}h^3}\frac{1}{r}F_2. \tag{4.13}$$

Comparing the results of the last two sections, we note that the fundamental equations and design formulas for a single-layer shell (Section 4) are derived from the corresponding equations and formulas for a symmetrically assembled laminar shell (Section 3) by elementary substitutions. In this connection, henceforth we shall be chiefly concerned with the general case, that is, with laminar shells.

SECTION 5. REDUCTION OF THE SYSTEM OF EQUATIONS IN (3.16) AND (3.17) TO A SINGLE EQUATION. A PARTICULAR SOLUTION OF THE INHOMOGENEOUS EQUATION.

We make the approximate assumption

$$\frac{C_{22}}{C_{11}} = \frac{D_{22}}{D_{11}} = \lambda, \tag{5.1}$$

where γ is an arbitrary constant.

However, as is easily shown, condition (5.1) is precisely satisfied in the case of a single-layer shell as well as in the case in which the ratio B^i_{11} for all layers of the shell has identical value. We return again to approximate condition (5.1) in the present section of this chapter.

Considering (5.1), for the equations of solution of the system (3.16) and (3.17) we have

$$L(V)+\frac{C_{12}}{C_{11}}\frac{1}{R_1R_2}V=\frac{\Omega}{C_{11}}\frac{1}{R_2}W+\Phi_1(s); \tag{5.2}$$

$$L(W)-\frac{D_{12}}{D_{11}}\frac{1}{R_1R_2}W=-\frac{1}{D_{11}}\frac{1}{R_2}V+\Phi_2(s), \tag{5.3}$$

where, for the sake of brevity, we have introduced the known linear operator

$$L=\frac{d^2}{ds^2}-\frac{\sin\vartheta}{r}\frac{d}{ds}-\lambda\frac{\sin^2\vartheta}{r^2}. \tag{5.4}$$

We then follow the reasoning presented by A. I. Lur'ye* in the case of an isotropic shell. Multiplying (5.2) by the still undetermined factor a and combining with (5.3), we obtain

$$L(W)-\left(\frac{D_{12}}{D_{11}}\frac{1}{R_1}+\frac{\Omega}{C_{11}}a\right)\frac{W}{R_2}+a\left[L(V)+\right.$$
$$\left.+\left(\frac{C_{12}}{C_{11}}\frac{1}{R_1}+\frac{1}{aD_{11}}\right)\frac{V}{R_2}\right]=\Phi_2+a\Phi_1. \tag{5.5}$$

Assuming

$$\left.\begin{aligned}\frac{D_{12}}{D_{11}}\frac{1}{R_1}+\frac{\Omega}{C_{11}}a&=-ik^2,\\ \frac{C_{12}}{C_{11}}\frac{1}{R_1}+\frac{1}{aD_{11}}&=ik^2,\end{aligned}\right\} \tag{5.6}$$

for determination of a we obtain the following quadratic equation:

$$a^2+\frac{C_{11}}{\Omega}\left(\frac{D_{12}}{D_{11}}+\frac{C_{12}}{C_{11}}\right)\frac{a}{R_1}+\frac{C_{11}}{D_{11}\Omega}=0. \tag{5.7}$$

Comparing one of the roots of this equation with (5.6), with an accuracy of $\frac{h}{R}$ (discarding terms of the order of $\frac{h}{R_1}$ in comparison with unity), we obtain

$$k^2=\sqrt{\frac{\Omega}{C_{11}D_{11}}},\quad a=-i\frac{C_{11}}{\Omega}\sqrt{\frac{\Omega}{C_{11}D_{11}}}. \tag{5.8}$$

---

*A. I. Lur'ye, see pp. 36-37 of work cited in footnote on p. 111.

Now, inserting the value of a from (5.8) into (5.5) and discarding terms of the order of $\frac{h}{R_1}$ in comparison with unity, we obtain the differential equation of solution of the problem

$$L(\sigma) + i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\,\frac{\sigma}{R_2} = \Phi(s), \tag{5.9}$$

where for the new sought function we have

$$\sigma = W - i\frac{C_{11}}{\Omega}\sqrt{\frac{\Omega}{C_{11}D_{11}}}\,V, \tag{5.10}$$

while for the load term we obtain

$$\Phi(s) = -\frac{F_2}{rD_{11}} - \frac{i}{r}\,\frac{\sqrt{\frac{\Omega}{C_{11}D_{11}}}}{\Omega}\left[C_{12}\frac{dF_1}{ds} - C_{22}\frac{\sin\vartheta}{r}F_1\right]. \tag{5.11}$$

Numerous investigations*,** have shown that a particular solution of equation (5.9) corresponding to the right side of the equation, with sufficiently smooth change in external load may be constructed from the membrane theory. Generally, for the loads ordinarily encountered in practice the particular solution of inhomogeneous equation (5.9) is obtained by various approximate methods***. A method of determining the particular integrals for certain special types of loading of a shell may be found in V. M. Darevskiy****.

The particular integral of the inhomogeneous equation (5.9) is obtained on the basis of membrane theory. Proceeding from the basic premises of this theory (Chapter III, Section 1), in the case of a symmetrically loaded shell of revolution we assume (3.1.7)

$$M_1 = 0, \quad M_2 = 0, \quad N = 0. \tag{5.12}$$

From (2.2), (3.14) and (3.15), by virtue of (5.12), as the particular integral of inhomogeneous system (5.2) and (5.3) we obtain

$$W^0 = 0, \quad V^0 = -\frac{F_2}{\cos\vartheta}. \tag{5.13}$$

---

*A. I. Lur'ye, see pp. 38-39 of work cited in footnote on p. 111.

**V. V. Novozhilov, Teoriya tonkikh obolochek, Sudpromgiz, 1951, pp. 180-183 and 247-255.

***A. L. Gol'denveyzer, Teoriya uprugikh tonkikh obolochek, Gostekhizdat, 1953, part V.

****V. M. Darevskiy, On a theory of cylindrical shells PMM, Vol. XV, No. 5, 1951.

Hence, on the basis of (5.10), the sought particular solution of inhomogeneous equation (5.9) has the form

$$\sigma_0 = i\frac{C_{11}}{\Omega}\sqrt{\frac{\Omega}{C_{11}D_{11}}}\frac{F_2}{\cos\vartheta}. \qquad (5.14)$$

The values of forces T_1 and T_2 corresponding to particular solution (5.14) are determined from (2.1). After certain transformations for T_1^0 and T_2^0

we obtain

$$T_1^0 = \frac{1}{R_2\cos^2\vartheta}\left[\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right]; \qquad (5.15)$$

$$T_2^0 = -\frac{1}{R_1\cos^2\vartheta}\left[\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right] + R_2Z. \qquad (5.16)$$

T_1^0 and T_2^0 may easily be derived directly from the corresponding formulas in the membrane theory of shells of revolution (Chapter III, Section 4). From (3.4.21) and (3.4.22), by means of (2.3)-(2.7) we obtain for T_1^0 and T_2^0 the same formulas as (5.15) and (5.16). As was to be expected, the formulas for T_1^0 and T_2^0 do not differ from the corresponding formulas derived for the case of an isotropic shell of revolution*.

SECTION 6. ASYMPTOTIC INTEGRATION OF THE EQUATION OF SOLUTION (5.9)

Assuming that the particular solution of inhomogeneous equation (5.9) is precisely or approximately derived, we shall here concern ourselves only with problems of determining the general solution of the corresponding homogeneous equation.

The principal method of obtaining the general solution of the homogeneous equation of a symmetrically loaded anisotropic shell of revolution will be considered to be the method of asymptotic integration, which, as correctly pointed out in the cited works by A. L. Gol'denveyzer, A. L. Lur'ye and V. V.

---

*A. I. Lur'ye, see footnote 1 on p. 123.

Novozhilov, can provide the required accuracy corresponding to the accuracy of equation of solution (5.9). We shall not concern ourselves with the other methods of solution of problems of anisotropic shells of revolution*.

We shall describe the method of asymptotic integration of the equation of solution (5.9) used by A. L. Gol'denveyzer**, in whose work many important and interesting details concerning asymptotic integration in the theory of shells are discussed.

In expanded form this homogeneous equation is as follows:

$$\frac{d^2\sigma}{ds^2}-\frac{\sin\vartheta}{r}\frac{d\sigma}{ds}-\lambda\frac{\sin^2\vartheta}{r^2}\sigma+i\sqrt{\frac{\Omega}{C_{11}D_{11}}}\frac{\sigma}{R_2}=0. \tag{6.1}$$

The coefficient of the last term of equation (6.1) is a large parameter

$$A=\sqrt{\frac{\Omega}{C_{11}D_{11}}}=\frac{1}{h_0}\sqrt{\frac{b}{a}}, \tag{6.2}$$

where a and b are certain constants depending on the elastic properties of the layer and on the ratios of the thicknesses of the individual layers to the half-thickness of the entire shell ($h=2h_0$). It is evident that these constants have the dimension of B_{jk} (see (3.1) and (3.2)). Introducing the notation

$$k^2=A,\quad k=\sqrt[4]{\frac{b}{a}}\,h_0^{-1/2}, \tag{6.3}$$

the integrals of equation (6.3) will be sought in the form

$$\sigma=\varphi(s,\dot{k})\,e^{kf(s)}, \tag{6.4}$$

where $\varphi(s,k)$ is a function of intensity and $f(s)$ is a function of variability***.

The intensity function may be represented in the form an asymptotic series

$$\varphi(s,k)\backsim\sigma_0(s)+\frac{\sigma_1(s)}{k}+\frac{\sigma_2(s)}{k^2}+\ldots+\frac{\sigma_n(s)}{k^n}+\ldots(\sigma_0\neq 0). \tag{6.5}$$

---

*Ye. F. Burmistrov, Symmetrical deformation of a near-cylindrical shell, PMM, Vol. XIII, No. 4, 1949.

A. L. Gol'denveyzer, see pp. 282-326 of work cited in footnote * on p. 123.

***A. L. Gol'denveyzer, see p. 287 of work cited in footnote ***, on p. 123.

in which $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ do not depend on k. Inserting the value of σ from (6.4) into (6.1), considering (6.3) and (6.5) and canceling e^{kf}, we obtain,

$$\sum_{n=0}^{\infty} \frac{1}{k^{n-2}}\left[\left(\frac{df}{ds}\right)^2 \sigma_n + \frac{i}{R_2}\sigma_n\right] + \sum_{n=0}^{\infty} \frac{1}{k^{n-1}}\left[\frac{d^2 f}{ds^2}\sigma_n + 2\frac{df}{ds}\frac{d\sigma_n}{ds} - \frac{\sin\vartheta}{r}\frac{df}{ds}\sigma_n\right] +$$
$$+ \sum_{n=0}^{\infty} \frac{1}{k^n}\left[\frac{d^2\sigma_n}{ds^2} - \frac{\sin\vartheta}{r}\frac{d\sigma_n}{ds} - \lambda\frac{\sin^2\vartheta}{r^2}\sigma_n\right] = 0. \tag{6.6}$$

Requiring that the coefficients in (6.6) for all powers of k vanish, we obtain an infinite system of recursion formulas for determination of $f, \sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$

$$\left.\begin{aligned} &\left(\frac{df}{ds}\right)^2 + \frac{i}{R_2} = 0; \\ &2\frac{df}{ds}\frac{d\sigma_0}{ds} + \left(\frac{d^2 f}{ds^2} - \frac{\sin\vartheta}{r}\frac{df}{ds}\right)\sigma_0 = 0; \\ &2\frac{df}{ds}\frac{d\sigma_n}{ds} + \left(\frac{d^2 f}{ds^2} - \frac{\sin\vartheta}{r}\frac{df}{ds}\right)\sigma_n = \\ &= -\left(\frac{d^2\sigma_{n-1}}{ds^2} - \frac{\sin\vartheta}{r}\frac{d\sigma_{n-1}}{ds} - \lambda\frac{\sin^2\vartheta}{r^2}\sigma_{n-1}\right) \\ &\qquad (n = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{6.7}$$

Limiting ourselves to the first approximation of asymptotic integration (that is, examining only those equations which are obtained when the coefficients of k^2 and k vanish), we obtain

$$\left(\frac{df}{ds}\right)^2 = -\frac{i}{R_2}; \tag{6.8}$$

$$2\frac{df}{ds}\frac{d\sigma_0}{ds} + \left(\frac{d^2 f}{ds^2} - \frac{\sin\vartheta}{r}\frac{df}{ds}\right)\sigma_0 = 0. \tag{6.9}$$

From (6.8) for the variability function we find

$$f_{1,2} = \pm\frac{1}{\sqrt{2}}(1 - i)\int_{s_0}^{s}\frac{ds}{\sqrt{R_2}}. \tag{6.10}$$

Inserting the values of $f(s)$ from (6.10) into (6.9), in order to determine $\sigma_0(s)$ we obtain the following differential first-order equation:

$$\frac{d\sigma_0}{ds} - \frac{\sin\vartheta}{4r}\left(1 + \frac{R_2}{R_1}\right)\sigma_0 = 0. \tag{6.11}$$

From (6.11) for σ_0 we have

$$\sigma_0 = R_2^{1/4} r^{-1/2}. \tag{6.12}$$

Inserting the values of σ_0 and f from (6.12) and (6.10), respectively, into (6.4) and considering (6.3), we obtain for σ

$$\sigma = \frac{\sqrt[4]{R_2}}{\sqrt{r}} \left[\bar{C}_1' e^{-(1-i)\beta} + \bar{C}_2' e^{(1-i)\beta}\right], \tag{6.13}$$

where

$$\beta = \sqrt{\frac{A}{2}} \int_{s_0}^{s} \frac{ds}{\sqrt{R_2}}; \tag{6.14}$$

$\bar{C}_1'$ and $\bar{C}_2'$ are arbitrary complex constants.

Inserting the value of r in terms of C_2 into (6.13), we finally obtain for σ

$$\sigma = \frac{1}{\sqrt[4]{R_2 \cos^2 \vartheta}} \left[\bar{C}_1' e^{-(1-i)\beta} + \bar{C}_2' e^{(1-i)\beta}\right]. \tag{6.15}$$

The solution (6.15) derived in this manner is the first approximation of the general solution of differential equation (6.1).

Examining the procedure for obtaining solution (6.15), it is easily seen that with the same accuracy the solution may be obtained directly from the initial homogeneous equations (3.16) and (3.17) if we disregard therein the third and fourth terms (that is, those terms which contain ratios C_{22}/C_{11} and D_{22}/D_{11}. By virtue of this it may be asserted that the limiting assumption in (5.1) loses its force; that is, if we limit ourselves to the accuracy of the first approximation of the asymptotic integration, condition (5.1) will not have any effect on the subsequent calculations for a shell of revolution in the general case of symmetrical lamination and orthotropic anisotropy.

By elementary reasoning it may be shown that the first approximation in asymptotic integration of the equations of solution of symmetrically loaded orthotropic shells of revolution has an error of the order of $\sqrt{h/R_j}$ in comparison with unity*, **.

---

*V. V. Novozhilov, see pp. 255-267 of work cited in footnote ** on p. 123.

**A. I. Lur'ye, see pp. 147-148 of work cited in footnote on p. 111.

In the present work we shall limit ourselves only to the first approximation, since for purposes of engineering calculation its accuracy is quite adequate*.

With these remarks in mind, it is easily shown that with an accuracy of the first approximation the factor in formula (6.15), $\frac{1}{\sqrt[4]{R_2\cos^2\vartheta}}$, upon differentiation, behaves as a constant. Then, including the mentioned factor in the integration constants, we obtain from (6.15)

$$\sigma=\bar{C}_1e^{-(1-i)\beta}+\bar{C}_2e^{(1-i)\beta}. \tag{6.16}$$

Considering that

$$\bar{C}_1=E_1+iF_1;\quad \bar{C}_2=E_2+iF_2;$$
$$e^{-(1-i)\beta}=e^{-\beta}(\cos\beta+i\sin\beta);\quad e^{(1-i)\beta}=e^{\beta}(\cos\beta-i\sin\beta).$$

we finally obtain for σ

$$\begin{aligned}\sigma={}&(E_1\cos\beta-F_1\sin\beta)e^{-\beta}+(E_2\cos\beta+F_2\sin\beta)e^{\beta}+\\&+i[(E_1\sin\beta+F_1\cos\beta)e^{-\beta}-(E_2\sin\beta-F_2\cos\beta)e^{\beta}].\end{aligned} \tag{6.17}$$

Since the general solution of the equation of solution (5.9) is the sum of solutions (5.14) and (6.17), on the basis of (5.10) we obtain

$$\begin{aligned}W-iA\frac{C_{11}}{\Omega}V={}&(E_1\cos\beta-F_1\sin\beta)e^{-\beta}+(E_2\cos\beta+F_2\sin\beta)e^{\beta}+\\&+i[(E_1\sin\beta+F_1\cos\beta)e^{-\beta}-(E_2\sin\beta-F_2\cos\beta)e^{\beta}]+\\&+i\frac{C_{11}}{\Omega}A\frac{F_2(s)}{\cos\vartheta}.\end{aligned} \tag{6.18}$$

Hence, separating the imaginary and real parts, we obtain

$$W=(E_1\cos\beta-F_1\sin\beta)e^{-\beta}+(E_2\cos\beta+F_2\sin\beta)e^{\beta}, \tag{6.19}$$

$$\begin{aligned}V=-\frac{\Omega}{AC_{11}}[(E_1\sin\beta+F_1\cos\beta)e^{-\beta}-\\-(E_2\sin\beta-F_2\cos\beta)e^{\beta}]-\frac{F_2(s)}{\cos\vartheta}.\end{aligned} \tag{6.20}$$

---

*If desired, proceeding from the system of equations in (3.16) and (3.17), we may obtain the second approximation, the accuracy of which will not be less than that of the initial system of equations.

Following A. I. Lur'ye*, the solution of equation (6.1) may be represented with the aid of

$$\sigma = (C_1 - iC_3)(\Omega_1 - 2i\Omega_3) + (C_2 - iC_4)(\Omega_1 - 2i\Omega_4), \tag{6.21}$$

where $C_1 \ldots C_4$ are arbitrary constants, $\Omega_j(\beta)$ are A. N. Krylov's functions (See Table 1), which are linear combinations of $e^{-(1-i)\beta}$ and $e^{(1-i)\beta}$. From (6.21) and (5.14), on the basis of (5.10), we obtain

$$W - iA\frac{C_{11}}{\Omega}V = (C_1 - iC_3)(\Omega_1 - 2i\Omega_3) + \\ + (C_2 - iC_4)(\Omega_2 - 2i\Omega_4) + i\frac{C_{11}}{\Omega}A\frac{F_2(s)}{\cos\vartheta}. \tag{6.22}$$

hence, by analogy with (6.19) and (6.20), we have

$$W = C_1\Omega_1 + C_2\Omega_2 - 2C_3\Omega_3 - 2C_4\Omega_4; \tag{6.23}$$

$$V = AD_{11}(C_3\Omega_1 + C_4\Omega_2 + 2C_1\Omega_3 + 2C_2\Omega_4) - \frac{F_2(s)}{\cos\vartheta}. \tag{6.24}$$

Table 1

A. N. Krylov's functions

k	$\Omega_k(\beta)$	$\frac{d\Omega_k}{d\beta}$	$\frac{d^2\Omega_k}{d\beta^2}$	$\frac{d^3\Omega_k}{d\beta^3}$	$\frac{d^4\Omega_k}{d\beta^4}$
1	$\operatorname{ch}\beta\cos\beta$	$-4\Omega_4$	$-4\Omega_3$	$-4\Omega_2$	$-4\Omega_1$
2	$\frac{1}{2}(\operatorname{ch}\beta\sin\beta + \operatorname{sh}\beta\cos\beta)$	Ω_1	$-4\Omega_4$	$-4\Omega_3$	$-4\Omega_2$
3	$\frac{1}{2}\operatorname{sh}\beta\sin\beta$	Ω_2	Ω_1	$-4\Omega_4$	$-4\Omega_3$
4	$\frac{1}{4}(\operatorname{ch}\beta\sin\beta - \operatorname{sh}\beta\cos\beta)$	Ω_3	Ω_2	Ω_1	$-4\Omega_4$

The solution of equation (5.9) may be presented in slightly different form**. Let us introduce the new variable (Figures 18 and 28)

$$\beta_1 = \sqrt{\frac{A}{2}}\int_{s_0}^{s_1}\frac{ds}{\sqrt{R_1}} - \sqrt{\frac{A}{2}}\int_{s_0}^{s}\frac{ds}{\sqrt{R_1}}, \tag{6.25}$$

---

*A. I. Lur'ye, see pp. 68-69 of work cited in footnote on p. 111.

**A. I. Lur'ye, see pp. 59-61 of work cited in footnote on p. 111.

considering (6.14) and introducing for the constant the notation

$$\alpha_0 = \sqrt{\frac{A}{2}} \int_{s_0}^{s_1} \frac{ds}{\sqrt{R_2}}, \tag{6.26}$$

Table 2

β	φ (β)	ψ (β)	ϑ (β)	ζ (β)
0	1,0000	1,0000	1,0000	0
0,1	0,9307	0,8100	0,9003	0,0903
0,2	0,9651	0,6398	0,8024	0,1627
0,3	0,9267	0,4888	0,7077	0,2189
0,4	0,8784	0,3564	0,6174	0,2610
0,5	0,8231	0,2415	0,5323	0,2908
0,6	0,7628	0,1431	0,4530	0,3099
0,7	0,6997	0,0599	0,3798	0,3199
0,8	0,6354	—0,0093	0,3131	0,3223
0,9	0,5712	—0,0657	0,2527	0,3185
1,0	0,5083	—0,1108	0,1988	0,3096
1,1	0,4467	—0,1457	0,1510	0,2967
1,2	0,3899	—0,1716	0,1091	0,2807
1,3	0,3355	—0,1807	0,0729	0,2626
1,4	0,2849	—0,2011	0,0419	0,2430
1,5	0.2384	—0,2068	0,0158	0,2226
1,6	0,1959	—0,2077	—0,0059	0,2018
1,7	0,1576	—0,2047	—0,0235	0,1812
1,8	0,1234	—0,1985	—0,0376	0,1610
1,9	0,0932	—0,1899	—0,0484	0,1415
2,0	0,0667	—0,1794	—0,0563	0,1230
2,1	0,0439	—0,1675	—0,0618	0,1057
2,2	0,0244	—0,1548	—0,0652	0,0895
2,3	0,0080	—0,1416	—0,0668	0,0748
2,4	—0,0056	—0,1282	—0,0669	0,0613
2,5	—0,0166	—0,1149	—0,0658	0,0492
2,6	—0,0254	—0,1019	—0,0636	0,0383
2,7	—0,0320	—0,0895	—0,0608	0,0287
2,8	—0,0369	—0,0777	—0,0573	0,0204
2,9	—0,0403	—0,0666	—0,0534	0,0132
3,0	—0,0423	—0,0563	—0,0493	0,0071
3,1	—0,0431	—0,0469	—0,0450	0,0019
3,2	—0,0431	—0,0383	—0,0407	—0,0024
3,3	—0,0422	—0,0306	—0,0364	—0,0058
3,4	—0,0408	—0,0237	—0,0323	—0,0085
3,5	—0,0389	—0,0177	—0,0283	—0,0106
3,6	—0,0366	—0,0124	—0,0245	—0,0121
3,7	—0,0341	—0,0079	—0,0210	—0,0131
3,8	—0,0314	—0,0040	—0,0177	—0,0137
3,9	—0,0286	—0,0008	—0,0147	—0,0140
4,0	—0,0258	0,0019	—0,0120	—0,0139
4,1	—0,0204	0,0057	—0,0074	—0,0131
4,2	—0,0204	0,0057	—0,0074	—0,0131
4,3	—0,0179	0,0070	—0,0054	—0,0125
4,4	—0,0155	0,0079	—0,0038	—0,0117
4,5	—0,0132	0,0085	—0,0023	—0,0108
4,6	—0,0111	0,0089	—0,0011	—0,0100
4,7	—0,0092	0,0090	—0,0001	—0,0091
4,8	—0,0075	0,0089	0,0007	—0,0082
4,9	—0,0059	0,0087	0,0014	—0,0073

Note: Comma represents decimal point.

Table 2 Continued

β	$\varphi(\beta)$	$\psi(\beta)$	$\theta(\beta)$	$\zeta(\beta)$
5,0	—0,0046	0,0084	0,0019	—0,0065
5,1	—0,0033	0,0080	0,0023	—0,0057
5,2	—0,0023	0,0075	0,0026	—0,0049
5,3	—0,0014	0,0069	0,0023	—0,0042
5,4	—0,0006	0,0064	0,0029	—0,0035
5,5	0,0000	0,0058	0,0029	—0,0029
5,6	0,0005	0,0052	0,0029	—0,0029
5,7	0,0010	0,0046	0,0028	—0,0018
5,8	0,0013	0,0041	0,0027	—0,0014
5,9	0,0015	0,0036	0,0026	—0,0010
6,0	0,0013	0,0031	0,0024	—0,0007

Note: Comma represents decimal point.

for the independent variable we obtain

$$\beta = \alpha_0 - \beta_1. \tag{6.27}$$

Inserting the value of β from (6.27) into (6.19), we obtain

$$W = (E_1 \cos \beta - F_1 \sin \beta) e^{-\beta} + [(E_2 \cos \alpha_0 + F_2 \sin \alpha_0) \cos \beta_1 + \\ + (E_2 \sin \alpha_0 - F_2 \cos \alpha_0) \sin \beta_1] e^{\alpha_0 - \beta_1}.$$

Introducing the new constants

$$\left. \begin{aligned} A_1 &= E_1; \quad A_2 = (E_2 \cos \alpha_0 + F_2 \sin \alpha_0) e^{\alpha_0}; \\ B_1 &= -F_1; \quad B_2 = (E_2 \sin \alpha_0 - F_2 \cos \alpha_0) e^{\alpha_0}, \end{aligned} \right\} \tag{6.28}$$

for the sought W we obtain

$$W = A_1 e^{-\beta} \cos \beta + B_1 e^{-\beta} \sin \beta + A_2 e^{-\beta} \cos \beta_1 + B_2 e^{-\beta_1} \sin \beta_1. \tag{6.29}$$

Similarly, from (6.20), for the sought V we obtain

$$V = -\frac{\Omega}{AC_{11}} (A_1 e^{-\beta} \sin \beta - B_1 e^{-\beta} \cos \beta + A_2 e^{-\beta_1} \sin \beta_1 - \\ - B_2 e^{-\beta_1} \cos \beta_1) - \frac{F_1(s)}{\cos \vartheta}. \tag{6.30}$$

For the tabulated functions let us introduce the following notation:

$$\left. \begin{aligned} \theta(\beta) &= e^{-\beta} \cos \beta, \quad \zeta(\beta) = e^{-\beta} \sin \beta, \\ \varphi(\beta) &= \theta(\beta) + \zeta(\beta), \quad \psi(\beta) = \theta(\beta) - \zeta(\beta). \end{aligned} \right\} \tag{6.31}$$

Table 2 lists the values of these functions for values of arguments $0 \leqslant \beta \leqslant 6$. (The table was taken from A. I. Lur'ye. See his frequently cited work.) Between these functions there exists the following relationships:

$$\left.\begin{aligned} &\frac{d}{d\beta}\theta(\beta) = -\varphi(\beta), \quad \frac{d}{d\beta}\zeta(\beta) = \psi(\beta), \\ &\frac{d^2}{d\beta^2}\theta(\beta) = 2\zeta(\beta), \quad \frac{d^2}{d\beta^2}\zeta(\beta) = -2\theta(\beta). \end{aligned}\right\} \tag{6.32}$$

Considering (6.31), from (6.29) and (6.60) we finally obtain for the sought functions

$$W = A_1\theta(\beta) + B_1\zeta(\beta) + A_2\theta(\beta_1) + B_2\zeta(\beta_1); \tag{6.33}$$

$$V = \frac{\Omega}{AC_{11}}[-A_1\zeta(\beta) + B_1\theta(\beta) - A_2\zeta(\beta_1) + B_2\theta(\beta_1)] - \frac{F_2(s)}{\cos\vartheta}. \tag{6.34}$$

SECTION 7. INTERNAL FORCES, MOMENTS, STRESSES AND DISPLACEMENTS

Inserting the value of the homogeneous part of V from (6.24) into homogeneous formulas (2.1) and adding the corresponding particular solutions (5.15) and (5.16), we obtain for internal forces T_1 and T_2

$$\begin{aligned} T_1 = -AD_{11}[C_3\Omega_1 + C_4\Omega_2 + 2C_1\Omega_3 + 2C_2\Omega_4]\frac{\sin\vartheta}{r} + \\ + \frac{1}{R_2\cos^2\vartheta}\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right); \end{aligned} \tag{7.1}$$

$$\begin{aligned} T_2 = D_{11}\sqrt{\frac{A^3}{2R_2}}[C_4\Omega_1 + 2C_1\Omega_2 + 2C_2\Omega_3 - 4C_3\Omega_4] - \\ - \frac{1}{R_1\cos^2\vartheta}\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right) + R_2Z. \end{aligned} \tag{7.2}$$

Inserting the value of V from (6.24) into (2.2), we obtain for the lateral force

$$N = AD_{11}[C_3\Omega_1 + C_4\Omega_2 + 2C_1\Omega_3 + 2C_2\Omega_4]\frac{\cos\vartheta}{r}. \tag{7.3}$$

Inserting the value of W from (6.23) into (3.14) and (3.15), and disregarding the terms of the order of $\sqrt{\frac{h}{R_i}}$, we obtain for the moments

$$M_1 = D_{11}\sqrt{\frac{A}{2R_2}}\,[-C_2\Omega + 2C_3\Omega_2 + 2C_4\Omega_3 - 4C_1\Omega_4]; \tag{7.4}$$

$$\begin{aligned} M_2 = D_{12}\sqrt{\frac{A}{2R_2}}\,&[-C_2\Omega_1 + 2C_3\Omega_2 + 2C_4\Omega_3 + 4C_1\Omega_4] + \\ &+ D_{22}[C_1\Omega_1 + C_2\Omega_2 - 2C_3\Omega_3 - 2C_4\Omega_4]\frac{\sin\theta}{r}. \end{aligned} \tag{7.5}$$

From (3.8) and (3.9), for the normal stresses we have

$$\sigma_s^i = -\frac{\Delta_{11}^i}{\Omega}\frac{dV}{ds} - \frac{\Delta_{12}^i}{\Omega}\left(V\frac{\sin\theta}{r} - \frac{F_1}{r}\right) - \gamma B_{11}^i\frac{dW}{ds}; \tag{7.6}$$

$$\sigma_\varphi^i = \frac{\Delta_{21}^i}{\Omega}\frac{dV}{ds} + \frac{\Delta_{22}^i}{\Omega}\left(V\frac{\sin\theta}{r} - \frac{F_1}{r}\right) - \gamma\left(B_{12}^i\frac{dW}{ds} - B_{22}^i\frac{\sin\theta}{r}W\right). \tag{7.7}$$

Considering (1.16), (1.19), (3.6) and (3.7), from (3.10) and (3.11), with the accuracy of calculations adopted here, we obtain for the shearing stresses in each layer of the shell:

for the outer layers $(i = m+1,\ m+2,\ \ldots,\ 2m+1)$

$$\begin{aligned} t^i = X^+ + \frac{\gamma^2}{2}B_{11}^{2m+2-i}\frac{d^2W}{ds^2} - \gamma\frac{C_{11}B_{11}^{2m+2-i}}{\Omega}&\times \\ \times\left(\frac{B_{12}^{2m+2-i}}{B_{11}^{2m+2-i}} - \frac{C_{12}}{C_{11}}\right)\frac{d^2V}{ds^2} - \frac{h_0^2}{2}P_{11}^{2m+2-i}\frac{d^2W}{ds^2} &+ \\ + h_0\frac{C_{11}R_{11}^{2m+2-i}}{\Omega}\left(\frac{R_{12}^{2m+2-i}}{R_{11}^{2m+2-i}} - \frac{C_{12}}{C_{11}}\right)\frac{d^2V}{ds^2}&; \end{aligned} \tag{7.8}$$

for the internal layer $(i = 1,\ 2,\ 3,\ \ldots,\ m+1)$

$$\begin{aligned} t^i = X^- + \frac{\gamma^2}{2}B_{11}^i\frac{d^2W}{ds^2} - \gamma\frac{C_{11}B_{11}^i}{\Omega}\left(\frac{B_{12}^i}{B_{11}^i} - \frac{C_{12}}{C_{11}}\right)\frac{d^2V}{ds^2} &- \\ - \frac{h_0^2}{2}P_{11}^i\frac{d^2W}{ds^2} - h_0\frac{C_{11}R_{11}^i}{\Omega}\left(\frac{R_{12}^i}{R_{11}^i} - \frac{C_{12}}{C_{11}}\right)\frac{d^2V}{ds^2}&. \end{aligned} \tag{7.9}$$

Inserting the values of ε_1 and ε_2 from (3.6) and (3.7) into formulas (2.26), we obtain for the two components of displacement

$$e_r = \frac{C_{11}}{\Omega}r\frac{dV}{ds} + \frac{C_{12}}{\Omega}V\sin\theta - \frac{C_{12}}{\Omega}F_1(s); \tag{7.10}$$

$$e_z = e_z^0 + \int_{s_0}^{s}\left[W\sin\theta - \frac{C_{12}}{\Omega}\cos\theta\frac{dV}{ds} - \frac{C_{22}}{\Omega}\left(V\frac{\sin\theta}{R_2} - \frac{F_1(s)}{R_2}\right)\right]ds. \tag{7.11}$$

The components of displacement may be represented in slightly different form. From (3.4) for the deformations we have

$$\varepsilon_1=\frac{C_{22}T_1-C_{12}T_2}{\Omega};\quad \varepsilon_2=\frac{C_{11}T_2-C_{12}T_1}{\Omega} \tag{7.12}$$

Inserting the values of ε_1 and ε_2 from (7.12) into (2.26), for the displacements we obtain

$$e_r=r\frac{C_{11}T_2-C_{12}T_1}{\Omega}; \tag{7.13}$$

$$e_z=e_z^0+\int\limits_{s_1}^{s}\left(W\sin\vartheta+\frac{C_{22}T_1-C_{12}T_2}{\Omega}\cos\vartheta\right)ds. \tag{7.14}$$

The values of functions V and W in design formulas (7.6)-(7.14) are not represented here due to unwieldiness of the formulas derived thereform. In order to facilitate calculations associated with the indicated design quantities let us present the final expressions for the required derivatives of the sought functions V and W

$$\frac{dV}{ds}=D_{11}\sqrt{\frac{A^3}{2R_2}}(C_4\Omega_1+2C_1\Omega_2+2C_2\Omega_3-4C_3\Omega_4)-$$

$$-\frac{1}{R_1\cos^2\vartheta}\left(\frac{P_z^0}{2\pi}-\int\limits_{s_0}^{s}rE_z\,ds\right)+R_2Z;$$

$$\frac{d^2V}{ds^2}=D_{11}\sqrt{\frac{A^3}{2R_2}}(2C_1\Omega_1+2C_2\Omega_2-4C_3\Omega_3-4C_4\Omega_4);$$

$$\frac{dW}{ds}=\sqrt{\frac{A}{2R_2}}(C_2\Omega_1-2C_3\Omega_2-2C_4\Omega_3-4C_1\Omega_4);$$

$$\frac{d^2W}{ds^2}=-\sqrt{\frac{A}{2R_2}}(2C_3\Omega_1+2C_4\Omega_2+2C_1\Omega_3+4C_2\Omega_4).$$

In deriving these formulas terms of the order of $\sqrt{\frac{h}{R_j}}$ were discarded in comparison with unity.

Proceeding from (6.33) and (6.34), then following the procedure given at the beginning of this section, for the design quantities of the problem we obtain the following formulas:

$$T_1=AD_{11}[A_1\zeta(\beta)-B_1\theta(\beta)+A_2\zeta(\beta_1)-B_2\theta(\beta_1)]\frac{\sin\vartheta}{r}+$$

$$+\frac{1}{R_2\cos^2\vartheta}\left(\frac{P_z^0}{2\pi}-\int\limits_{s_0}^{s}rE_z\,ds\right); \tag{7.15}$$

$$T_2=D_{11}\sqrt{\frac{A^3}{2R_2}}[-A_1\psi(\beta)-B_1\varphi(\beta)+A_2\psi(\beta_1)+B_2\varphi(\beta_1)]-$$

$$-\frac{1}{R_1\cos^2\vartheta}\left(\frac{P_z^0}{2\pi}-\int\limits_{s_0}^{s}rE_z\,ds\right)+R_2Z; \tag{7.16}$$

$$N = AD_{11}[B_1\theta(\beta) - A_1\zeta(\beta) + B_2\theta(\beta_1) - A_2\zeta(\beta_1)]\frac{\cos\vartheta}{r}; \tag{7.17}$$

$$M_1 = D_{11}\sqrt{\frac{A}{2R_2}}[A_1\varphi(\beta) - B_1\psi(\beta) - A_2\varphi(\beta_1) + B_2\psi(\beta_1)]; \tag{7.18}$$

$$M_2 = D_{12}\sqrt{\frac{A}{2R_2}}[A_1\varphi(\beta) - B_1\psi(\beta) - A_2\varphi(\beta_1) + B_2\psi(\beta_1)] +$$

$$+ D_{22}[A_1\theta(\beta) + B_1\zeta(\beta) + A_2\theta(\beta_1) + B_2\zeta(\beta_1)]\frac{\sin\vartheta}{r}. \tag{7.19}$$

The formulas for determining the stresses and displacements (7.6)-(7.14) remain unchanged. As before, let us present the final expressions for the required derivatives of the sought functions V and W:

$$\frac{dV}{ds} = D_{11}\sqrt{\frac{A^3}{2R_2}}[-A_1\psi(\beta) - B_1\varphi(\beta) + A_2\psi(\beta_1) + B_2\varphi(\beta_1)] -$$

$$- \frac{1}{R_1\cos^2\vartheta}\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s} rE_z\,ds\right) + R_2Z;$$

$$\frac{d^2V}{ds^2} = \frac{Q}{C_{11}R_2}[A_1\theta(\beta) + B_1\zeta(\beta) + A_2\theta(\beta_1) + B_2\zeta(\beta_1)];$$

$$\frac{dW}{ds} = -\sqrt{\frac{A}{2R_2}}[A_1\varphi(\beta) - B_1\psi(\beta) - A_2\varphi(\beta_1) + B_2\psi(\beta_1)];$$

$$\frac{d^2W}{ds^2} = \frac{A}{R_2}[A_1\zeta(\beta) - B_1\theta(\beta) + A_2\zeta(\beta_1) - B_2\theta(\beta_1)].$$

In deriving these formulas we have discarded terms of the order of $\sqrt{\frac{h}{R_j}}$ in comparison with unity.

We will note that we cannot expect that from the design formulas derived here all points of the shell will have an accuracy of the first approximation of asymptotic integration. At singular points of equation (6.1) the design formulas may yield unacceptable results. The radii of curvature R_1 and R_2 of the discussed shells must be sufficiently smooth functions of coordinate s.

In the special case of a cylindrical shell, wherein $1/R_1 = 0$, $R_2 = R =$ const and $\vartheta = 0$, the derived design formulas in the first approximation of asymptotic integration are precise, that is, they have the accuracy of the initial equations. This is easily seen by introducing appropriate changes in the course of deriving all of the equations and design formulas of the present chapter.

SECTION 8. EDGE EFFECT IN ANISOTROPIC SHELLS*

If the point of reckoning of coordinate s coincides with one edge of the shell and the length of the shell over the meridian arc is designated as L, for values of β corresponding to the edges of the shell we will have (6.14) and (6.26)

$$\beta = 0,\ \beta = \sqrt{\frac{A}{2}} \int_0^L \frac{ds}{\sqrt{R_2}} = \alpha_0. \tag{8.1}$$

Then, as is easily seen from (6.27), the coordinate of the current cross-section of the shell reckoned from edge s = L (Figure 31) will be expressed in terms of $\beta_1 = \alpha_0 - \beta$.

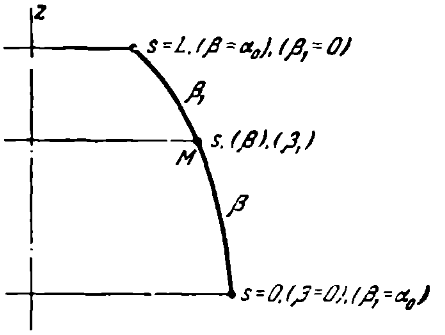

Figure 31

Examining the functions entering into solutions (6.33) and (6.34), it is easily seen that functions $\theta(\beta)$ and $\zeta(\beta)$ decrease with distance from edge s = 0 and functions $\theta(\beta_1)$ and $\zeta(\beta_1)$ decrease with distance from edge s = L.

As is seen from Table 2, these functions are of the order of 0.04 even with $\beta = \pi$. Hence, if the investigated point lies on the periphery of a cross-section of the shell which is sufficiently removed from the edge of the shell that with the given elastic constants $\beta \geq \pi$, the influence of this edge on the stress of points of the investigated periphery or more remote points, within the accuracy of engineering calculations, may be disregarded. This condition may be expressed in slightly different terms: if the elastic constants (C_{11}, C_{22}, C_{12}, D_{11}) of the material of the shell are such that $\beta \geq \pi$ with given s, then the influence of the edge on the stress of the part of the shell corresponding to s may be disregarded.

---

*S.A. Ambartsumyan, Long anisotropic shells, Izvestiya AN ArmSSR (FMYeIT nauki), Vol. IV, No. 6, 1951.

If in expression (6.14) we disregard the variability of R_2, for β we obtain

$$\beta = \sqrt{\frac{A}{2R_1}}\, s. \tag{8.2}$$

Inserting π instead of β, we obtain the length (over the meridian arc) of the zone of extension of the edge effect

$$s^* = \pi \sqrt{\frac{2R_1}{A}}. \tag{8.3}$$

Hence, if the length of shell over a meridian arc $L \geq s^*$, the interaction of the edges may be disregarded. This means that with $L \geq s^*$ the load applied to the edge $\beta = 0$ has practically no effect on the stress of the other edge ($\beta = \alpha_0$) and vice versa.

For homogeneous anisotropic shells, on the basis of (6.2), (2.11), (4.1) and (2.14.9), we have

$$A = \frac{2}{h}\sqrt{3\left(\frac{E_\varphi}{E_s} - \nu_2^2\right)}. \tag{8.4}$$

Inserting the value of A from (8.4) into (8.3), we obtain the length of the zone of edge effect in the case of a homogeneous orthotropic shell of thickness h:

$$s^* = \pi R_2 \sqrt{\frac{h}{R_1}}\,\frac{1}{\sqrt[4]{3\left(\frac{E_\varphi}{E_s} - \nu_2^2\right)}} \tag{8.5}$$

In the special case of an anisotropic shell, considering that $E_\varphi = E_s$ and $\nu_2 = \frac{1}{m}$, for the zone of edge effect we obtain the well-known formula of A.I. Lur'ye*

$$s^* = \pi R_2 \sqrt{\frac{h}{R_1}} \sqrt{\frac{m^2}{3(m^2 - 1)}}. \tag{8.6}$$

Although formulas (8.3)-(8.6) are precise for a circular cylindrical shell, they elsewhere introduce a certain error for reasons of simplicity and in order to obtain general formulas for all types of shells of revolution. However, where necessary, in each special case the quadrature in formula (6.14) may be calculated precisely and we may find the required value of length of the edge effect zone.

---

*A.I. Lur'ye, see p. 61 of work cited in footnote on p. 111.

Comparing formulas (8.3) and (8.6), we note that in distinction from isotropic shells, by appropriate selection of the elastic constants in the case of anisotropic shells we may substantially change the effect of the edge conditions (with a given length of shell) on the overall stress of the shell.

Let us examine two examples.

First example:

$$\frac{E_\varphi}{E_s} = 12, \quad \nu_2 = 0, \quad \frac{h}{R_2} = 0{,}04, \quad \frac{L}{R_2} = 0{,}4;$$

from formula (8.5) we obtain for s*

$$s^* = \frac{3{,}14 \cdot 0{,}2}{\sqrt[4]{36}} R_2 = 0{,}256R_2.$$

Second example:

$$\frac{E_\varphi}{E_s} = 0{,}083, \quad \nu_2 = 0, \quad \frac{h}{R_2} = 0{,}04, \quad \frac{L}{R_2} = 0{,}8;$$

from formula (8.5) we obtain for s*

$$s^* = \frac{3{,}14 \cdot 0{,}2}{\sqrt[4]{0{,}25}} R_2 = 0{,}88R_2.$$

It is seen from these examples that in the first case the length of the edge effect is less than the length of the shell $L = 0.4\ R_2 > 0.256R_2$, while in the second case, conversely, the length of the edge effect is greater than the length of the shell $L = 0.8R_2 < 0.88R_2$. Hence, it is evident that in the first case the edges of the shell will have no effect on one another and interaction may be disregarded, whereas in the second case the edges of the shell will effect one another and there will be no justification for disregarding interaction of the edges.

In shell theory long shells are considered to be those shells for which interaction of the edges may be disregarded and short shells are those for which the interaction of the edges cannot be disregarded. From this standpoint, in discussing anisotropic shells we notice a certain contradiction between the geometric and mechanical concepts of length of a shell.

If from the geometric standpoint an anisotropic shell is considered long (or short), this still does not mean that from the point of view of mechanics it may be regarded as a long (or short) shell; this is also true of the converse condition.

This is easily seen by examination of the examples given above. Despite the fact that from the geometric point of view the length of the shell in the first case is half the length of the shell in the second case, we must consider the shell in the first case to be long and in the second case to be short.

The results derived in the previous sections are wholly acceptable and sufficient for solution of the various problems for long shells as well as for short shells. However, in subsequent discussion we shall deal only with long shells as being most important from the standpoint of application. An understanding of the procedures for calculation of short shells may be obtained from an examination of the work by A.I. Lur'ye.*

SECTION 9. LONG SHELLS OF REVOLUTION**

The anisotropic shells of revolution used in engineering practice are largely long shells (for which the interaction of the edges may be disregarded).

Let us discuss a long shell of revolution along the edges (s = 0, s = L) of which there act the bending moments M_1^0 and M_1^L and the intersecting forces N^0 and N^L (Figure 32). We will note that M_1^0 and M_2^L are positive if they increase the curvature of the coordinate surface of a shell with positive Gaussian curvature or if the stresses caused by them at the outer grain of the shell are tensile forces and on the inner grain of the shell are compressive; the intersecting forces N^0 and N^L are positive if, being applied to areas with positive external normals, they are directed toward the positive normal to the coordinate surface. The positive directions of the forces and moments are shown in Figure 32.

In consideration of the basic property of long shells, we shall discard functions β_1 in calculating quantities referring to edge s = 0 and functions β in calculating quantities referring to edge s = L.

With s = 0 (which, on the basis of (6.14) and (8.1), is equivalent to assuming $\beta = 0$) for the transverse forces and moments we obtain from (7.17) and (7.18) (Figure 33)

$$N = N^0 = AD_{11}\frac{\cos\vartheta_0}{r_0}B_1 = \frac{AD_{11}}{R_2^0}B_1; \qquad (9.1)$$

---

*A.I. Lur'ye, see pp. 68-80 of work cited in footnote on p. 111.

**S.A. Ambartsumyan, see footnote on p. 136.

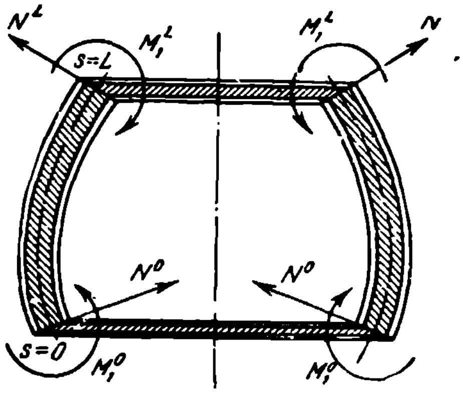

Figure 32

$$M_1 = M_1^0 = D_{11}\sqrt{\frac{A}{2R_2^0}}(A_1 - B_1). \tag{9.2}$$

With s = L (which is equivalent to $\beta_1 = 0$) for the transverse forces and moments we obtain from (7.17) and (7.18)

$$N = N^L = \frac{AD_{11}}{R_2^L}B_2; \tag{9.3}$$

$$M_1 = M_1^L = D_{11}\sqrt{\frac{A}{2R_2^L}}(B_2 - A_2). \tag{9.4}$$

By simultaneous solution of (9.1), (9.2) and (9.3), (9.4) for constants A_1, B_1, A_2 and B_2 we obtain new values which are represented by means of the initial parameters

$$B_1 = \frac{R_2^0}{AD_{11}}N^0; \quad A_1 = \frac{R_2^0}{D_{11}A}N^0 + \frac{1}{D_{11}}\sqrt{\frac{2R_2^0}{A}}M_1^0; \tag{9.5}$$

$$B_2 = \frac{R_2^L}{AD_{11}}N^L; \quad A_2 = \frac{R_2^L}{D_{11}A}N^L - \frac{1}{D_{11}}\sqrt{\frac{2R_2^L}{A}}M_1^L. \tag{9.6}$$

Inserting the values of A_1 . . . B_2 into (6.33), (6.34) and (7.15)-(7.19), for the principle required functions and internal forces we obtain the following formulas:

$$W = \frac{1}{AD_{11}}\left[R_2^0 N^0 \varphi(\beta) + \sqrt{2R_2^0 A}M_1^0 \theta(\beta) + \right.$$
$$\left. + R_2^L N^L \varphi(\beta_1) - \sqrt{2R_2^L A}M_1^L \theta(\beta_1)\right]; \tag{9.7}$$

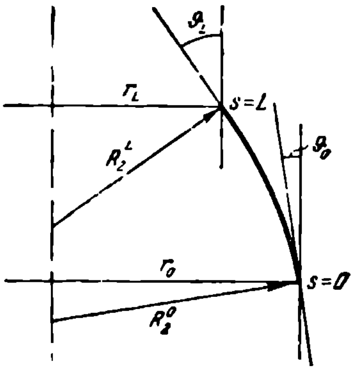

Figure 33

$$V=\left[R_2^0N^0\psi(\beta)-\sqrt{2R_2^0A}M_1^{0\zeta}(\beta)+\right.$$
$$\left.+R_2^LN^L\psi(\beta_1)+\sqrt{2R_2^LA}M_1^{L\zeta}(\beta_1)\right]-\frac{F_2(s)}{\cos\vartheta}; \tag{9.8}$$

$$T_1=\left[-R_2^0N^0\psi(\beta)+\sqrt{2R_2^0A}M_1^{0\zeta}(\beta)-\right.$$
$$\left.-R_2^LN^L\psi(\beta_1)-\sqrt{2R_2^LA}M_1^{0\zeta}(\beta_1)\right]\frac{\sin\vartheta}{r}+$$
$$+\frac{1}{R_2\cos^2\vartheta}\left(\frac{P_z^0}{2\pi}-\int_{s_L}^{s}rE_z\,ds\right); \tag{9.9}$$

$$T_2=\sqrt{\frac{A}{2R_2}}\left[-2R_2^0N^0\theta(\beta)-\sqrt{2R_2^0A}M_1^0\psi(\beta)-\right.$$
$$\left.-2R_2^LN^L\theta(\beta_1)-\sqrt{2R_2^LA}M_1^L\psi(\beta_1)\right]+R_2Z-$$
$$-\frac{1}{R_1\cos^2\vartheta}\left(\frac{P_z^0}{2\pi}-\int_{s_0}^{s}rE_z\,ds\right); \tag{9.10}$$

$$N=\left[R_2^0N^0\psi(\beta)-\sqrt{2R_2^0A}M_1^{0\zeta}(\beta)+\right.$$
$$\left.+R_2^LN^L\psi(\beta_1)+\sqrt{2R_2^LA}M_1^{L\zeta}(\beta_1)\right]\frac{\cos\vartheta}{r}; \tag{9.11}$$

$$M_1=2R_2^0\sqrt{\frac{1}{2R_2A}}N^0\zeta(\beta)+\sqrt{\frac{R_2^0}{R_2}}M_1^0\varphi(\beta)-$$
$$-2R_2^L\sqrt{\frac{1}{2R_2A}}N^L\zeta(\beta_1)+\sqrt{\frac{R_2^L}{R_2}}M_1^L\varphi(\beta_1); \tag{9.12}$$

$$M_2=\frac{D_{12}}{D_{11}}M_1+\frac{D_{22}}{D_{11}A}\frac{\sin\vartheta}{r}\left[R_2^0N^0\varphi(\beta)+\right.$$
$$\left.+\sqrt{2R_2^0A}M_1^0\theta(\beta)+R_2^LN^L\varphi(\beta_1)-\sqrt{2R_2^LA}M_1^L\theta(\beta_1)\right]. \tag{9.13}$$

The formulas for determining the stresses and displacements (7.6)-(7.14) remain unchanged. In determining the design stresses and displacements we shall also be interested in the following quantities:

$$\begin{aligned}\frac{dV}{ds} = \sqrt{\frac{A}{2R_2}}\Big[&-2R_2^0 N^0 \theta(\beta) - \sqrt{2R_2^0 A}\, M_1^0 \psi(\beta) - \\ &- 2R_2^L N^L \theta(\beta_1) - \sqrt{2R_2^L A}\, M_1^l \psi(\beta_1)\Big] + R_2 Z - \\ &- \frac{1}{R_1 \cos^2 \vartheta}\left(\frac{P_z^0}{2\pi} - \int_{s_0}^{s} r E_z\, ds\right),\end{aligned} \tag{9.14}$$

$$\begin{aligned}\frac{dW}{ds} = &-2\frac{R_2^0}{D_{11}} \sqrt{\frac{1}{2R_2 A}}\, N^0 \zeta(\beta) - \frac{1}{D_{11}}\sqrt{\frac{R_2^0}{R_2}}\, M_1^0 \varphi(\beta) + \\ &+ 2\frac{R_2^L}{D_{11}}\sqrt{\frac{1}{2R_2 A}}\, N^L \zeta(\beta_1) - \frac{1}{D_{11}}\sqrt{\frac{R_2^l}{R_2}}\, M_1^L \varphi(\beta_1).\end{aligned} \tag{9.15}$$

As in the general case, the boundary conditions do not differ from those for an isotropic shell of revolution.* Let us present the three basic variants of boundary conditions (see Chapter II, Section 10):

edge $\beta = 0$ is free (2.10.1)

$$N^0 = 0, \quad M_1^0 = 0; \tag{9.16}$$

edge $\beta = 0$ is hinged (2.10.3)

$$M_1^0 = 0, \quad w = 0; \tag{9.17}$$

edge $\beta = 0$ is fixed (2.10.6)

$$R_2^0 N^0 + \sqrt{2R_2^0 A}\, M_1^0 = 0, \quad w = 0. \tag{9.18}$$

The corresponding boundary conditions for edge $\beta_1 = 0$ have the form

$$N^L = 0, \quad M_1^L = 0; \tag{9.19}$$

$$M_1^L = 0, \quad w = 0; \tag{9.20}$$

$$R_2^L N^L - \sqrt{2R_2^L A}\, M_1^L = 0, \quad w = 0. \tag{9.21}$$

For the normal displacement w, by virtue of (2.27), (7.13), and (7.14), we have

---

*A.I. Lur'ye, see p. 64 of work cited in footnote on p. 111.

$$w = \frac{C_{11}T_2 - C_{12}T_1}{\Omega} r\cos\vartheta +$$

$$+ \left\{ e_z^0 + \int\limits_{s_1}^{s} \left(W \sin\vartheta + \frac{C_{22}T_1 - C_{12}T_2}{\Omega} \cos\vartheta \right) ds \right\} \sin\vartheta. \qquad (9.22)$$

SECTION 10. EXAMPLES OF CALCULATION OF LONG SHELLS OF REVOLUTION

In all the examples we shall limit ourselves to only the first approximation of the asymptotic integration.

1. A long cylindrical shell possesses two nondeformable end plates which may be displaced in the axial direction. The edges of the shell are fixed in the end plates. The shell is loaded by a uniformly distributed normal load of intensity q. The radius of curvature of the shell is R. Using (2.5)-(2.7), for the load terms we have

$$\left. \begin{aligned} &E_r = q, \quad E_z = 0, \quad P_z^0 = \pi R^2 q, \\ &F_1 = \frac{\pi R^2}{2\pi} q = \frac{R^2}{2} q, \quad F_2 = -qRs, \end{aligned} \right\} \qquad (10.1)$$

since for a cylindrical shell $\vartheta = 0$ and $r = R$; for the lower limit of integration we take $s_0 = 0$.

From (9.9) and (9.10), considering that the edges of the sheel are fixed in the end plates (that is, with consideration of (9.18) and (9.21)) and that for a cylindrical shell $\sin\vartheta = 0$, for the internal tangential forces we obtain

$$T_1 = \frac{P_z^0}{2\pi R} = \frac{R}{2} q; \qquad (10.2)$$

$$T_2 = -\sqrt{\frac{A}{2R}} R [N^0\varphi(\beta) - N^L\varphi(\beta_1)] + Rq. \qquad (10.3)$$

Then from (9.22) we have

$$w = R\frac{C_{11}T_2 - C_{12}T_1}{\Omega} = \frac{R^2 q (C_{11} - 0{,}5C_{12})}{\Omega} -$$

$$- \frac{R^2 C_{11}}{\Omega} \sqrt{\frac{A}{2R}} [N^0\varphi(\beta) - N^L\varphi(\beta_1)]. \qquad (10.4)$$

Assuming a shell length L, for the values of β corresponding to the edges of the shell we obtain from (8.1)

$$\beta = 0, \quad \beta = \sqrt{\frac{A}{2R}} L = \alpha_0.$$

Since with these values of β the normal displacement ω becomes zero, we obtain for the initial parameters N^0 and N^L

$$N^0 = -\frac{C_{11}-0{,}5C_{12}}{C_{11}}\sqrt{\frac{2R}{A}}\,q, \qquad N^l = -\frac{C_{11}-0{,}5C_{12}}{C_{11}}\sqrt{\frac{2R}{A}}\,q. \tag{10.5}$$

Further, from (9.18) and (9.19), for the initial parameters M_1^0 and M_1^L we have

$$M_1^0 = -\frac{Rq}{A}\,\frac{C_{11}-0{,}5C_{12}}{C_{11}}, \tag{10.6}$$

$$M_1^L = -\frac{Rq}{A}\,\frac{C_{11}-0{,}5C_{12}}{C_{11}}. \tag{10.7}$$

Using the values of N and M_1 in (10.5) and (10.6), from (9.7) and (9.8) we obtain for the principal required functions

$$W = \frac{Rq}{AD_{11}}\sqrt{\frac{2R}{A}}\,\frac{C_{11}-0{,}5C_{12}}{C_{11}}\,[\zeta(\beta)-\zeta(\beta_1)]; \tag{10.8}$$

$$V = qR\sqrt{\frac{2R}{A}}\,\frac{C_{11}-0{,}5C_{12}}{C_{11}}\,[\theta(\beta)-\theta(\beta_1)]+Rqs. \tag{10.9}$$

Inserting the values of W and V from (10.8) and (10.9) into (7.6) and (7.7), we obtain for the normal stresses

$$\sigma_s^i = Rq\,\frac{C_{11}-0{,}5C_{12}}{C_{11}}\left\{\frac{\Delta_{11}^i}{\Omega}[\varphi(\beta)+\varphi(\beta_1)] - \frac{\gamma B_{11}^i}{AD_{11}}[\psi(\beta)+\psi(\beta_1)]\right\} - \frac{Rq}{2}\left(\frac{2\Delta_{11}^i-\Delta_{12}^i}{\Omega}\right); \tag{10.10}$$

$$\sigma_\varphi^i = -Rq\,\frac{C_{11}-0{,}5C_{12}}{C_{11}}\left\{\frac{\Delta_{21}^i}{\Omega}[\varphi(\beta)+\varphi(\beta_1)] + \frac{\gamma B_{12}^i}{AD_{11}}[\psi(\beta)+\psi(\beta_1)]\right\} - \frac{Rq}{2}\left(\frac{\Delta_{22}^i-2\Delta_{21}^i}{\Omega}\right). \tag{10.11}$$

From (9.22) we finally obtain for the normal displacement

$$w = \frac{R^2q}{\Omega}(C_{11}-0{,}5C_{12})\{1-[\varphi(\beta)+\varphi(\beta_1)]\}. \tag{10.12}$$

Let us discuss an example of a three-layer shell. Let the outer layers by isotropic ($E_s^{[1]} = E_\varphi^{[1]} = E_s^{[3]} = E_\varphi^{[3]} = E$), while the inner layer is orthotropic ($E_s^{[2]}$, $E_\varphi^{[2]}$). For simplicity we shall assume that the Poisson ratios of all layers are equal to zero. We shall consider that $h_1 = h_3 = h_0$, $h_2 = kh_0$ (Figures 15, 27, 34). From (3.1) and (3.2) we obtain for the stiffnesses

$$C_{11} = 2Eh_0(1 - k + m_s k); \qquad C_{22} = 2Eh_0(1 - k + m_\varphi k);,$$
$$D_{11} = \frac{2}{3} Eh_0^3(1 - k^3 + m_s k^3),$$

with the following notation:

$$m_s = \frac{E_s^{[2]}}{E}; \qquad m_\varphi = \frac{E_\varphi^{[2]}}{E}.$$

For the stresses in each layer we obtain

$$\sigma_s^i = RqE_s^i\left\{\frac{1}{2C_{11}} - \frac{\gamma}{AD_{11}}[\psi(\beta) + \psi(\beta_1)]\right\};$$
$$\sigma_\varphi^i = Rq\frac{E_\varphi^i}{C_{22}}\{1 - [\varphi(\beta) + \varphi(\beta_1)]\}.$$

Comparing these formulas with the corresponding formulas in the membrane theory, it is easily seen that changes in the normal stresses, associated with torsional phenomena (the second terms in the formulas), are local in nature and at a certain distance from the edges these changes may be disregarded. In view of this it will be of interest to examine the stresses at the edge of the shell, where in addition to stresses of the membrane state we have greater stresses from bending.

Considering that with $\beta = 0$, $\psi(\beta) = \varphi(\beta) = 1.0$, $\psi(\beta_1) \approx \varphi(\beta_1) \approx 0$, we obtain for σ_s^i

$$\sigma_s^i = \frac{qR}{4h_0}\frac{E_s^i}{E}\left[\frac{1}{(1 - k + m_s k)} - \frac{\gamma}{h_0}\frac{2\sqrt{3}}{\sqrt{(1 - k^3 + m_s k^3)(1 - k + m_\varphi k)}}\right].$$

With $\beta = 0$ stresses σ_φ^i are equal to zero.

For the sake of clarity we shall discuss a numerical example. Let all the layers of the shell be isotropic and for the ratios of the elastic moduli of the layers we have $m_s = m_\varphi = 0.1$. Let us further assume that $k = 0.8$.

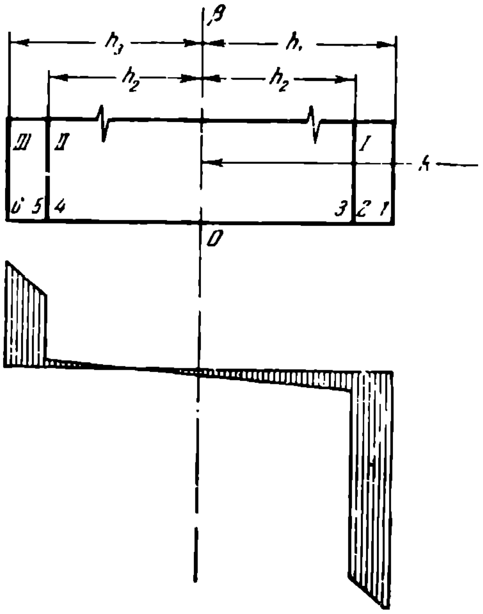

Figure 34

Representing the normal stress σ_s^i in the form

$$\sigma_s^i = \lambda_i \frac{Rq}{4h_0},$$

for individual points of the end section of the shell (Figure 34) and for λ_i we obtain the following values:

Layer	I		II		III	
Points	1	2	3	4	5	6
λ_i	12.45	10.67	1.07	- 0.35	- 3.53	- 5.31

The results of these calculations are represented graphically in Figure 34.

2. An infinitely long cylindrical shell is loaded by forces and moments uniformly distributed over the periphery of the cross-section. The load intensity is q kg/cm and the intensity of moment m kg (Figure 35).

Let us first examine the case wherein the shell is loaded only by uniform transversely peripheral distributed forces q. Let the origin for the reckoning of coordinates s and β coincide with the plane of the loaded cross-section.

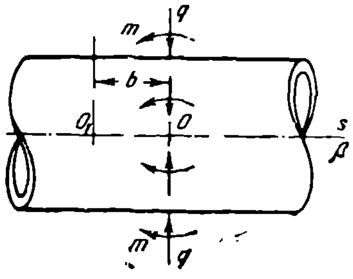

Figure 35.

Let us examine the part of the shell to the right of the loaded section. Then, with $\beta = 0$, for the right portion of the shell we have

$$W = 0, \quad N^0 = \frac{1}{2} q; \tag{10.13}$$

further, from (9.19), we obtain

$$M_1^0 = -\frac{Rq}{2\sqrt{2RA}}. \tag{10.14}$$

Inserting the values of N^0 and M_1^0 into formulas (9.7)-(9.13), for the design quantities we obtain $(0 \leqslant \beta \leqslant \infty)$

$$\left.\begin{aligned} &V = \frac{Rq}{2}\theta(\beta); \quad W = \frac{Rq}{2AD_{11}}\zeta(\beta); \\ &M_1 = -\frac{Rq}{2\sqrt{2RA}}\psi(\beta); \quad N = \frac{q}{2}\theta(\beta); \\ &T_2 = -\frac{Rq}{2}\sqrt{\frac{A}{2R}}\varphi(\beta); \quad w = -\frac{R^2q}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\varphi(\beta). \end{aligned}\right\} \tag{10.15}$$

Similarly, for the left portion of the shell we obtain $(0 \geqslant \beta \geqslant -\infty)$

$$\left.\begin{aligned} &V = -\frac{Rq}{2}\theta_1(\beta); \quad W = \frac{Rq}{2AD_{11}}\zeta_1(\beta); \\ &M_1 = -\frac{Rq}{2\sqrt{2RA}}\psi_1(\beta); \quad N = -\frac{q}{2}\theta_1(\beta); \\ &T_2 = -\frac{Rq}{2}\sqrt{\frac{A}{2R}}\varphi_1(\beta); \quad w = -\frac{R^2q}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\varphi_1(\beta). \end{aligned}\right\} \tag{10.16}$$

where we introduce the notation:

$$\left.\begin{aligned} &\theta_1(\beta) = e^{\beta}\cos\beta; \quad \zeta_1(\beta) = e^{\beta}\sin\beta; \\ &\varphi_1(\beta) = \theta_1(\beta) - \zeta_1(\beta); \quad \psi_1(\beta) = \theta_1(\beta) + \zeta_1(\beta). \end{aligned}\right\} \tag{10.17}$$

Among these functions we have the following relationships:

$$\left.\begin{aligned}&\frac{d}{d\beta}\theta_1(\beta)=\varphi_1(\beta), && \frac{d}{d\beta}\zeta_1(\beta)=\psi_1(\beta),\\ &\frac{d^2}{d\beta^2}\theta_1(\beta)=-2\zeta_1(\beta), && \frac{d^2}{d\beta^2}\zeta_1(\beta)=2\theta_1(\beta).\end{aligned}\right\} \qquad (10.18)$$

In the case where the shell is loaded only by moments with uniform distribution over the transverse periphery, all the design quantities of the problem are derived in elementary fashion on the basis of the solutions obtained in the first case (that is, when the shell is loaded only by uniformly distributed forces q).

Assuming that the shell is loaded by two systems of uniformly distributed annular forces (Figure 36) (each of intensity q) which act in opposite directions and as η approaches zero form a distributed pair with intensity $m=q\cdot\eta$, from (10.15) we obtain* for the right portion of the shell

$$V=\frac{Rq}{2}\left\{\theta\left[\sqrt{\frac{A}{2R}}\,s\right]-\theta\left[\sqrt{\frac{A}{2R}}\,(s+\eta)\right]\right\}=$$

$$=-\frac{Rm}{2}\,\frac{\theta\left[\sqrt{\frac{A}{2R}}\,s\right]-\theta\left[\sqrt{\frac{A}{2R}}\,(s+\eta)\right]}{\eta}=-\frac{Rm}{2}\,\frac{d}{ds}\,\theta\left[\sqrt{\frac{A}{2R}}\,s\right];$$

$$W=\frac{Rq}{2AD_{11}}\left\{\zeta\left[\sqrt{\frac{A}{2R}}\,s\right]-\zeta\left[\sqrt{\frac{A}{2R}}\,(s+\eta)\right]\right\}=$$

$$=\frac{Rm}{2AD_{11}}\,\frac{\zeta\left[\sqrt{\frac{A}{2R}}\,s\right]-\zeta\left[\sqrt{\frac{A}{2R}}\,(s+\eta)\right]}{\eta}=\frac{Rm}{2AD_{11}}\,\frac{d}{ds}\,\zeta\left[\sqrt{\frac{A}{2R}}\,s\right],$$

or, finally,

$$V=-\frac{Rm}{2}\sqrt{\frac{A}{2R}}\,\varphi(\beta),\quad W=\frac{Rm}{2D_{11}\sqrt{2RA}}\,\psi(\beta). \qquad (10.19)$$

Similarly, from (10.15) we obtain for the remaining quantities

$$\left.\begin{aligned}&M_1=\frac{m}{2}\,\theta(\beta); && N=-\frac{m}{2}\sqrt{\frac{A}{2R}}\,\varphi(\beta);\\ &T_2=\frac{m}{2}\,A\zeta(\beta); && w=\frac{Rm}{2}\,\frac{C_{11}}{\Omega}\,A\zeta(\beta).\end{aligned}\right\} \qquad (10.20)$$

---

*S. P. Timeshenko, Soprotivleniye materialov, Vol. II, Gostekhizdat, 1946, p. 17.

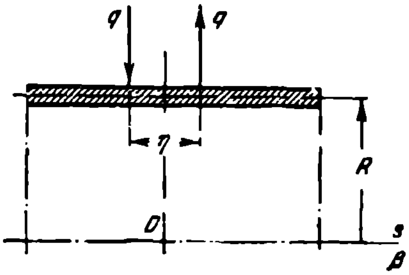

Figure 36.

In like manner, from (10.16) we obtain for the left part of the shell:

$$V=-\frac{Rm}{2}\sqrt{\frac{A}{2R}}\varphi_1(\beta);\quad W=\frac{Rm}{2D_{11}\sqrt{2RA}}\psi_1(\beta);$$

$$\left.\begin{aligned}M_1&=-\frac{m}{2}\theta_1(\beta); & B&=-\frac{m}{2}\sqrt{\frac{A}{2R}}\varphi_1(\beta);\\ T_2&=\frac{m}{2}A\zeta_1(\beta); & w&=\frac{Rm}{2}\frac{C_{11}}{\Omega}A\zeta_1(\beta).\end{aligned}\right\}\tag{10.21}$$

Now let the shell be loaded both by an annular force q and distributed moment m. We also assume that the origin of coordinate s (as of β) lies not in the plane of the loaded cross-section, but to the left by a distance b from the loaded section* (Figure 35).

Assuming that in the coordinate system $(O_1-\beta)$ to the quantity b there corresponds $t=b\sqrt{A/2R}$, for the sought W and V, from (10.15), (10.16), (10.19) and (10.21), we obtain:

in the sector $0\leqslant\beta\leqslant t$

$$\left.\begin{aligned}W&=-\frac{qR}{2AD_{11}}\zeta(t-\beta)+\frac{mR}{2D_{11}\sqrt{2RA}}\psi(t-\beta);\\ V&=-\frac{qR}{2}\theta(t-\beta)-\frac{mR}{2}\sqrt{\frac{A}{2R}}\varphi(t-\beta);\end{aligned}\right\}\tag{10.22}$$

in the sector $t\leqslant\beta\leqslant\infty$

$$\left.\begin{aligned}W&=\frac{qR}{2AD_{11}}\zeta(\beta-t)+\frac{mR}{2D_{11}\sqrt{2RA}}\psi(\beta-t);\\ V&=\frac{qR}{2}\theta(\beta-t)-\frac{mR}{2}\sqrt{\frac{A}{2R}}\varphi(\beta-t).\end{aligned}\right\}\tag{10.23}$$

---

*S. A. Ambartsumyan, Toward calculation of anisotropic cylindrical shells of revolution fastened with transverse ribs, Izvestiya OTN AN SSSR, No. 12, 1955.

Inserting the values of W and V from (10.22) and (10.23) into formulas (2.1), (2.2), (3.14), (3.15) and (7.10) or by direct use of formulas (10.15)-(10.21), we obtain for the design quantities:

in the sector $0 \leqslant \beta \leqslant t$

$$w = -\frac{qR^2}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\,\varphi(t-\beta) - \frac{mR}{2}\frac{C_{11}}{\Omega}A\zeta(t-\beta); \tag{10.24}$$

$$M_1 = -\frac{qR}{2\sqrt{2RA}}\,\psi(t-\beta) - \frac{m}{2}\,\theta(t-\beta); \tag{10.25}$$

$$N = -\frac{q}{2}\,\theta(t-\beta) - \frac{m}{2}\sqrt{\frac{A}{2R}}\,\varphi(t-\beta); \tag{10.26}$$

$$T_2 = -\frac{qR}{2}\sqrt{\frac{A}{2R}}\,\varphi(t-\beta) - \frac{m}{2}A\zeta(t-\beta); \tag{10.27}$$

in the sector $t \leqslant \beta \leqslant \infty$

$$w = -\frac{qR^2}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\,\varphi(\beta-t) + \frac{mR}{2}\frac{C_{11}}{\Omega}A\zeta(\beta-t); \tag{10.28}$$

$$M_1 = -\frac{qR}{2\sqrt{2RA}}\,\psi(\beta-t) + \frac{m}{2}\,\theta(\beta-t); \tag{10.29}$$

$$N = \frac{q}{2}\,\theta(\beta-t) - \frac{m}{2}\sqrt{\frac{A}{2R}}\,\varphi(\beta-t); \tag{10.30}$$

$$T_2 = -\frac{qR}{2}\sqrt{\frac{A}{2R}}\,\varphi(\beta-t) + \frac{m}{2}A\zeta(\beta-t). \tag{10.31}$$

Let us analyze the stress of the shell in the case where the shell is loaded only by a uniformly distributed annular force of intensity q. From (7.6) and (7.7), in virtue of (10.15), (2.24) and (1.12.2), we obtain for the normal stresses in the shell layers:

$$\sigma_s^i = -\frac{E_s^i}{1-\nu_1^i\nu_2^i}\sqrt{\frac{A}{2R}}\left[\frac{C_{11}}{\Omega}\left(\nu_2^i - \frac{C_{12}}{C_{11}}\right)\varphi(\beta) + \frac{\gamma}{AD_{11}}\psi(\beta)\right]\frac{Rq}{2}, \tag{10.32}$$

$$\sigma_\varphi^i = -\frac{E_\varphi^i}{1-\nu_1^i\nu_2^i}\sqrt{\frac{A}{2R}}\left[\frac{C_{11}}{\Omega}\left(1 - \nu_1^i\frac{C_{12}}{C_{11}}\right)\varphi(\beta) + \gamma\frac{\nu_1^i}{AD_{11}}\psi(\beta)\right]\frac{Rq}{2}. \tag{10.33}$$

In the vicinity of the sections where the load q is applied we have for the normal stresses

$$\left.\begin{aligned} \sigma_s^i &= -\frac{E_s^i}{1-\nu_1^i\nu_2^i}\sqrt{\frac{A}{2R}}\left[\frac{C_{11}}{\Omega}\left(\nu_2^i - \frac{C_{12}}{C_{11}}\right) + \frac{\gamma}{AD_{11}}\right]\frac{Rq}{2}; \\ \sigma_\varphi^i &= -\frac{E_\varphi^i}{1-\nu_1^i\nu_2^i}\sqrt{\frac{A}{2R}}\left[\frac{C_{11}}{\Omega}\left(1 - \nu_1^i\frac{C_{12}}{C_{11}}\right) + \frac{\gamma\nu_1^i}{AD_{11}}\right]\frac{Rq}{2}. \end{aligned}\right\} \tag{10.34}$$

Let the orthotropic shell be a single-layer shell and the Poisson ratio of the shell material be zero. Then, for the stresses entering into (10.34), we obtain from (4.1)

$$C_{11}=E_s h,\quad C_{22}=E_\varphi h,\quad D_{11}=\frac{E_s h^3}{12},\quad C_{12}=0,$$

and further, from (6.2), for A we obtain

$$A=\frac{2}{h}\sqrt{\frac{3E_\varphi}{E_s}}.$$

On the basis of the above relationships for the maximum normal stresses σ_s and σ_φ we obtain from (10.34)

$$\sigma_s=-\frac{3q}{2h^2}\sqrt{Rh}\sqrt[4]{\frac{E_s}{3E_\varphi}};$$

$$\sigma_\varphi=-\frac{q}{2h^2}\sqrt{Rh}\sqrt[4]{\frac{3E_\varphi}{E_s}}.$$

By means of these formulas it is easy to establish the nature of the change in normal stresses σ_s and σ_φ in accordance with $e=E_\varphi/E_s$, that is, depending on the ratio of the elastic moduli of the principal directions. Modifying the formulas, we obtain

$$\sigma_s=-\frac{q\sqrt{Rh}}{h^2}\,1{,}14e^{-1/4}=-\frac{q\sqrt{Rh}}{h^2}\lambda_s,$$

$$\sigma_\varphi=-\frac{q\sqrt{Rh}}{h^2}\,0{,}66e^{1/4}=-\frac{q\sqrt{Rh}}{h^2}\lambda_\varphi.$$

The coefficients of these formulas λ_s and λ_φ, are represented in the following form:

e	0,1	0,25	0,5	1,0	2,0	3,0	4,0	8,0	10,0
λ_s	2,03	1,61	1,36	1,14	0,96	0,87	0,81	0,67	0,64
λ_φ	0,37	0,47	0,55	0,66	0,79	0,87	0,93	1,12	1,17

For the sake of convenience the data of this table are represented in graphical form (Figure 37), whence it is easily established that for the discussed problem from the standpoint of strength it is more convenient to have greater stiffnesses in the annular direction of the shell. It is also of interest that with a sufficiently wide range of change in e (from two to five) the normal stresses σ_s and σ_φ have sufficiently close values.

3. A semi-inifinite cylindrical shell is loaded by forces and moments uniformly distributed over the periphery of a loaded cross-section at a distance $b\,(t = b\sqrt{A/2R})$ from the free end of the shell. The origin of coordinates s and β is located in the plane of the cross-section of the free end of the shell (Figure 38).

The solution of this problem may be constructed by the method of superposition. That is to say, on the solution of the problem of an infinitely long cylindrical shell loaded at any intermediate cross-section by forces and moments uniformly distributed over the periphery of the loaded section (Figure 35) we superpose the solution of a semi-infinite cylindrical shell loaded by the following uniformly distributed (over the periphery of the end section) bending moments and intersecting forces (Figure 39):

$$M_1^0 = \frac{qR}{2\sqrt{2RA}}\psi(t) + \frac{m}{2}\theta(t); \tag{10.35}$$

$$N^0 = \frac{q}{2}\theta(t) + \frac{m}{2}\sqrt{\frac{A}{2R}}\varphi(t). \tag{10.36}$$

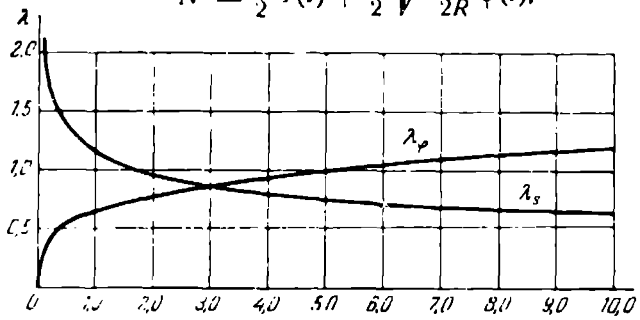

Figure 37.

which with opposite sign are the values (10.25) and (10.26) when

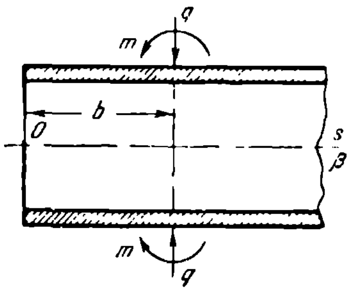

Figure 38.

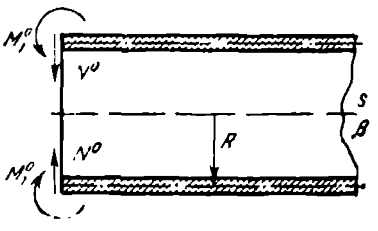

Figure 39.

From (9.7)-(9.13) and (9.22) for the second auxiliary problem (the problem of a semi-infinite shell with end loads) we obtain the following design formulas:

$$W = \frac{1}{AD_{11}}\left[RN^0\varphi(\beta) + \sqrt{2RA}\,M_1^0\theta(\beta)\right]; \tag{10.37}$$

$$V = RN^0\psi(\beta) - \sqrt{2RA}\,M_1^0\zeta(\beta); \tag{10.38}$$

$$w = -R\frac{C_{11}}{\mu}\sqrt{\frac{A}{2R}}\left[2RN^0\theta(\beta) + \sqrt{2RA}\,M_1^0\psi(\beta)\right]; \tag{10.39}$$

$$M_1 = \frac{2R}{\sqrt{2RA}}N^0\zeta(\beta) + M_1^0\varphi(\beta); \tag{10.40}$$

$$N = N^0\psi(\beta) - \frac{\sqrt{2RA}}{R}M_1^0\zeta(\beta); \tag{10.41}$$

$$T_2 = -\sqrt{\frac{A}{2R}}\left[2RN^0\theta(\beta) + \sqrt{2RA}\,M_1^0\zeta(\beta)\right]. \tag{10.42}$$

Considering (10.35) and (10.36), from (10.22)-(10.31) and (10.37)-(10.42) we obtain the following design formulas for the problem under consideration:

over sector

$$\begin{aligned} w = {} & -q\frac{R^2}{2}\frac{C_{11}}{\mu}\sqrt{\frac{A}{2R}}\left[\varphi(t-\beta) + 2\theta(\beta)\theta(t) + \psi(\beta)\psi(t)\right] - \\ & - m\frac{R}{2}\frac{C_{11}}{\mu}A\left[\zeta(t-\beta) + \theta(\beta)\varphi(t) + \psi(\beta)\theta(t)\right]; \end{aligned} \tag{10.43}$$

$$\begin{aligned} W = {} & q\frac{R}{2}\frac{1}{AD_{11}}\left[-\zeta(t-\beta) + \varphi(\beta)\theta(t) + \theta(\beta)\psi(t)\right] + \\ & + m\frac{R}{2D_{11}\sqrt{2RA}}\left[\psi(t-\beta) + \varphi(\beta)\varphi(t) + 2\theta(\beta)\theta(t)\right]; \end{aligned} \tag{10.44}$$

$$\begin{aligned} V = {} & -q\frac{R}{2}\left[\theta(t-\beta) - \psi(\beta)\theta(t) + \zeta(\beta)\psi(t)\right] - \\ & - m\frac{R}{2}\sqrt{\frac{A}{2R}}\left[\varphi(t-\beta) - \psi(\beta)\varphi(t) + 2\zeta(\beta)\theta(t)\right]; \end{aligned} \tag{10.45}$$

$$\begin{aligned} M_1 = {} & q\frac{R}{2\sqrt{2RA}}\left[-\psi(t-\beta) + 2\zeta(\beta)\theta(t) + \varphi(\beta)\psi(t)\right] - \\ & - m\frac{1}{2}\left[\theta(t-\beta) - \zeta(\beta)\varphi(t) - \varphi(\beta)\theta(t)\right]; \end{aligned} \tag{10.46}$$

$$\begin{aligned} N = {} & -\frac{q}{2}\left[\theta(t-\beta) - \psi(\beta)\theta(t) + \zeta(\beta)\psi(t)\right] - \\ & - m\frac{1}{2}\sqrt{\frac{A}{2R}}\left[\varphi(t-\beta) - \psi(\beta)\varphi(t) + 2\zeta(\beta)\theta(t)\right]; \end{aligned} \tag{10.47}$$

$$T_2 = -q\frac{R}{2}\sqrt{\frac{A}{2R}}[\varphi(t-\beta) + 2\theta(\beta)\theta(t) + \psi(\beta)\psi(t)] -$$
$$-m\frac{A}{2}[\zeta(t-\beta) + \theta(\beta)\varphi(t) + \psi(\beta)\theta(t)]; \tag{10.48}$$

over sector $t \leqslant \beta \leqslant \infty$

$$\omega = -q\frac{R^2}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}[\varphi(\beta-t) + 2\theta(\beta)\theta(t) + \psi(\beta)\psi(t)] +$$
$$+m\frac{R}{2}\frac{C_{11}}{\Omega}A[\zeta(\beta-t) - \theta(\beta)\varphi(t) - \psi(\beta)\theta(t)]; \tag{10.49}$$

$$W = q\frac{R}{2}\frac{1}{AD_{11}}[\zeta(\beta-t) + \varphi(\beta)\theta(t) + \theta(\beta)\psi(t)] +$$
$$+m\frac{R}{2D_{11}\sqrt{2RA}}[\psi(\beta-t) + \varphi(\beta)\varphi(t) + 2\theta(\beta)\theta(t)]; \tag{10.50}$$

$$V = q\frac{R}{2}[\theta(\beta-t) + \psi(\beta)\theta(t) - \zeta(\beta)\psi(t)] -$$
$$-m\frac{R}{2}\sqrt{\frac{A}{2R}}[\varphi(\beta-t) - \psi(\beta)\varphi(t) + 2\zeta(\beta)\theta(t)]; \tag{10.51}$$

$$M_1 = q\frac{R}{2\sqrt{2RA}}[-\psi(\beta-t) + 2\zeta(\beta)\theta(t) + \varphi(\beta)\psi(t)] +$$
$$+m\frac{1}{2}[\theta(\beta-t) + \zeta(\beta)\varphi(t) + \varphi(\beta)\theta(t)]; \tag{10.52}$$

$$N = \frac{q}{2}[\theta(\beta-t) + \psi(\beta)\theta(t) - \zeta(\beta)\psi(t)] -$$
$$-m\frac{1}{2}\sqrt{\frac{A}{2R}}[\varphi(\beta-t) - \psi(\beta)\varphi(t) + 2\zeta(\beta)\theta(t)]; \tag{10.53}$$

$$T_2 = -q\frac{R}{2}\sqrt{\frac{A}{2R}}[\varphi(\beta-t) + 2\theta(\beta)\theta(t) + \psi(\beta)\psi(t)] +$$
$$+m\frac{A}{2}[\zeta(\beta-t) - \theta(\beta)\varphi(t) - \psi(\beta)\theta(t)]. \tag{10.54}$$

4. A vertically positioned cylindrical reservoir with the upper edge free is filled to the top with liquid. The lower edge of the reservoir is closed.

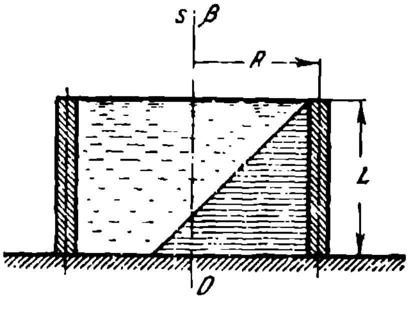

Figure 40.

We shall assume that at the lower edge $s=0$, $(\beta=0)$, and at the upper edge $s=L$, $(\beta=\sqrt{A/2R}\,s=l)$ (Figure 40).

On the basis of (2.5) and (2.6), we have

$$E_r=Z=\rho(L-s), \quad E_z=0,$$

where p is the volumetric weight of the liquid. From (2.3) and (2.4) we obtain

$$F_1=0, \quad F_2=-\rho Rs\left(L-\frac{s}{2}\right).$$

From (5.13), (5.15) and (5.16), for the particular solutions corresponding to the given load we obtain

$$T_1^0=0, \quad T_2^0=\rho R(L-s), \quad V=\rho Rs\left(L-\frac{s}{2}\right). \tag{10.55}$$

For the free upper edge of the shell we have, after (9.19), $M_1^L=0$, and $N^L=0$; then from (9.8) we obtain the general solution for the sought V

$$V=RN^0\psi(\beta)-\sqrt{2RA}\,M_1^0\zeta(\beta)+\rho Rs\left(L-\frac{s}{2}\right). \tag{10.56}$$

For the normal displacement, from (9.22) and on the basis of (9.10), we have

$$\varpi=-R\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\left[2RN^0\theta(\beta)+\sqrt{2RA}\,M_1^0\psi(\beta)\right]+\rho R^2\frac{C_{11}}{\Omega}(L-s); \tag{10.57}$$

whence by virtue of (1.7) we obtain for the second sought W

$$W=\frac{1}{AD_{11}}\left[RN^0\varphi(\beta)+\sqrt{2RA}\,M_1^0\theta(\beta)\right]-\rho R^2\frac{C_{11}}{\Omega}. \tag{10.58}$$

For the lower closed edge, from (9.18),

$$\text{for } \beta=0; \quad W=0, \quad \varpi=0.$$

For initial parameters M_1^0, N^0 we obtain

$$M_1^0=-\frac{L-\sqrt{2R/A}}{A}\rho R; \tag{10.59}$$

$$N^0=\left(\frac{\sqrt{2RA}}{R}L-1\right)\frac{\rho R}{A}. \tag{10.60}$$

Then, we finally obtain

$$V=\rho R\frac{1}{A}\left[R\left(\frac{\sqrt{2RA}}{R}L-1\right)\psi(\beta)+\right.$$
$$\left.+\sqrt{2RA}\left(L-\sqrt{2R/A}\right)\zeta(\beta)\right]+\rho Rs\left(L-\frac{s}{2}\right); \tag{10.61}$$

$$W = \rho R \frac{1}{A^3 D_{11}} \left[R \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \varphi(\beta) - \right.$$
$$\left. - \sqrt{2RA} \left(L - \sqrt{2R/A} \right) \theta(\beta) \right] - \rho R^2 \frac{C_{11}}{\Omega}; \qquad (10.62)$$

$$w = \rho R^2 \frac{C_{11}}{A\Omega} \sqrt{\frac{A}{2R}} \left[-2R \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \theta(\beta) + \right.$$
$$\left. + \sqrt{2RA} \left(L - \sqrt{2R/A} \right) \psi(\beta) \right] + \rho R^2 \frac{C_{11}}{\Omega} (L - s). \qquad (10.63)$$

From (7.6) and (7.7) we obtain the formulas for determining the normal stresses in the shell layers

$$\sigma_s^l = \frac{E_s^l}{1 - \nu_1^l \nu_2^l} \left\{ \frac{C_{11}}{\Omega} \left(\nu_2^l - \frac{C_{12}}{C_{11}} \right) \sqrt{\frac{A}{2R}} \left[-\frac{2R}{A} \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \theta(\beta) + \right. \right.$$
$$\left. + \sqrt{\frac{2R}{A}} \left(L - \sqrt{\frac{2R}{A}} \right) \psi(\beta) \right] + \frac{C_{11}}{\Omega} \left(\nu_2^l - \frac{C_{12}}{C_{11}} \right) (L - s) -$$
$$- \gamma \frac{1}{AD_{11}} \sqrt{\frac{A}{2R}} \left[-\frac{2R}{A} \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \zeta(\beta) + \right.$$
$$\left. \left. + \sqrt{\frac{2R}{A}} \left(L - \sqrt{\frac{2R}{A}} \right) \varphi(\beta) \right] \right\} \rho R; \qquad (10.64)$$

$$\sigma_\varphi^l = \frac{E_\varphi^l}{1 - \nu_1^l \nu_2^l} \left\{ \frac{C_{11}}{\Omega} \left(1 - \nu_1^l \frac{C_{12}}{C_{11}} \right) \sqrt{\frac{A}{2R}} \left[-\frac{2R}{A} \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \theta(\beta) + \right. \right.$$
$$\left. + \sqrt{\frac{2R}{A}} \left(L - \sqrt{\frac{2R}{A}} \right) \psi(\beta) \right] + \frac{C_{11}}{\Omega} \left(1 - \nu_1^l \frac{C_{12}}{C_{11}} \right) (L - s) -$$
$$- \gamma \frac{\nu_1^l}{AD_{11}} \sqrt{\frac{A}{2R}} \left[-\frac{2R}{A} \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \zeta(\beta) + \right.$$
$$\left. \left. + \sqrt{\frac{2R}{A}} \left(L - \sqrt{\frac{2R}{A}} \right) \varphi(\beta) \right] \right\} \rho R. \qquad (10.65)$$

In the case where the Poisson ratios of all the layers are equal to zero the design formulas are simplified; in this case, from (7.8) or (7.9), we obtain for the maximum tangential stress

$$t_{\max}^{m+1} = \frac{h_0^2}{2} \frac{P_{11}^{m+1}}{D_{11}} \left[\frac{R}{A} \left(\frac{\sqrt{2RA}}{R} L - 1 \right) \psi(\beta) + \right.$$
$$\left. + \sqrt{\frac{2R}{A}} \left(L - \sqrt{\frac{2R}{A}} \right) \zeta(\beta) \right] \rho. \qquad (10.66)$$

Examining formula (10.58), it is easily seen that in distinction from the previous examples W contains a component corresponding to the particular solution of inhomogeneous equation (5.9). If we were to retain the premises of Section 5 of the present chapter, for W, instead of (10.58), we would obtain

$$W = \frac{1}{AD_{11}} \left[RN^0 \varphi(\beta) + \sqrt{2RA}\, M_1^0 \theta(\beta) \right].$$

Then from the conditions for the lower closed edge the initial parameters and M_1^0 will be N^0 будут

$$M_1^0 = -\rho R \frac{L}{A}, \quad N^0 = \rho L \sqrt{\frac{2R}{A}}. \qquad (10.67)$$

Comparing (10.67) with (10.59) and (10.60), we note that in the expressions for the support moment (10.59) and the intersecting support force (10.60) components not containing L are associated with the presence of the last term in (10.58), which represents the solution of inhomogeneous equation (5.9). In comparison with the principal components in (10.67), these components are of the order of $1/L$ and are proportional to the thickness of the shell*. By examination of the design formulas in (10.64)-(10.66) it is not difficult to see that at a certain distance from the closed edge the stresses in the shell layers (determined by means of the homogeneous equation) will contain an error of the order of h/R. In considering the local stresses such an approach to the problem may result in an error of the order of $\sqrt{h/R}$ *.

By way of example let us examine the calculation of a 7-layer container (Figure 41). Let R = 500 cm, L = 500 cm, ρ = 0.001 kg/cm^3. The elastic and geometric characteristic of the layers are listed in the table below. It is seen from the table that the central and outer layers are isotropic and the other layers are orthotropic.

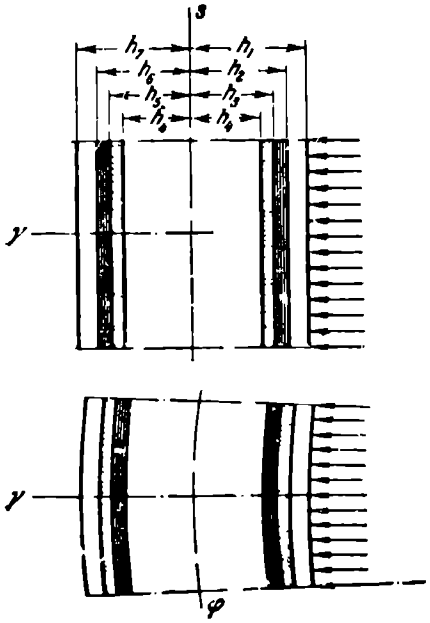

Figure 41.

---

*A. I. Lur'ye, see p. 77 of work cited in footnote on p. 111.

Layers	h_m (cм)	E_s^i	E_φ^i	ν_j^i
I & VII	10,0	E	E	0,0
II & VI	8,0	k_1E	E	0,0
III & V	7,0	E	k_2E	0,0
IV	6,0	E	E	0,0

Assuming that $k_1 = 2{,}08$, $k_2 = 2{,}35$, from (3.1) and (3.2) we obtain for the stiffnesses

$$C_{11} = 22{,}2E \text{ kg/cм}, \quad C_{22} = 22{,}7E \text{ kg/cм},$$
$$C_{12} = 0, \quad D_{11} = 788{,}3E \text{ kg cм}.$$

The stresses in each layer of the shell are determined by the use of formulas (10.64)-(10.66). The results of calculation are listed in the tables.

Table of Normal Stresses σ_s^i (kg/cм²)

s	Layer I		Layer II	Layer III	Layer IV begin.
	Start	End			
0	15,7	12,6	26,1	11,0	9,4
$L/10$	1,0	0,8	1,7	0,7	0,4
$L/5$	−3,6	−2,9	−6,0	−2,5	−2,2
$L/3$	−2,7	−2,2	−4,6	−1,9	−1,6
$L/2$	−0,4	−0,3	−0.7	−0,28	−0,24

The stresses σ_s^i in the remaining layers (VII, VI, V and at the end of layer IV) are correspondingly equal but of opposite sign. Stresses σ_s^i in the second and third layers vary inconsiderably over the thickness of the layer, hence the table lists only one value of stress in each of the mentioned layers.

With a zero Poisson ratio for the thickness of each layer stresses σ_φ^i remain unchanged.

Table of Normal Stresses σ_φ^i (kg/cм²)

s	Layer I	Layer II	Layer III	Layer IV
0	0,0	0,0	0,0	0,0
$L/10$	2,4	2,4	5,6	2,4
$L/5$	5,7	5,7	13,4	5,7
$L/3$	7,2	7,2	16,9	7,2
$L/2$	6,0	6,0	14,1	6,0

Stresses σ_φ^i in the remaining layers (VII, VI, V) will be of corresponding value.

Examining the tables of stresses σ_s^i and σ_φ^i, it is easily seen that in cross-sections $s > L/2$ the normal stresses σ_s^i may be disregarded and calculation of stresses σ_φ^i may be achieved from the membrane theory, that is, from the following formula:

$$\sigma_\varphi^i = E_\varphi^i \frac{pR}{C_{22}} (L - s). \tag{10.68}$$

On the basis of (10.66), considering that in the case under discussion $P_{11}^{(i+1-i)} = 1{,}2E$ kg/cм², for the maximum shearing stress (in cross-section $s = 0$) we obtain $\tau_{\max}^{(IV)} = 2{,}7$ kg/cм².

5. A closed conical shell is loaded by a bending moment of intensity m kg uniformly distributed over the support contour ($s = 0$) (Figures 20 and 42).

For the median surface of a circular conical shell we have, from (3.5.32),

$$\left.\begin{array}{l} R_1 = \infty, \quad r = (s' - s)\sin\alpha, \\ \dfrac{1}{R_2} = \dfrac{\cos\alpha}{r} = \dfrac{\operatorname{ctg}\alpha}{(s' - s)}, \quad \vartheta = \alpha, \end{array}\right\} \tag{10.69}$$

where s' is the length of the generatrix of the shell (that is, the value of s for the vertex of the cone), s is reckoned from the support section of the shell; is the vertex angle.

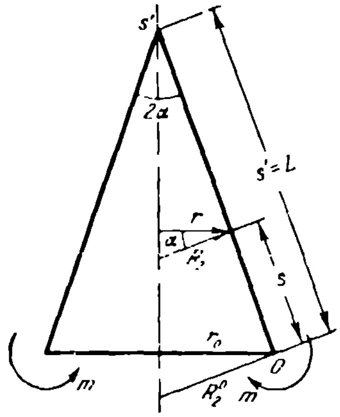

Figure 42.

From (6.14) we obtain for β

$$\beta = -\sqrt{2A\,\mathrm{ctg}\,\alpha}\left(\sqrt{s'-s}-\sqrt{s'}\right).$$

In view of the fact that the surface of the discussed shell is free of load and its vertex is thus not loaded, we have:

$$E_r = 0, \quad E_z = 0, \quad P_z^0 = 0, \quad F_1 = 0, \quad F_2 = 0.$$

For the discussed shell the initial parameters have the following values*, **:

$$M_1^0 = m, \quad N^0 = 0, \quad M_1^L = 0, \quad N^L = 0.$$

Then from (9.7)-(9.13) we obtain for the fundamental design quantities

$$\left.\begin{aligned} W &= \frac{m}{AD_{11}}\sqrt{2As'\,\mathrm{tg}\,\alpha}\,\theta(\beta); \\ V &= -m\sqrt{2As'\,\mathrm{tg}\,\alpha}\,\zeta(\beta); \end{aligned}\right\} \qquad (10.70)$$

$$\left.\begin{aligned} T_1 &= \frac{m}{s'-s}\sqrt{2As'\,\mathrm{tg}\,\alpha}\,\zeta'(\beta); \\ T_2 &= -mA\sqrt{\frac{s'}{s'-s}}\,\psi(\beta); \end{aligned}\right\} \qquad (10.71)$$

$$N = -\frac{m\,\mathrm{ctg}\,\alpha}{s'-s}\sqrt{2As'\,\mathrm{tg}\,\alpha}\,\zeta(\beta); \qquad (10.72)$$

$$\left.\begin{aligned} M_1 &= m\sqrt{\frac{s'}{s'-s}}\,\varphi(\beta); \\ M_2 &= \frac{D_{12}}{D_{11}}M_1 + \frac{m}{s'-s}\,\frac{D_{11}}{D_{22}}\,\frac{\sqrt{2As'\,\mathrm{tg}\,\alpha}}{A}\,\theta(\beta). \end{aligned}\right\} \qquad (10.73)$$

---

*A. I. Lur'ye see pp. 94-107 of work cited in footnote on p. 111.

**V. V. Novozhilov, see p. 265 of work cited in footnote ** on p. 123.

The stresses in the layers and the displacements are determined by means of formulas (7.6)-(7.14).

We shall not discuss other problems of conical shells, since they will be solved in precisely the same manner as the similar problems for cylindrical shells. It is necessary only to consider that for a conical shell relationships (10.69) apply. If the shell is closed (contains a conical vertex), then it is necessary to assume that

$$B_2 = 0, \quad A_2 = 0,$$

or, which is the same thing, to consider that initial parameters M^L & N^L are equal to zero (Figure 42).

We will note also that in the vicinity of the vertex of the shell the proposed theory may not provide reliable values for the design quantities of the problem.

6. A spherical shell is loaded by a uniformly distributed, normally applied load of intensity q kg/cм². The edge of the shell ($s = 0$) is fixed. Let us designate the radius of the sphere as R and introduce into the discussion angle φ, which is reckoned along the meridian from the pole (Figure 43). Let $2\varphi_0$ be the central angle corresponding to the edge of the shell.

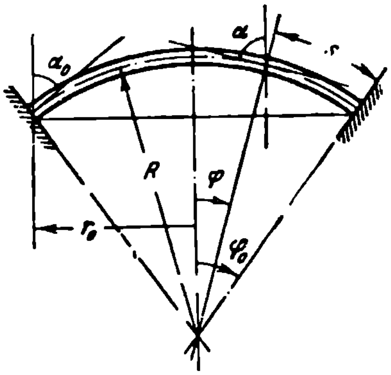

Figure 43.

For the discussed shell we have

$$\left.\begin{aligned} R_1 = R_2 = R, \quad \vartheta = \frac{\pi}{2} - \varphi, \\ r = R\sin\varphi, \quad R\,d\varphi = -ds; \end{aligned}\right\} \qquad (10.74)$$

$$Z = -q, \quad X = 0, \quad E_s = -q\cos\varphi, \quad E_z = -q\sin\varphi.$$

From (5.15) and (5.16) we obtain for T_1^0 and T_2^0

$$\left.\begin{aligned} T_1^0 &= \frac{1}{\sin^2\varphi}\left[\frac{P_z^0}{2\pi R} + R\int_{\varphi_0}^{\varphi} E_z \sin\varphi\, d\varphi\right], \\ T_2^0 &= -\frac{1}{\sin^2\varphi}\left[\frac{P_z^0}{2\pi R} + R\int_{\varphi_0}^{\varphi} E_z \sin\varphi\, d\varphi\right] + RZ. \end{aligned}\right\} \tag{10.75}$$

In the case of a closed shell and in the absence of a concentrated force at the vertex, internal forces T_1^0 and T_2^0, calculated from (10.75), must be finite*. Then for P_z^0 we obtain

$$P_z^0 = 2\pi R^2 \int_0^{\varphi_0} E_z \sin\varphi\, d\varphi = -\pi R^2 q_0 \sin^2\varphi_0. \tag{10.76}$$

From (10.75) for the particular solutions we find

$$T_1^0 = -\frac{qR}{2} \qquad T_2^0 = -\frac{qR}{2}. \tag{10.77}$$

In the case of the homogeneous problem, from (9.9) and (9.10) for the internal forces we have $(N^L = M_1^L = 0)$

$$\left.\begin{aligned} T_1 &= \left[-RN^0\psi(\beta) + \sqrt{2RA}\, M_1^0\zeta(\beta)\right]\frac{\sin\vartheta}{r}; \\ T_2 &= \sqrt{\frac{A}{2R}}\left[-2RN^0\theta(\beta) - \sqrt{2RA}\, M_1^0\psi(\beta)\right]. \end{aligned}\right\} \tag{10.78}$$

Considering that with $s = 0$ $\left(\beta = \sqrt{\frac{A}{2R}}\, s\right)$, $\psi = 1$, $\theta = 1$, $\zeta = 0$, from (7.13) and by virtue of (10.77) and (10.78) for the radial displacement of the shell edge we obtain with accuracy of the order of the first approximation

$$e_r = -\left\{\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\, R\left[\sqrt{2RA}\, M_1^0 + 2RN^0\right] + \frac{C_{11} - C_{12}}{\Omega}\,\frac{R^2 q}{2}\right\}\cos\vartheta_0.$$

On the basis of (9.18), assuming that for the fixed edge

$$C_{11}\sqrt{\frac{A}{2R}}\left[\sqrt{2RA}\, M_1^0 + 2RN^0\right] + (C_{11} - C_{12})\frac{Rq}{2} = 0,$$
$$RN^0 + \sqrt{2RA}\, M_1^0 = 0,$$

---

*A. I. Lur'ye, see p. 110 of work cited in footnote on p. 111.

we obtain for the initial parameters

$$\left.\begin{aligned} M_1^0 &= \frac{C_{11}-C_{12}}{AC_{11}}\,\frac{Rq}{2}; \\ N^0 &= -\frac{C_{11}-C_{12}}{AC_{11}}\sqrt{\frac{2A}{R}}\,\frac{Rq}{2}. \end{aligned}\right\} \tag{10.79}$$

Then, finally, for the principal unknowns we have from (9.7) and (9.8)

$$\left.\begin{aligned} W &= -\sqrt{2RA}\,\frac{C_{11}-C_{12}}{\Omega}\,\frac{Rq}{2}\,\zeta(\beta), \\ V &= -\frac{\sqrt{2RA}}{A}\,\frac{C_{11}-C_{12}}{C_{11}}\,\frac{Rq}{2}\,\theta(\beta) - \frac{F_2(s)}{\cos\vartheta}. \end{aligned}\right\} \tag{10.80}$$

From (9.9)-(9.13), by virtue of (10.77) and (10.79) we obtain for the internal forces and moments

$$\left.\begin{aligned} T_1 &= \sqrt{\frac{2A}{R}}\,\frac{C_{11}-C_{12}}{AC_{11}}\,\frac{Rq}{2}\,\theta(\beta)\operatorname{tg}\vartheta - \frac{Rq}{2}; \\ T_2 &= \frac{C_{11}-C_{12}}{C_{11}}\,\frac{Rq}{2}\,\varphi(\beta) - \frac{Rq}{2}; \end{aligned}\right\} \tag{10.81}$$

$$N = -\sqrt{\frac{2A}{R}}\,\frac{C_{11}-C_{12}}{AC_{11}}\,\frac{Rq}{2}\,\theta(\beta); \tag{10.82}$$

$$\left.\begin{aligned} M_1 &= \frac{C_{11}-C_{12}}{AC_{11}}\,\frac{Rq}{2}\,\psi(\beta); \\ M_2 &= \frac{D_{12}}{D_{11}}M_1 - \frac{D_{22}}{D_{11}}\sqrt{\frac{2A}{R}}\,\frac{C_{11}-C_{12}}{A^2C_{11}}\,\frac{Rq}{2}\,\zeta(\beta)\operatorname{tg}\vartheta. \end{aligned}\right\} \tag{10.83}$$

The stresses within the layers may be determined by means of formulas (7.6) and (7.7). In the special case where the Poisson ratios of all layers are equal to zero, the formulas for the stresses are simplified and take the form

$$\left.\begin{aligned} \sigma_s^l &= -E_s^l\left(\frac{V}{C_{11}}\,\frac{\sin\vartheta}{r} + \gamma\frac{dW}{ds^2}\right) + \frac{E_s^l}{C_{11}}\,\frac{F_1(s)}{r}; \\ \sigma_\varphi^l &= E_\varphi^l\left(\frac{1}{C_{22}}\,\frac{dV}{ds} + \gamma W\,\frac{\sin\vartheta}{r}\right). \end{aligned}\right\} \tag{10.84}$$

Hence, limiting discussion to the stresses of the edge effect, that is, the stresses which arise from M_1^0 and N^0, we obtain with an accuracy of $\sqrt{h/R}$,

$$\bar{\sigma}_s^l = \frac{E_s^l}{C_{11}}A\gamma\,\frac{Rq}{2}\,\psi(\beta), \quad \bar{\sigma}_\varphi^l = \frac{E_\varphi^l}{C_{22}}\,\frac{Rq}{2}\,\varphi(\beta). \tag{10.85}$$

Let it be pointed out that in order to determine the total stresses it is necessary to add to the stresses in (10.85) the nominal stresses (stresses of the membrane state), which with known T_1^0 and T_2^0 (10.77) may be determined by elementary means by use of the formulas in (3.9.7).

Let us discuss a numerical example. Let a three-layer spherical shell $t_i = h/3 = 1{,}0$ см, and, further, let $\varphi_0 = 35°$. $R/h = 30$, $q = 1{,}0$ kg/см². Moreover, for reasons of simplicity we shall assume that the Poisson ratios of all layers are equal to zero. Three different ratios between the elastic moduli of the layers are examined, namely:

I. The elastic modulus of the middle layer is ten times greater than the elastic modulus of the outer layers.

II. The elastic modulus of the middle layer is ten times smaller than the elastic modulus of the outer layers.

III. The elastic moduli of all layers are equal, that is, we have a homogeneous isotropic shell.

The elastic characteristics of the layers and the stiffness of each of the shells are listed in the following table.

Case	Elastic Moduli of Layers			C	D	A
	Layer I	Layer II	Layer III			
I	E	$10E$	E	$12E$	$3E$	2,0
II	$10E$	E	$10E$	[illegible]	$21{,}8E$	0,98
III	E	E	E	$3E$	$2{,}25E$	1,15

The values of stiffnesses $C = C_{11} = C_{22}$, $D = D_{11} = D_{22}$ and coefficients A are calculated, respectively, by means of formulas (3.1), (3.2) and (6.2).

By means of formulas (10.85) it is easy to calculate the edge-effect stresses in all layers of the shell. The table of stresses lists only the maximum values of edge-effect stress. It is easily seen in (10.85) that the edge-effect stresses $\bar{\sigma}_s^i$, and $\bar{\sigma}_\varphi^i$ reach their maximum in section $s = 0$ ($\beta = 0$), where the functions $\psi(\beta) = \varphi(\beta) = 1{,}0$, that is, they have maximum value.

From (3.9.7) and by virtue of (3.9.4), (3.9.5) and (10.77) we have for the nominal stresses

$$[\sigma_s^i] = [\sigma_\varphi^i] = -\frac{E^i}{C}\,\frac{qR}{2}\,. \tag{10.86}$$

Table of stresses $\bar{\sigma}_s^i$ (kg/cм²) и $\bar{\sigma}_\varphi^i$ (kg/cм²) in cross-section.

Stresses	Case	Layer I		Layer II		Layer III	
		Point 1	Point 2	Point 3	Point 4	Point 5	Point 6
$\bar{\sigma}_s^i$	I	—11,25	—3,75	—37,50	37,50	3,75	11,25
	II	—31,50	—10,50	—1,05	1,05	10,50	31,50
	III	—25,98	—8,66	—8,66	8,66	8,66	—25,98
$\bar{\sigma}_\varphi^i$	I	3,75	3,75	37,5	37,5	3,75	3,75
	II	21,4	21,4	2,14	2,14	21,4	21,4
	III	15,0	15,0	15,0	15,0	15,0	15,0

The values of nominal stresses with opposite sign coincide with the values of $\bar{\sigma}_s^i$ in cross-section $s = 0$. The edge-effect stresses $\bar{\sigma}_s^i$ and $\bar{\sigma}_\varphi^i$ in section $s = 0$ are represented in Figure 44.

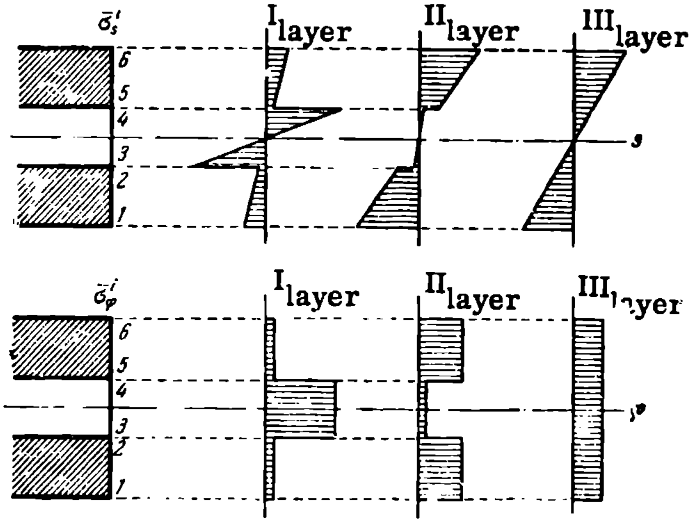

Figure 44.

The problems discussed in this section may also be solved without particular difficulties in the second approximation of asymptotic integration (for a cylindrical shell, as has been stated, this is not necessary). However, the second approximation is of no interest to us here, since, on the one hand, the formulas of the second approximation are extremely unwieldy and, on the other hand, our knowledge of the materials and structural characteristics of shells are such that for purposes of practical calculation the first approximation is adequate*, **.

SECTION 11. SOLUTION OF A FEW PROBLEMS OF SHELLS OF REVOLUTION OF ZERO GAUSSIAN CURVATURE CONSISTING OF AN ARBITRARY NUMBER OF LAYERS***

Here we shall investigate shells consisting of orthotropic layers whose planes of elastic symmetry are mutually parallel and whose principal directions of elasticity coincide with the principal directions of the coordinate surface of the shell (see Section 1 of the present chapter).

Assuming $R_1 = \infty$, from (2.17) and (2.18) we obtain the following system of differential equations of solution.

$$\left.\begin{aligned} &\frac{d^2V}{ds^2} - \frac{\sin\vartheta}{r}\frac{dV}{ds} - \frac{C_{22}}{C_{11}}\frac{\sin^2\vartheta}{r^2}V = \frac{P_1}{C_{11}}\frac{d^2W}{ds^2} + \\ &\quad + \frac{P_2 - P_1}{C_{11}}\frac{\sin\vartheta}{r}\frac{dW}{ds} + \left(\frac{\Omega}{C_{11}}\frac{1}{R_2} - \frac{P_3}{C_{11}}\frac{\sin^2\vartheta}{r^4}\right)W + \Phi_1(s), \\ &\frac{d^2W}{ds^2} - \frac{\sin\vartheta}{r}\frac{dW}{ds} - \frac{D_{22} - D_{22}^0}{D_{11} - D_{11}^0}\frac{\sin^2\vartheta}{r^2}W = \\ &\quad = \frac{P_1}{\Omega\left(D_{11} - D_{11}^0\right)}\frac{d^2V}{ds^2} + \frac{P_2 + P_1}{\Omega\left(D_{11} - D_{11}^0\right)}\frac{\sin\vartheta}{r}\frac{dV}{ds} + \\ &\quad + \left[\frac{P_3}{\Omega\left(D_{11} - D_{11}^0\right)}\frac{\sin^2\vartheta}{r^2} - \frac{1}{D_{11} - D_{11}^0}\frac{1}{R_2}\right]V + \Phi_2(s). \end{aligned}\right\} \tag{11.1}$$

Proceeding as in Section 5 of this chapter, let us assume

$$\frac{D_{22}}{D_{11}} = \frac{K_{22}}{K_{11}} = \frac{C_{22}}{C_{11}} = \lambda; \tag{11.2}$$

---

*V. V. Novozhilov, see pp. 260-261 of work cited in footnote on p. 11

**A. I. Lur'ye, see pp. 77 and 125 of work cited in footnote on p. 111.

***D. V. Peshtmaldzhyan, Toward calculation of symmetrically loaded anisotropic shells of revolution, Izvestiya AN Arm SSR (seriya fiz.-mat. nauk), Vol. X, No. 2, 1957.

then the system of differential equations in (11.1) may be reduced to a single second-order differential equation of solution of the following form:

$$\frac{d^2\sigma}{ds^2}\left(1-\frac{iP_1}{\sqrt{C_{11}\Omega(D_{11}-D_{11}^0)}}\right)-\frac{d\sigma}{ds}\left(1-\frac{iP_1}{\sqrt{C_{11}\Omega(D_{11}-D_{11}^0)}}\right)\frac{\sin\vartheta}{r}-$$
$$-\sigma\left[\left(\lambda-\frac{iP_3}{\sqrt{C_{11}\Omega(D_{11}-D_{11}^0)}}\right)\frac{\sin^2\vartheta}{r^2}+\frac{i\sqrt{\Omega}}{\sqrt{C_{11}(D_{11}-D_{11}^0)}}\frac{\cos\vartheta}{r}\right]=$$
$$=\Phi_2(s)+i\sqrt{\frac{C_{11}}{\Omega(D_{11}-D_{11}^0)}}\,\Phi_1(s). \tag{11.3}$$

where for the sought function, by analogy with (5.10), we have

$$\sigma=W+i\sqrt{\frac{C_{11}}{\Omega(D_{11}-D_{11}^0)}}\,V. \tag{11.4}$$

Thus, the problem of an asymmetrically assembled multi-layer shell (upon fulfilment of condition (11.2)) reduces to the solution of differential equation (11.3). Having the value of the sought σ, from (11.4) it is easy to determine the values of the principal unknown W and V through which all the design quantities of the problem are expressed.

As in the above cited work, let us present a detailed discussion of cases of conical and cylindrical shells. Therein we will examine only the corresponding homogeneous equations, since determination of the particular solutions corresponding to the right members of the inhomogeneous equations presents no particular difficulties (see Section 5 of this chapter).

A. Conical Shell. Considering that for a conical shell we have (10.69), that is,

$$\left.\begin{aligned}&R_1=\infty, \quad r=(s'-s)\sin\alpha,\\ &\frac{1}{R_2}=\frac{\cos\alpha}{r}=\frac{\operatorname{ctg}\alpha}{(s'-s)}, \quad \vartheta=\alpha,\end{aligned}\right\} \tag{11.5}$$

where s' is the length of the generatrix of the shell and 2α is the vertex angle (Figure 42), we obtain from (11.3)

$$\frac{d^2\sigma}{ds^2}-\frac{1}{s'-s}\frac{d\sigma}{ds}-\left[\lambda\frac{1}{(s'-s)^2}+\beta r_0\frac{\operatorname{ctg}\alpha}{s'-s}\right]\sigma=\Psi(s), \tag{11.6}$$

where

$$\beta=-\frac{1}{r_0}\frac{\Omega}{C_{11}\Omega(D_{11}-D_{11}^0)+P_1^2}\left[P_1-i\sqrt{C_{11}\Omega(D_{11}-D_{11}^0)}\right]. \tag{11.7}$$

r_0 is the radius of the base of the cone,

$$\Psi(s) = \frac{1}{C_{11}\Omega(D_{11} - D_{11}^0) + P_1^2}\{C_{11}\Omega(D_{11} - D_{11}^0)\Phi_2(s) - \\ - C_{11}P_1\Phi_1(s) + i\sqrt{C_{11}\Omega(D_{11} - D_{11}^0)}[C_{11}\Phi_1(s) + P_1\Phi_2(s)]\}. \tag{11.8}$$

Following A. I. Lur'ye* and assuming

$$x = \frac{2\sqrt{(s' - s)\operatorname{ctg}\alpha}}{\sqrt{\delta}}, \quad \tau = \sqrt{x}\,\sigma, \tag{11.9}$$

from (11.5) we obtain

$$\frac{d^2\tau}{dx^2} - \left[\left(4\lambda - \frac{1}{4}\right)\frac{1}{x^2} + \delta r_0\beta\right]\tau = 0, \tag{11.10}$$

δ is the total thickness of the shell.

Thus, we obtain one of the forms of the Bessel equation, the general integral of which may be represented in the following form**:

$$\tau = \sqrt{x}\,[C_1 I_{2\sqrt{\lambda}}(i\sqrt{\delta r_0\beta}\,x) + C_2 Y_{2\sqrt{\lambda}}(i\sqrt{\delta r_0\beta}\,x)], \tag{11.11}$$

where I_ν and Y_ν are Bessel functions of the first and second kind, respectively, with index ; C_1, C_2 are complex constants.

Let us limit discussion to shells with small angle α and deal only with points which are sufficiently removed from the vertex of the cone. Then, together with x the argument of the Bessel functions takes large values and for them asymptotic expansions will be valid. In order to obtain the latter let us examine the expansion of Hankel functions in terms of which the Bessel functions are represented as follows:

$$I_\nu = \frac{1}{2}(H_\nu^{(1)} + H_\nu^{(2)}), \quad Y_\nu = \frac{1}{2i}(H_\nu^{(1)} - H_\nu^{(2)});$$

whereupon for the Hankel functions we have the following expansions***:

---

*A. I. Lur'ye see p. 83 of work cited in footnote on p. 111.

**G. N. Watson, A treatise on the theory of Bessel functions, Cambridge, 1944, p. 97.

***G. N. Watson, A treatise on the theory of Bessel functions, Cambridge, 1944, p. 198.

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)} \left[\sum_{m=0}^{p-1} \frac{(-1)^m (\nu, m)}{(2iz)^m} + O(z^{-p})\right];$$

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)} \left[\sum_{m=0}^{p-1} \frac{(\nu, m)}{(2iz)^m} + O(z^{-p})\right],$$

where

$$(\nu, m) = \frac{[4\nu^2 - 1^2][4\nu^2 - 3^2] \ldots [4\nu^2 - (2m-1)^2]}{2^{2m} m!},$$

and the symbol $O(z^{-p})$ designates the function of the order of z^{-p}.

In calculating the Hankel functions, depending on the values of argument with consideration of the required accuracy, we may limit ourselves to a definite number of terms of expansion. Thus, with an accuracy of $1/z$, for the Bessel functions we obtain

$$\left.\begin{aligned} J_\nu(z) &= \sqrt{2/\pi z}\cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right), \\ Y_\nu(z) &= \sqrt{2/\pi z}\sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right). \end{aligned}\right\} \tag{11.12}$$

Then, by virtue of (11.11), (11.9) and (11.4), for the sought functions W and V we obtain

$$\begin{aligned} W = {} & E_1 \frac{1}{\sqrt{x}} \cos\left(-b\sqrt{r_0\delta}\,x - \frac{\pi}{4} - \sqrt{\lambda}\,\pi\right) \operatorname{ch}\left(a\sqrt{r_0\delta}\,x\right) + \\ & + F_1 \frac{1}{\sqrt{x}} \sin\left(-b\sqrt{r_0\delta}\,x - \frac{\pi}{4} - \sqrt{\lambda}\,\pi\right) \operatorname{sh}\left(a\sqrt{r_0\delta}\,x\right) + \\ & + E_2 \frac{1}{\sqrt{x}} \sin\left(-b\sqrt{r_0\delta}\,x - \frac{\pi}{4} - \sqrt{\lambda}\,\pi\right) \operatorname{ch}\left(a\sqrt{r_0\delta}\,x\right) - \\ & - F_2 \frac{1}{\sqrt{x}} \cos\left(-b\sqrt{r_0\delta}\,x - \frac{\pi}{4} - \sqrt{\lambda}\,\pi\right) \operatorname{sh}\left(a\sqrt{r_0\delta}\,x\right) \end{aligned}$$

$$\begin{aligned} V = {} & \sqrt{\frac{2(D_{11} - D_{11}^0)}{C_{11}}} \left\{F_1 \frac{1}{\sqrt{x}} \cos\left(-b\sqrt{r_0\delta}\,x - \frac{\pi}{4} - \sqrt{\lambda}\pi\right) \times \right. \\ & \times \operatorname{ch}\left(a\sqrt{r_0\delta}\,x\right) - E_1 \frac{1}{\sqrt{x}} \sin\left(-b\sqrt{r_0\delta}\,x - \frac{\pi}{4} - \right. \\ & \left. - \sqrt{\lambda}\pi\right) \operatorname{sh}\left(a\sqrt{r_0\delta}\,x\right) + F_2 \frac{1}{\sqrt{x}} \sin\left(-b\sqrt{r_0\delta}\,x - \right. \\ & \left. - \frac{\pi}{4} - \sqrt{\lambda}\pi\right) \operatorname{ch}\left(a\sqrt{r_0\delta}\,x\right) + E_2 \frac{1}{\sqrt{x}} \cos\left(-b\sqrt{r_0\delta}\,x - \right. \\ & \left. \left. - \frac{\pi}{4} - \sqrt{\lambda}\pi\right) \operatorname{sh}\left(a\sqrt{r_0\delta}\,x\right)\right\}. \end{aligned}$$

where

$$\left.\begin{aligned} a &= \sqrt{\frac{\omega\left[\sqrt{C_{11}\omega(D_{11}-D_{11}^0)} + P_1' - P_1\right]}{2r_0\left[C_{11}\omega(D_{11}-D_{11}^0) + P_1'\right]}}; \\ b &= \sqrt{\frac{\omega\left[\sqrt{C_{11}\omega(D_{11}-D_{11}^0)} + P_1' + P_1\right]}{2r_0\left[C_{11}\omega(D_{11}-D_{11}^0) + P_1'\right]}}, \end{aligned}\right\} \qquad (11.13)$$

since it is assumed that

$$\sqrt{\beta} = a + ib, \qquad (11.14)$$

E_j and F_j are real integraion constants which are determined from the boundary conditions.

Having the values of W and V, from formulas (2.1), (2.2), (2.12), (2.13), (2.22), (2.23), (2.26) and (2.27) we may determine all the design quantities of the problem.

Let us discuss the case in which the conical shell consists of isotropic layers. In this case condition (11.2) is identically satisfied. Then in the general case of a conical shell consisting of an arbitrary number of isotropic layers solution (11.11) is represented by means of Bessel functions with an index of 2 $(\lambda = 1)$

$$z = \sqrt{x}\left[C_1 I_2(i\sqrt{\delta r_0\beta x}) + C_2 Y_2(i\sqrt{\delta r_0\beta x})\right]. \qquad (11.15)$$

Bessel functions of index 2 may be represented as Bessel functions of indexes 1 and 0 by means of the following recursion formulas:

$$I_2(z) = \frac{2}{z} I_1(z) - I_0(z), \quad Y_2(z) = \frac{2}{z} Y_1(z) + Y_0(z), \qquad (11.16)$$

while functions $I_1(z)$, $I_0(z)$, $Y_1(z)$, $Y_0(z)$ for complex argument $z = i\sqrt{\delta r_0\beta}x$ have been tabulated in sufficient detail.

With large values of argument we may take expressions (11.12) for the Bessel functions; then for the sought functions W and V we obtain

$$\begin{aligned} W = {} & E_1' \frac{1}{\sqrt{x}} \cos(-b\sqrt{r_0\delta x})\,\mathrm{ch}(a\sqrt{\delta r_0 x}) + \\ & + F_1' \frac{1}{\sqrt{x}} \sin(-b\sqrt{\delta r_0 x})\,\mathrm{sh}(a\sqrt{\delta r_0 x}) + \\ & + E_2' \frac{1}{\sqrt{x}} \sin(-b\sqrt{\delta r_0 x})\,\mathrm{ch}(a\sqrt{\delta r_0 x}) - \\ & \qquad - F_2' \frac{1}{\sqrt{x}} \cos(-b\sqrt{\delta r_0 x})\,\mathrm{sh}(a\sqrt{\delta r_0 x}); \end{aligned} \qquad (11.17)$$

$$V = \sqrt{\frac{2(D_{11}-D_{11}^{0})}{C_{11}}}\left[F_1'\frac{1}{\sqrt{x}}\cos(-b\sqrt{\delta r_0}x)\,\mathrm{ch}\,(a\sqrt{\delta r_0}x) - \right.$$
$$- E_1'\frac{1}{\sqrt{x}}\sin(-b\sqrt{\delta r_0}x)\,\mathrm{sh}\,(a\sqrt{\delta r_0}x) +$$
$$+ F_2'\frac{1}{\sqrt{x}}\sin(-b\sqrt{\delta r_0}x)\,\mathrm{ch}\,(a\sqrt{\delta r_0}x) + \qquad (11.18)$$
$$\left. + E_2'\frac{1}{\sqrt{x}}\cos(-b\sqrt{\delta r_0}x)\,\mathrm{sh}\,(a\sqrt{\delta r_0}x)\right],$$

where E_j', F_j' are new integration constants which are related to the old constants as follows:

$$E_1' = \frac{\sqrt{2}}{2}(E_2 - E_1),\ E_2' = -\frac{\sqrt{2}}{2}(E_1 + E_2);$$
$$F_1' = \frac{\sqrt{2}}{2}(F_2 - F_1),\ F_2' = -\frac{\sqrt{2}}{2}(F_1 + F_2).$$

A similar problem has been examined by E. I. Grigolyuk* in the special case of a two-layer shell; however, it is assumed therein that the Poisson ratios of the material of the layers are identical.

By way of example let us examine the problem of a closed conical shell consisting of isotropic layers and sustaining an edgewise load in the form of a distributed moment with intensity m (kg) and distributed thrust with intensity q (kg/cm) (Figure 45).

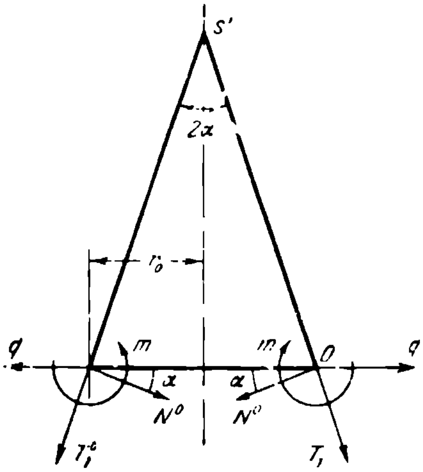

Figure 45.

---

*E. I. Grigolyuk, Equations of axially symmetrical bimetallic elastic shells, Inzh. sbornik, XVIII, 1954.

At the edge $s=0$ we have the conditions

$$M_1^0 = m, \quad N^0 \cos\alpha - T_1^0 \sin\alpha = q. \tag{11.19}$$

By virtue of (11.9) we obtain from (11.15) for the sought function σ

$$\sigma = C_1 I_2(i\sqrt{\delta r_0\beta}\,x) + C_2 Y_2(i\sqrt{\delta r_0\beta}\,x). \tag{11.20}$$

In order for the solution to remain finite over the entire extent of the of the shell, constant C_2 must be taken equal to zero, otherwise solution of (11.20) at the vertex of the shell, that is, with $x=0$, reverts to infinity, since with $x=0$; $Y_2(i\sqrt{\delta r_0\beta}\,x)=\infty$. Then, considering that C_1 is a complex number, from (11.19) we obtain

$$\sigma = (A - iB)\,I_2(i\sqrt{\delta r_0\beta}\,x). \tag{11.21}$$

Representing the Bessel function of index 2 in terms of I_0 and I_1, by means of formula (11.16) and assuming

$$\begin{aligned} z &= i\sqrt{\delta r_0\beta}\,x = \rho e^{i\varphi}, \\ \rho &= \sqrt{\delta r_0 (a^2+b^2)}\,x, \\ \varphi &= \operatorname{arctg}\left(-\frac{a}{b}\right), \\ &\left.\begin{aligned} I_0(\rho e^{i\varphi}) &= u_0(\rho, \varphi) + i v_0(\rho, \varphi), \\ I_1(\rho e^{i\varphi}) &= u_1(\rho, \varphi) + i v_1(\rho, \varphi), \end{aligned}\right\} \end{aligned} \tag{11.22}$$

we finally obtain for the sought functions W and V:

$$\begin{aligned} W = A\left[\frac{2}{\rho} u_1 \cos\varphi + \frac{2}{\rho} v_1 \cos\varphi - u_0\right] + \\ + B\left[\frac{2}{\rho} v_1 \cos\varphi - \frac{2}{\rho} u_1 \sin\varphi - v_0\right]; \end{aligned} \tag{11.23}$$

$$\begin{aligned} V = \sqrt{\frac{\omega(D_{11} - D_{11}^0)}{C_{11}}}\left[A\left(\frac{2}{\rho} v_1 \cos\varphi - \frac{2}{\rho} u_1 \sin\varphi - v_0\right) - \right. \\ \left. - B\left(\frac{2}{\rho} u_1 \cos\varphi + \frac{2}{\rho} v_1 \sin\varphi - u_0\right)\right]. \end{aligned} \tag{11.24}$$

Inserting the values of W and V from (11.23) and (11.24) into (2.12), (2.1) and (2.2), therein considering that $F_1(s)=0$, $F_2(s)=0$ and inserting the resulting values of moment and force into (11.19), for determination of constants A and B we obtain the following system of equations:

$$A\left(\frac{2}{\rho}v_1\cos\varphi-\frac{2}{\rho}u_1\sin\varphi-v_0\right)-B\left(\frac{2}{\rho}u_1\cos\varphi+\frac{2}{\rho}v_1\sin\varphi-u_0\right)=$$
$$=-\sqrt{\frac{C_{11}}{\Omega(D_{11}-D_{11}^0)}}\,qs'\sin\varphi; \tag{11.25}$$

$$A\left\{(D_{11}-D_{11}^0)\frac{d\rho}{ds}\left[\left(1-\frac{4}{\rho^2}\right)u_1\cos\varphi-\left(1+\frac{4}{\rho^2}\right)v_1\sin\varphi+\frac{2}{\rho}u_0\right]-\right.$$
$$-(D_{12}-D_{12}^0)\frac{1}{s'}\left[\frac{2}{\rho}u_1\cos\varphi+\frac{2}{\rho}v_1\sin\varphi-u_0\right]+$$
$$+\frac{K_{11}C_{12}-K_{12}C_{11}}{\Omega}\sqrt{\frac{\Omega(D_{11}-D_{11}^0)}{C_{11}}}\frac{d\rho}{ds}\left[\left(1-\frac{4}{\rho^2}\right)v_1\cos\varphi+\right.$$
$$\left.+\left(1+\frac{4}{\rho^2}\right)u_1\sin\varphi+\frac{2}{\rho}v_0\right]+$$
$$+\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}\frac{1}{s'}\sqrt{\frac{\Omega(D_{11}-D_{11}^0)}{C_{11}}}\left[\frac{2}{\rho}v_1\cos\varphi-\right.$$
$$\left.\left.-\frac{2}{\rho}u_1\sin\varphi-v_0\right]\right\}+B\left\{(D_{11}-D_{11}^0)\frac{d\rho}{ds}\left[\left(1-\frac{4}{\rho^2}\right)v_1\cos\varphi+\right.\right.$$
$$\left.+\left(1+\frac{4}{\rho^2}\right)u_1\sin\varphi+\frac{2}{\rho}v_0\right]-$$
$$-(D_{12}-D_{12}^0)\frac{1}{s'}\left[\frac{2}{\rho}v_1\cos\varphi-\frac{2}{\rho}u_1\sin\varphi-v_0\right]-$$
$$-\frac{K_{11}C_{12}-K_{12}C_{11}}{\Omega}\sqrt{\frac{\Omega(D_{11}-D_{11}^0)}{C_{11}}}\frac{d\rho}{ds}\left[\left(1-\frac{4}{\rho^2}\right)u_1\cos\varphi-\right.$$
$$\left.-\left(1+\frac{4}{\rho^2}\right)v_1\sin\varphi+\frac{2}{\rho}u_0\right]-$$
$$-\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}\frac{1}{s'}\sqrt{\frac{\Omega(D_{11}-D_{11}^0)}{C_{11}}}\left[\frac{2}{\rho}u_1\cos\varphi+\right.$$
$$\left.\left.+\frac{2}{\rho}v_1\sin\varphi-u_0\right]\right\}=-m, \tag{11.26}$$

where after transformations it is necessary everywhere to assume $s=0$; while u_1, u_0 and v_0, v_1, as follows from (11.22), are the real and imaginary parts of Bessel functions of index 0 and 1, respectively.

B. Cylindrical Shell. For a cylindrical shell we have:

$$R_1=\infty,\ r=R_2=R=\text{const},\ \vartheta=0. \tag{11.27}$$

Then the initial equation of solution (11.3) is obtained without the assumptions in (11.2). Consequently, for any combination of elastic constants of the layers we obtain from (11.3) the following equation of solution of a cylindrical shell:

$$\frac{d^4\sigma}{ds^4}-\beta\sigma=\Psi(s). \tag{11.28}$$

Considering (11.14), solution of the corresponding homogeneous equation is obtained in the form

$$\sigma = (E_1 + iF_1)e^{a+ib} + (E_2 + iF_2)e^{-(a+ib)}; \tag{11.29}$$

whence, in virtue of (11.4), for the principal unknowns V and W, after elementary transformations, we obtain

$$V = \sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}}\,[F_1\chi_1(s) + E_1\eta_1(s) + F_2\chi(s) - E_2\eta(s)]; \tag{11.30}$$

$$W = E_1\chi_1(s) - F_1\eta_1(s) + E_2\chi(s) + F_2\eta(s), \tag{11.31}$$

where

$$\left.\begin{aligned} \chi(s) &= e^{-as}\cos bs, & \eta(s) &= e^{-as}\sin bs,\\ \chi_1(s) &= e^{as}\cos bs, & \eta_1(s) &= e^{as}\sin bs. \end{aligned}\right\} \tag{11.32}$$

Inserting the values of V and W from (11.30) and (11.31) into formulas (2.1), (2.2), (2.12), (2.13) and (2.27), we obtain for the design quantities the following homogeneous formulas:

$$T_1 = 0; \tag{11.33}$$

$$\begin{aligned} T_2 = \sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}}\,&[(aE_1 - bF_1)\eta_1(s) + (aF_1 + bE_1)\chi_1(s) +\\ &+ (aE_2 - bF_2)\eta(s) - (aF_2 + bE_2)\chi(s)], \end{aligned} \tag{11.34}$$

$$N = -\frac{1}{R}\sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}}\,[F_1\chi_1(s) + E_1\eta_1(s) + F_2\chi(s) - E_2\eta(s)]; \tag{11.35}$$

$$\begin{aligned} M_j = &\Bigg[(D_{1j} - D_{1j}^0)(aE_1 - bF_1) + \frac{K_{1j}C_{12} - K_{j2}C_{11}}{\Omega} \times\\ &\times \sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}}(bE_1 + aF_1)\Bigg]\chi_1(s) + \Bigg[(D_{1j} - D_{1j}^0)(aF_1 + bE_1) -\\ &- \frac{K_{1j}C_{12} - K_{j2}C_{11}}{\Omega}(aE_1 - bF_1)\sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}}\Bigg]\eta_1(s) +\\ &+ \Bigg[\frac{K_{1j}C_{12} - K_{j2}C_{11}}{\Omega}(bE_2 + aF_2)\sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}} +\\ &+ (D_{11} - D_{11}^0)(aE_2 - bF_2)\Bigg]\chi(s) + \Bigg[(D_{11} - D_{11}^0)(aF_2 + bE_2) -\\ &- \frac{K_{1j}C_{12} - K_{j2}C_{11}}{\Omega}(aE_2 - bF_2)\sqrt{\frac{\Omega(D_{11} - D_{11}^0)}{C_{11}}}\Bigg]\eta(s); \end{aligned} \tag{11.36}$$

$$
\begin{aligned}
w &= \frac{R}{\Omega}\Big\{\left[\sqrt{C_{11}\Omega(D_{11}-D_{11}^{0})}\,(bE_1+aF_1)-P_1(aE_1-bF_1)\right]\chi_1(s)+\\
&+\left[\sqrt{C_{11}\Omega(D_{11}-D_{11}^{0})}\,(aE_1-bF_1)+P_1(bE_1+aF_1)\right]\eta_1(s)-\\
&-\left[\sqrt{C_{11}\Omega(D_{11}-D_{11}^{0})}\,(bE_2+aF_2)-P_1(aE_2-bF_2)\right]\chi_2(s)+\\
&+\left[\sqrt{C_{11}\Omega(D_{11}-D_{11}^{0})}\,(aE_2-bF_2)+P_1(bE_2+aF_2)\right]\eta_2(s)\Big\}.
\end{aligned}
\tag{11.37}
$$

In the formula for moments (11.36) $j — 1$ or $j — 2$. Let us discuss two examples of a cylindrical shell.

1. We have a long cylindrical shell whose length L is greater than the zone of extension of the edge-effect. Proceeding as in Section 8 of this chapter, for the zone of extension of the edge-effect we obtain

$$
s^{*}=\pi\sqrt{\frac{2R\left[C_{11}\Omega(D_{11}-D_{11}^{0})+P_1^{2}\right]}{\Omega\left[\sqrt{C_{11}\Omega(D_{11}-D_{11}^{0})+P_1^{2}}-P_1\right]}}.
\tag{11.38}
$$

Thus, it is assumed that $L > s^{*}$.

Let us consider that along the edges of the shell ($s=0$, $s=L$) there act transverse forces N^{0}, N^{L} and bending moments M_1^{0}, M_1^{L} (Figure 46).

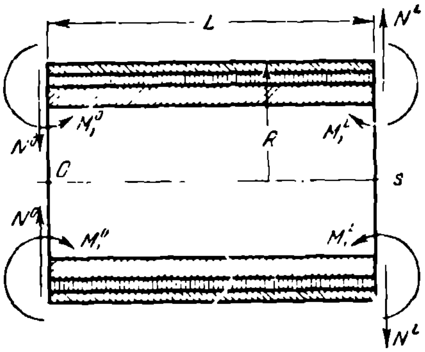

Figure 46.

Proceeding as in the previous sections of this chapter and omitting interaction of the edges (since $L > s^{*}$); for constants E_j and F_j we obtain

$$
\left.
\begin{aligned}
E_1 &= A_1'N^{L}-A_1''M_1^{L}, \quad E_2 = A_2'N^{0}-A_2''M_1^{0},\\
F_1 &= B_1'N^{L}-B_1''M_1^{L}, \quad F_2 = B_2'N^{0}.
\end{aligned}
\right\}
\tag{11.39}
$$

where

$$A'_j = R\sqrt{\frac{C_{11}}{\omega(D_{11}-D''_{11})}}\left[\frac{\sqrt{C_{11}\omega(D_{11}-D''_{11})}\,[a\eta(s)+b\zeta(s)]}{a\sqrt{C_{11}\omega(D_{11}-D''_{11})}+bP_1}-\frac{P_1\,[a\zeta(s)-b\eta(s)]}{a\sqrt{C_{11}\omega(D_{11}-D''_{11})}+bP_1}\right].$$

$$A''_j = \sqrt{\frac{C_{11}\omega}{D_{11}-D''_{11}}}\,\frac{(-1)^{j+1}\zeta(s)}{a\sqrt{C_{11}\omega(D_{11}-D''_{11})}+bP_1},$$

$$B'_j = R\sqrt{\frac{C_{11}}{\omega(D_{11}-D''_{11})}}\,\frac{1}{\zeta_1(s)}-\frac{\eta(s)}{\zeta(s)}A'_j,$$

$$B''_j = -\frac{\eta(s)}{a\sqrt{C_{11}\omega(D_{11}-D''_{11})}+bP_1}\sqrt{\frac{C_{11}\omega}{D_{11}-D''_{11}}}.$$

In these formulas, when $j=1$ it is necessary to insert $s=L$ and when $j=2$ it is necessary to insert $s=0$.

Inserting the values of E_j and F_j from (11.39) into (11.30) and (11.31), we obtain for the sought functions:

$$V' = \sqrt{\frac{\omega(D_{11}-D''_{11})}{C_{11}}}\left\{[B'_1\zeta_1(s)+A'_1\eta_1(s)]N^l - [B''_1\zeta_1(s)+A''_1\eta_1(s)]M^l_1 + [B'_2\zeta(s) - A'_2\eta(s)]N^0 + A''_2\eta(s)M^0_1\right\}, \tag{11.40}$$

$$W' = [A'_1\zeta_1(s)-B'_1\eta_1(s)]N^l - [A''_1\zeta_1(s)-B''_1\eta_1(s)]M^l_1 + [A'_2\zeta(s)+B'_2\eta(s)]N^0 - A''_2\zeta(s)M^0_1 \tag{11.41}$$

whence, with $M^L_1 = N^l = 0$ we obtain the solution for a semi-infinite cylindrical shell with loaded edge.

2. Let us examine a long cylindrical shell which in cross-section $s=0$ is loaded by peripherally uniform forces q (kg/cm) and moments m (kg) (Figure 35).

Omitting the previously derived conclusions, let us write the final values of the required functions of the problem:

when $s>0$

$$\left.\begin{aligned} V &= \frac{qR}{2}\zeta(s)-\frac{mR}{2}[a\zeta(s)+b\eta(s)],\\ W &= \sqrt{\frac{C_{11}}{\omega(D_{11}-D''_{11})}}\left\{\frac{qR}{2}\eta(s)+\frac{mR}{2}[b\zeta(s)-a\eta(s)]\right\}; \end{aligned}\right| \tag{11.42}$$

when $s < 0$

$$\left.\begin{aligned} V &= -\frac{qR}{2}\chi_1(s) - \frac{mR}{2}[a\chi_1(s) - b\eta_1(s)], \\ W &= \sqrt{\frac{C_{11}}{2(D_{11} - D^{0}_{11})}}\left\{\frac{qR}{2}\eta_1(s) + \frac{mR}{2}[b\chi_1(s) + a\eta_1(s)]\right\}. \end{aligned}\right\} \quad (11.43)$$

The design formulas are not derived here due to their unwieldiness. However, as is seen from the preceding section, having the values of W and V, the design formulas for all of the discussed problems may be written in elementary fashion.

SECTION 12. ANISOTROPIC CYLINDRICAL SHELLS OF REVOLUTION REINFORCED BY LATERAL RIBS*

Let us discuss laterally rib-reinforced shells consisting of an odd number of orthotropic layers symmetrically arranged relative to the median surface of the shell.

The ribs are sufficiently thin and narrow, in connection with which it is assumed that: (a) the contact forces m and q between any rib and the shell are uniformly distributed over the circle coinciding with the mean plane of the given rib; b the radius of curvature of the rib is the same as the radius of curvature of the shell, that is, $R + h_0 + {}^1/_2H \approx R$ (Figure 47).

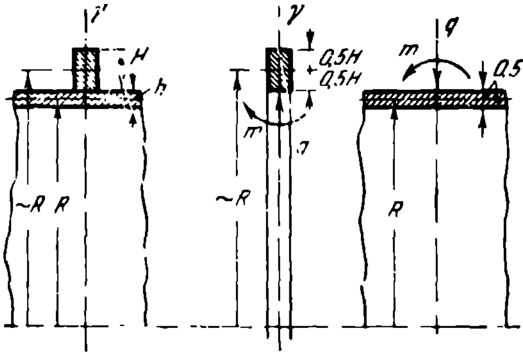

Figure 47.

Then for the change in radius of curvature and angle of rotation of the lateral sections of the rib relative to the axis of the rib under the action of uniformly distributed forces q and m we have**

---

*S. A. Ambartsumyan, see footnote on p. 149.

**S. P. Timoshenko see pp. 77-83 and 163-167 of work cited in footnote on p. 148.

$$\omega_p = \frac{qR^2}{E_1F}, \quad W_p = \frac{mR^2}{E_1J_\gamma}, \tag{12.1}$$

where E_1 is the elastic modulus of the rib material; F and J_γ are the area and moment of intertia relative to the γ axis of the lateral section of the rib.

Let us discuss a few examples.

1. A semi-infinite cylindrical shell reinforced by a single lateral rib is loaded by forces N^0 and moments M_1^0, uniformly distributed over the periphery of the end section ($s = 0$). The rib is located at a distance b ($t = b\sqrt{A/2R}$) from the loaded end of the shell (Figure 48).

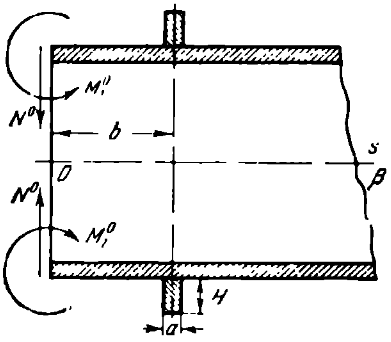

Figure 48.

From the conditions of simultaneous functioning of the shell and the rib, on the basis of (10.37), (10.39), (10.43), (10.44) and (12.1) we obtain

$$R\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\left[2RN^0\theta(t) + \sqrt{2RA}\,M_1^0\psi(t)\right] + qR^2\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\,\eta_1(t) + \frac{qR^2}{E_1F} = 0; \tag{12.2}$$

$$-\frac{1}{AD_{11}}\left[RN^0\varphi(t) + \sqrt{2RA}\,M_1^0\theta(t)\right] + m\frac{R}{AD_{11}}\sqrt{\frac{A}{2R}}\,\eta_2(t) + q\frac{R}{AD_{11}}\theta(t)^2 + \frac{mR^2}{E_1J_\gamma} = 0, \tag{12.3}$$

where

$$\left.\begin{aligned} \eta_1(t) &= \frac{1 + 2\theta(t)^2 + \psi(t)^2}{2}, \\ \eta_2(t) &= \frac{1 + 2\theta(t)^2 + \varphi(t)^2}{2}. \end{aligned}\right\} \tag{12.4}$$

By simultaneous solution of (12.2) and (12.3) for the unknown m and q, we obtain

$$m = \frac{\sqrt{2RA}\,\eta(t)^3 - R\psi(t)\,Z_1(t)}{RZ_1(t)\,Z_2(t) - A\eta(t)^4}\,N^0 - \\ - \frac{\sqrt{2RA}\,\eta(t)\,Z_1(t) - A\eta(t)^2\,\psi(t)}{RZ_1(t)\,Z_2(t) - A\eta(t)^4}\,M_1^0; \qquad (12.5)$$

$$q = \frac{R\eta(t)^2\,\psi(t) - \sqrt{2RA}\,\eta(t)\,Z_2(t)}{RZ_1(t)\,Z_2(t) - A\eta(t)^4}\,N^0 - \\ - \frac{A\psi(t)\,Z_2(t) - \sqrt{2RA}\,AR\eta(t)^3}{RZ_1(t)\,Z_2(t) - A\eta(t)^4}\,M_1^0. \qquad (12.6)$$

where

$$\left.\begin{aligned} Z_1 &= \sqrt{A/2R}\,\eta_1(t) + \frac{\Omega}{C_{11}E_1F}; \\ Z_2 &= \sqrt{A/2R}\,\eta_2(t) + \frac{D_{11}A}{E_1J_T}\,R. \end{aligned}\right\} \qquad (12.7)$$

Having the values of m and q, we may regard the problem as solved since, with consideration of (12.5) and (12.6), by appropriate summation of (10.37)-(10.42) with (10.43)-(10.54) we obtain two groups of formulas (the sectors $0 \leqslant \beta \leqslant t$ и $t \leqslant \beta \leqslant \infty$) for all design quantities of the problem.

In the special case where a long cylindrical shell is reinforced by an end former, that is, when $c = 0$ $(t = 0)$, the problem reduces to calculation of a semi-infinite cylindrical shell loaded by end forces $(N^0 + q)$ and $(M_1^0 + m)$, uniformly distributed over the periphery of the end section. In this case for m^0 and q^0 we have from (12.5) and (12.6)

$$m^0 = \frac{\left[A + \frac{\Omega\sqrt{2RA}}{C_{11}E_1F}\right]M_1^0 + \frac{R\Omega}{C_{11}E_1F}\,N^0}{A - R\left[2\sqrt{\frac{A}{2R}} + \frac{\Omega}{C_{11}E_1F}\right]\left[2\sqrt{\frac{A}{2R}} + \frac{AD_{11}}{E_1J_T}\,R\right]}; \qquad (12.8)$$

$$q^0 = \frac{R\,\frac{A^2D_{11}}{E_1J_T}\,M_1^0 + \left[A + R\sqrt{2RA}\,\frac{AD_{11}}{E_1J_T}\right]N^0}{A - R\left[2\sqrt{\frac{A}{2R}} + \frac{\Omega}{C_{11}E_1F}\right]\left[2\sqrt{\frac{A}{2R}} + \frac{AD_{11}}{E_1J_T}\,R\right]}. \qquad (12.9)$$

When the rib is infinitely removed from the loaded edge (i.e., $t = \infty$) or the rib stiffnesses is equal to zero (i.e., $E_1F = 0$, $E_1J_T = 0$) it is easily shown that $m = 0$, and $q = 0$, and the problem reduces to the elementary calculation of a semi-infinite shell loaded by end forces M_1^0, N^0. If the rib is absolutely stiff (that is, $E_1F = \infty$, $E_1J_1 = \infty$), regardless of whether $t > 0$, we have the problem of a cylindrical shell in which one edge of the shell is rigidly fixed and to the other edge there are applied end forces M_1^0 and N^0. In this case for m and q we obtain

$$m = \frac{\sqrt{\frac{2R}{A}}\left[\theta(t)^3 - \frac{1}{2}\varphi(t)\,\eta_1(t)\right]}{{}^1/_2\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4} N^0 - \frac{\theta(t)\,\eta_{11}(t) - \theta(t)^2\,\psi(t)}{{}^1/_2\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4} M_1^0; \tag{12.10}$$

$$q = \frac{\theta(t)^2\,\varphi(t) - \theta(t)\,\eta_{12}(t)}{{}^1/_2\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4} N^0 - \frac{\sqrt{\frac{A}{2R}}\left[\psi(t)\,\eta_{12}(t) - 2\theta(t)^4\right]}{{}^1/_2\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4} M_1^0. \tag{12.11}$$

In examining the derived results it is easily seen that with a decrease in rib stiffness or upon increasing the distance of the rib from the loaded edge of the shell its influence on the maximum values of the design parameters decreases. In this connection there arises the problem of determining that minimum value of t at which we may begin to disregard the influence of the rib on the maximum values of the design parameters.

As a characteristic design parameter let us take the maximum value of w, therein considering that only the lateral end force N^0 is applied to the shell. In this case, as our calculations have shown, maximum error occurs from disregarding the influence of the rib, that is, due to discarding terms containing m and q.

Let us discuss the pessimium case. Let the stiffness of the rib be infinite, that is, $E_1F = \infty$, $E_1J_1 = \infty$. Considering that in this case we obtain for a sufficiently large value (of the order of 9.5π), in subsequent discussion we may assume with high accuracy that

$$\eta_{11}(t) = \frac{1}{2}, \quad \eta_{12}(t) = \frac{1}{2} \tag{12.12}$$

and, with the same accuracy, from (10.39) and (10.43), we obtain for the $w(\beta \;\; 0)$ maximum value of

$$w_{\max} = -2R^2 \frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\,N^0 - qR\sqrt{2RA}\,\frac{C_{11}}{\Omega}\,\theta(t) - mRA\,\frac{C_{11}}{\Omega}\,\varphi(t). \tag{12.13}$$

where for m and q we have from (12.10) and (12.11)

$$m = 8\sqrt{\frac{2R}{A}}\,[\theta(t)^2 - \tfrac{1}{4}\varphi(t)]\,N^0, \quad q = -4\theta(t)\,N^0. \tag{12.14}$$

Disregarding the influence of the rib, from (12.13) we obtain for w max

$$w_{\max} = -2R^2\frac{C_{11}}{2}\sqrt{\frac{A}{2R}}\,N^0. \tag{12.15}$$

With this omission the assumed error will be

$$\Delta\% = -\frac{4\theta(t)^2 - 8\varphi(t)\theta(t)^3 + 2\varphi(t)^2}{1 - [4\theta(t)^2 - 8\varphi(t)\theta(t)^3 + 2\varphi(t)^2]}\cdot 100. \tag{12.16}$$

Below we list a few values of this error, depending on the location of the stiffening rib along the length of the shell. Examining these values, we see that with an accuracy of 10 percent the influence of the rib on the maximum values of design quantities may already be disregarded when $t = \frac{\pi}{2}$. A further small increase in the value of t substantially decreases the error assumed in disregarding the influence of the rib. Considering that the edge effect extends to t = π (see Section 8 of this chapter), we conclude that an absolutely stiff rib located at the center of the edge-effect zone has little effect on the maximum design quantities of the shell.

t		1,6	1,7	1,8	1,9
$-\Delta\%$	9,5	8,3	5,3	3,7	2,7

For a semi-infinite cylindrical shell reinforced by a single rib and loaded by an end force N^0 assume that we have:

for the shell

$$E_\theta = E_s = E, \quad \nu_{12} = \nu_{21} = 0, \quad \frac{h}{R} = \frac{1}{100}, \quad h = 2\delta;$$

for the rib

$$H = 2h = 4\delta, \quad a = h = 2\delta, \quad E_1 = mE,$$

where m takes the following values: m = 1, 10, 100. Let us discuss the case where the rib is located at a distance $b_1 = 0$, $b_2 = 0.125s^*$, $b_3 = 0.25s^*$, $b_4 = 0.5s^*$ from the loaded end of the shell. s* is the zone of extension of the edge effect and has the following form (8.6):

$$s^* = \pi R \sqrt{\frac{h}{R\sqrt{3(1-\nu^2)}}} \approx 0{,}24R.$$

Considering that $t_j = b_j\sqrt{A\cdot 2R}$, for each location of the rib we obtain:

$t_1 = 0,\ t_2 = 0{,}4,\ t_3 = 0{,}8,\ t_4 = 1{,}6$ (here, as an approximate assumption for the extent of the edge effect, instead of $t = 3{,}14$, we have $t = 3{,}2$).

By means of the above formulas the values of normal displacement w at point $\beta = 0$ were calculated. The calculations were performed using the precise and approximate formulas. In the table below we list a few values of error

$$\Delta\% = \frac{w_{\max} - w^*_{\max}}{w_{\max}}\,100,$$

where $w_{\max}$ is the precise value of normal displacement at point $\beta = 0$ as calculated by means of (10.39), (10.43), (12.5) and (12.6), while $w^*_{\max}$ is the value of normal displacement at point $\beta = 0$ as calculated from the approximate formula (12.15). No data are given in the table for large values of error.

m \ t	0,0	0,4	0,8	1,6
1,0		—14,0	—4,6	0,0
10,0			-25,3	0,0
100,0				—0,3

In examining the results of the calculation we conclude that for almost all practical problems we may disregard the influence of the rib on the maximum values of design quantities with an accuracy of 5-10 percent in the case where the rib is located at a distance t = 0.5π from the loaded edge of the shell, that is, when the rib is located at the center of the edge-effect zone. It must be pointed out that only extremely flexible ribs (m = 1) located at a quarter of the edge-effect zone have negligible influence on the maximum values of the design quantities.

2. A semi-infinite cylindrical shell is reinforced by one lateral rib and loaded by an internal pressure of intensity p. We assume that the shell has end plates which may be displaced in the axial direction and do not interfere with normal displacement and rotation of the end section $\beta = 0$ (Figure 49).

For the load terms of such a shell without a rib we have (10.1)

$$\left.\begin{aligned} &E_r = q,\quad E_x = 0,\quad P_x^0 = \pi R^2 p,\\ &F_1 = \frac{1}{2}R^2 p,\quad F_2(s) = -pRs. \end{aligned}\right\} \qquad (12.17)$$

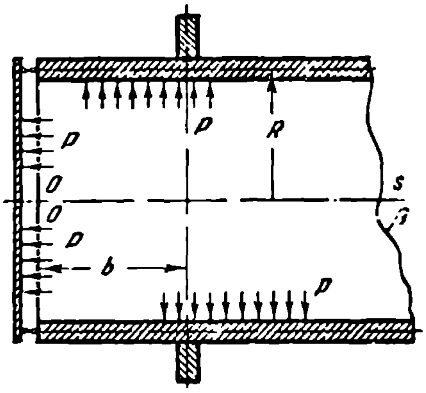

Figure 49.

Considering that for the discussed problem $M_1^0 = 0$, $M_1^L = 0$, $N^0 = 0$, $N^L = 0$, and also that for a cylindrical shell $\vartheta = 0$, $R_1 = \infty$, we obtain from (9.7)-(9.13) for a shell without rib

$$\begin{gathered} W = 0, \quad N = 0, \\ M_1 = 0, \quad M_2 = 0, \end{gathered} \tag{12.18}$$

$$\begin{gathered} V = pRs, \quad T_1 = \frac{1}{2} pR, \\ T_2 = pR. \end{gathered} \tag{12.19}$$

Inserting the values of T_1 and T_2 from (12.19) into (9.22), we obtain for the normal displacement

$$w = \frac{C_{11} - 0{,}5C_{12}}{\Omega} R^2 p. \tag{12.20}$$

From the conditions of simultaneous functioning of the shell and rib and on the basis of (10.43), (10.44), (12.1), (12.18) and (12.20), we obtain

$$\begin{gathered} -\frac{C_{11} - 0{,}5C_{12}}{\Omega} R^2 p + \frac{C_{11}}{\Omega} \sqrt{\frac{A}{2R}} \eta_1(t) R^2 q + \\ + \frac{C_{11}}{\Omega} A\theta(t)^2 Rm + \frac{R^3 q}{E_1 F} = 0; \end{gathered} \tag{12.21}$$

$$\frac{\theta(t)^2}{AD_{11}} Rq + \frac{\eta_2(t)}{D_{11}\sqrt{2RA}} Rm + \frac{R^3 m}{E_1 J_1} = 0. \tag{12.22}$$

whence

$$m = -\frac{C_{11} - 0{,}5C_{12}}{C_{11}} \frac{\theta(t)^3}{\chi_1 \chi_2 - \frac{A}{R} \theta(t)^4} p; \tag{12.23}$$

$$q = \frac{C_{11} - 0{,}5C_{12}}{C_{11}} \frac{\gamma_2}{\gamma_1\gamma_2 - \frac{A}{R}\theta(t)^4} p. \qquad (12.24)$$

Inserting the values of m and q into (10.43)-(10.54) and adding to the derived formulas the corresponding quantities from (12.18)-(12.20), we obtain two groups of formulas (for sectors $0 \leqslant \beta \leqslant t$ and $t \leqslant \beta \leqslant \infty$). In view of the elementary nature of the operations we shall not derive the sum formulas.

In the special case where the shell is reinforced by an end rib, that is, when $t = 0$, we obtain

$$m^0 = -\frac{\frac{C_{11} - 0{,}5C_{12}}{C_{11}}}{\left[2\sqrt{\frac{A}{2R}} + \frac{\Omega}{C_{11}E_1F}\right]\left[2\sqrt{\frac{A}{2R}} + \frac{D_{11}A}{E_1J_T}R\right] - \frac{A}{R}} p, \qquad (12.25)$$

$$q^0 = \frac{\frac{C_{11} - 0{,}5C_{12}}{C_{11}}\left[2\sqrt{\frac{A}{2R}} + \frac{D_{11}A}{E_1J_T}R\right]}{\left[2\sqrt{\frac{A}{2R}} + \frac{\Omega}{C_{11}E_1F}\right]\left[2\sqrt{\frac{A}{2R}} + \frac{D_{11}A}{E_1J_T}R\right] - \frac{A}{R}} p. \qquad (12.26)$$

In the other special case, where $t = \infty$, we obtain the well-known result of a uniformly loaded infinite tube reinforced by a single rib

$$m^\infty = 0, \quad q^\infty = \frac{\frac{C_{11} - 0{,}5C_{12}}{C_{11}}}{\frac{1}{2}\sqrt{\frac{A}{2R}} + \frac{\Omega}{C_{11}E_1F}} p. \qquad (12.27)$$

When the rib stiffness is equal to zero, that is, when $E_1F = 0$, $E_1J_T = 0$ we obtain

$$m = 0, \quad q = 0. \qquad (12.28)$$

If the rib is absolutely stiff, that is, if $E_1F = \infty$ and $E_1J_T = \infty$ we have

$$m = -\frac{C_{11} - 0{,}5C_{12}}{C_{11}} \frac{R}{A} \frac{\theta(t)^2}{\frac{1}{2}\eta_1(t)\eta_2(t) - \theta(t)^4} p; \qquad (12.29)$$

$$q = \frac{C_{11} - 0{,}5C_{12}}{C_{11}} \frac{R}{A} \sqrt{\frac{A}{2R}} \frac{\eta_2(t)}{\frac{1}{2}\eta_1(t)\eta_2(t) - \theta(t)^4} p. \qquad (12.30)$$

In analyzing the results we note that with increasing distance of the rib from the free edge of the shell (that is, with an increase in t) there is, in the first place, a decrease in the influence of contact moment m on the stress of the shell and, in the second place, the contact force q approaches the value

q^{∞} (i.e., the value of q in the case where $t=\infty$. In this connection there arises the problem of determining the minimum value of t beginning with which we may, with required accuracy, assume $q=q^{\infty}$ or disregard the influence of moment m.

As before, let us discuss the case in which $E_1F=\infty$ and $E_1J_T=\infty$.

From (12.27) for q^{∞} we have

$$q^{\infty}=4\frac{C_{11}-0{,}5C_{12}}{C_{11}}\frac{R}{A}\sqrt{\frac{A}{2R}}p. \tag{12.31}$$

Comparing (12.30) and (12.31), we obtain the formula for determining the error which results upon replacing q with q^{∞}:

$$\Delta\%=\frac{\eta_2(t)-2\eta_1(t)\eta_3(t)+4\theta(t)^4}{\eta_2(t)}100\%; \tag{12.32}$$

we list a few values for this error:

t	0,9	1,0	1,2	1,6
$-\Delta\%$	11,0	8,2	5,2	4,3

From (10.43), considering (12.29) and (12.30) as well as (12.20), we obtain for the normal displacement ω at point $\beta = 0$

$$w=\frac{C_{11}-0{,}5C_{12}}{\Omega}R^2p\left[1-\frac{\eta_2(t)\theta(t)+\varphi(t)\theta(t)^2}{\frac{1}{2}\eta_1(t)\eta_3(t)-\theta(t)^4}\right]. \tag{12.33}$$

If we disregard the influence of m, for approximate ω we obtain

$$w^*=\frac{C_{11}-0{,}5C_{12}}{\Omega}R^2p\left[1-\frac{\eta_3(t)\theta(t)}{\frac{1}{2}\eta_1(t)\eta_2(t)-\theta(t)^4}\right]. \tag{12.34}$$

By means of formulas (12.33) and (12.34) we easily derive formula (12.35), from which we may determine the error resulting from omission of the influence of contact moment m

$$\Delta\%=\frac{\varphi(t)\theta(t)^2}{\frac{1}{2}\eta_1(t)\eta_3(t)-\theta(t)^4-\eta_2(t)\theta(t)+\varphi(t)\theta(t)^2}100; \tag{12.35}$$

a few results of calculation from this formula are given below:

t	1,0	1,1	1,2	1,6
Δ%	29,4	12,4	4,8	0,0

In calculating the value of ω at point $\beta = 0$ it is evident that we may also disregard the influence of q (that is, we may generally disregard the influence of the rib); however, while this additional omission is possible in calculating ω at point $\beta = 0$, it is not admissible in investigating the value of ω at points lying close to the rib or in investigating other design quantities of the problem.

For example, if in investigating the bending moment M_1^0 we introduce the additional omission of the influence of q, we shall obtain for M_1 zero values at all all points of the shell, which does not correspond to the facts.

Let us determine the feasibility of disregarding the influence of m in calculating the bending moment. From (10.46), considering (12.29) and (12.30), we obtain for the bending moment M_1^0 at point x = t

$$M_1 = \frac{P}{2}\,\frac{C_{11} - 0{,}5C_{12}}{C_{11}}\,\frac{R}{A}\left[\frac{[1 - \zeta(t)\varphi(t) - \zeta(t)\theta(t)]\,\theta(t)^2}{\frac{1}{2}\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4} - \frac{\frac{1}{2}[1 - 2\zeta(t)\theta(t) - \varphi(t)\psi(t)]\,\eta_{12}(t)}{\frac{1}{2}\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4}\right]. \tag{12.36}$$

Omitting the influence of m, we obtain for M_1

$$M_1' = -\frac{P}{4}\,\frac{C_{11} - 0{,}5C_{12}}{C_{11}}\,\frac{R}{A}\,\frac{[1 - 2\zeta(t)\theta(t) - \varphi(t)\psi(t)]\,\eta_{12}(t)}{\frac{1}{2}\eta_{11}(t)\,\eta_{12}(t) - \theta(t)^4}. \tag{12.37}$$

As in the previous case, in order to determine the error we obtain

$$\Delta\% = -\frac{[1 - \varphi(t)^2]\,\theta(t)^2 \cdot 100}{\frac{1}{2}[1 - \zeta(t)\theta(t) - \varphi(t)\psi(t)]\,\eta_{12}(t) - [1 - \varphi(t)^2]\,\theta(t)^2}, \tag{12.38}$$

from which we obtain the following results of calculation

t	1,0	1,1	1,2	1,6
$-\Delta\%$	11,9	6,4	3,5	0,0

Examining the results of calculation of error by means of formulas (12.32), (12.35) and (12.38), we note that if from the free end of the shell ($\beta = 0$) the rib is located at a distance of 0.4-0.5 of the edge-effect zone, then with an accuracy of 5-10% we may assume $q = q^{\infty}$ and it is permissible to disregard the influence of m on the stress of the shell.

The above examples are of importance in themselves and may be used in calculations for anisotropic laminar cylindrical shells reinforced by a single lateral rib. They may also be used in calculations for cylindrical shells reinforced by many lateral ribs. Let us discuss two examples from this series of problems.

3. An infinitely long cylindrical shell is reinforced by lateral ribs of constant cross-section $F = aH$ and with spacing b. The shell is loaded by a uniform internal pressure of intensity p (Figure 50). The analogous problem in the case of an isotropic shell has been solved by other authors*, **, *** by another method.

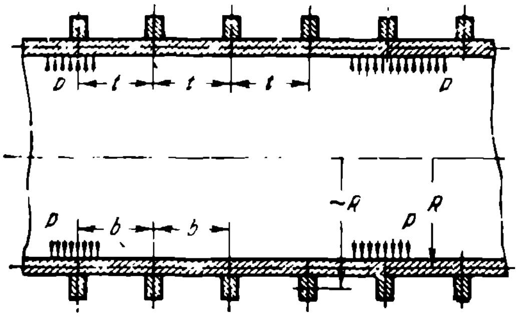

Figure 50.

In engineering practice stiffening ribs are generally placed no closer than 0.8-0.9 of the edge-effect zone. In this case, considering the above result, the problem may be solved by simple but somewhat approximate means.

---

*Z. V. Kantorovich, Osnovy rasheta khimicheskikh mashin i apparatov[Design fundamentals for chemical equipment], Mashgiz, 1946, pp. 89-127.

**S. P. Timoshenko, Plastinki i obolochki [plates and shells], Gostekhizdat, 1948, p. 382.

***E. I. Grigolyuk, Strength and stability of bimetallic cylindrical shells, Inzh. sbornik, XVI, 1953.

Assuming that the stress of the shell between any two ribs is affected only by these two adjacent ribs (that is, the ribs located at the beginning and the end of the examined segment of the shell), from (10.15)-(10.31) we obtain design formulas corresponding to a contact force q which is approximately equal to q^{∞} (12.27):

$$V=\frac{qR}{2}\left[\theta(\beta)-\theta(t-\beta)\right]; \tag{12.39}$$

$$W=\frac{qR}{2AD_{11}}\left[\zeta(\beta)-\zeta(t-\beta)\right]; \tag{12.40}$$

$$w=-\frac{qR^2}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\left[\varphi(\beta)+\varphi(t-\beta)\right]; \tag{12.41}$$

$$M_1=-\frac{qR}{2}\sqrt{\frac{1}{2RA}}\left[\psi(\beta)+\psi(t-\beta)\right], \tag{12.42}$$

$$T_2=-\frac{qR}{2}\sqrt{\frac{A}{2R}}\left[\varphi(\beta)+\varphi(t-\beta)\right]. \tag{12.43}$$

Adding to (12.39)-(12.43) the corresponding quantities from (12.18)-(12.20), we obtain the final design formulas for the problem.

For the sake of comparison, let us present the precise values of the design quantities of the problem. Omitting procedural details, let us write the final results:

$$V=\frac{qR}{2}\,\frac{\cos\beta\,\mathrm{ch}\,(t-\beta)-\cos\,(t-\beta)\,\mathrm{ch}\,\beta}{\mathrm{ch}\,t-\cos t}; \tag{12.44}$$

$$W=\frac{qR}{2AD_{11}}\,\frac{\sin\beta\,\mathrm{ch}\,(t-\beta)-\sin\,(t-\beta)\,\mathrm{sh}\,\beta}{\mathrm{ch}\,t-\cos t}; \tag{12.45}$$

$$w=-\frac{qR^2}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\left[\frac{\cos\,(t-\beta)\,\mathrm{sh}\,\beta+\sin\,(t-\beta)\,\mathrm{ch}\,\beta}{\mathrm{ch}\,t-\cos t}+\right.$$
$$\left.+\frac{\cos\beta\,\mathrm{ch}\,(t-\beta)+\sin\beta\,\mathrm{ch}\,(t-\beta)}{\mathrm{ch}\,t-\cos t}\right]; \tag{12.46}$$

$$M_1=\frac{qR}{2}\sqrt{\frac{1}{2RA}}\left[\frac{\sin\,(t-\beta)\,\mathrm{ch}\,\beta-\cos\,(t-\beta)\,\mathrm{sh}\,\beta}{\mathrm{ch}\,t-\cos t}+\right.$$
$$\left.+\frac{\sin\beta\,\mathrm{ch}\,(t-\beta)-\cos\beta\,\mathrm{sh}\,(t-\beta)}{\mathrm{ch}\,t-\cos t}\right]; \tag{12.47}$$

$$T_2=-\frac{qR}{2}\sqrt{\frac{A}{2R}}\left[\frac{\cos\,(t-\beta)\,\mathrm{sh}\,\beta+\sin\,(t-\beta)\,\mathrm{ch}\,\beta}{\mathrm{ch}\,t-\cos t}+\right.$$
$$\left.+\frac{\cos\beta\,\mathrm{sh}\,(t-\beta)+\sin\beta\,\mathrm{ch}\,(t-\beta)}{\mathrm{ch}\,t-\cos t}\right]. \tag{12.48}$$

where for the contact force we have

$$q = \frac{C_{11} - 0{,}5C_{12}}{C_{11}} \left[\frac{\omega}{C_{11}E_1F} + \frac{1}{2} \sqrt{\frac{A}{2R}} \frac{\sin t + \operatorname{sh} t}{\operatorname{ch} t - \cos t} \right]^{-1} . \tag{12.49}$$

Here, as in the case of the approximate problem, the final design formulas may be derived by summation of (12.44)-(12.48) with the corresponding quantities from (12.18)-(12.20).

By way of example, let us compare the total values of ω as calculated by the approximate (12.41), (12.20) and precise (12.46), (12.20) methods. In comparing these values it is assumed that $E_1F = \infty$ and $\beta = \frac{1}{2}t$. We present a few of the results of comparative calculations performed in a manner similar to that of the previous examples.

t	2,1	2,6	2,8	3,0	3,4	3,6	3,8
$\Delta\%$	19,1	5,24	−0,74	−2,91	−2,97	−2,38	−1,87

By examining the results of the calculations we observe that the maximum permissible error with $t > 0{,}8\pi$ does not exceed 5-10%.

Considering that the edge-effect zone is determined by the quantity $t = \pi$, we conclude that if the stiffening ribs are located no closer than 0.8-0.9 of the edge-effect zone, the problem may be solved with sufficient accuracy by means of approximate formulas (12.39)-(12.43).

4. A semi-infinite cylindrical shell is reinforced by lateral ribs of constant cross-section $F = aH$ with spacing of $b > 0{,}8s$' The shell is loaded by a uniform internal pressure of intensity p (Figure 51).

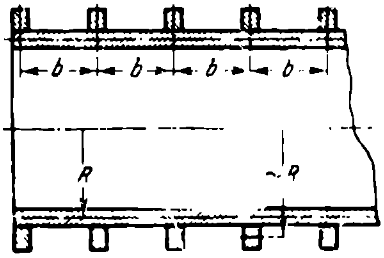

Figure 51.

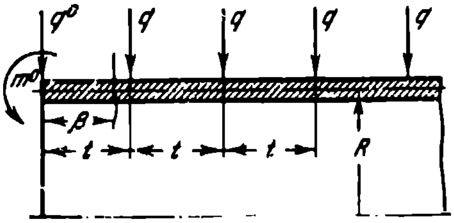

Figure 52.

Considering the results obtained in the previous examples, we may simplify the design procedure for the discussed problem (Figure 52) by limiting ourselves to the shell sector between the first and second ribs (that is, the sector for which $0 < \beta < l$. For the remaining portion of the shell ($l \leqslant x$), we have a repetition of the previous problem, that is, the problem of an infinitely long shell reinforced by ribs. On the basis of formulas (10.37)-(10.42), (10.43)-(10.48) and (10.18)-(12.20), we obtain for the design quantities

$$V = R^2 p - \frac{Rq}{2}[\theta(t-\beta) - \psi(\beta)\theta(t) + \zeta(\beta)\psi(t)] + Rq^0\psi(\beta) - \sqrt{2RA}\, m^0\zeta(\beta); \tag{12.50}$$

$$W' = \frac{Rq}{2}\frac{1}{AD_{11}}[-\zeta(t-\beta) + \varphi(\beta)\theta(t) + \theta(\beta)\psi(t)] + \frac{1}{AD_{11}}\left[Rq^0\varphi(\beta) + \sqrt{2RA}\, m^0\theta(\beta)\right]; \tag{12.51}$$

$$w = \frac{C_{11}-0{,}5C_{12}}{\Omega}R^2 p - \frac{R^2 q}{2}\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}[\varphi(t-\beta) + 2\theta(\beta)\theta(t) + \psi(\beta)\psi(t)] - R\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\left[2Rq^0\theta(\beta) + \sqrt{2RA}\, m^0\psi(\beta)\right]; \tag{12.52}$$

$$M_1 = -\frac{Rq}{2\sqrt{2RA}}[\psi(t-\beta) - 2\zeta(\beta)\theta(\beta) - \varphi(\beta)\psi(t)] + \frac{2R}{\sqrt{2RA}}q^0\zeta(\beta) + m^0\varphi(\beta); \tag{12.53}$$

$$T_2 = Rp - \frac{Rq}{2}\sqrt{\frac{A}{2R}}[\varphi(t-\beta) + 2\theta(\beta)\theta(t) + \psi(\beta)\psi(t)] - \sqrt{\frac{A}{2R}}\left[2Rq^0\theta(\beta) + \sqrt{2RA}\, m^0\psi(\beta)\right]. \tag{12.54}$$

In these formulas q represents the contact force between the shell and the ribs (beginning with the second rib) and is determined by means of (12.27).

Forces m^0 and Q^0 are determined from the conditions of contact of the outer (first) rib with the shell. The contact conditions have the form

$$\frac{C_{11}-0{,}5C_{12}}{C_{11}}R^2 p + 2R^2 q\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\theta(t) + R\frac{C_{11}}{\Omega}\sqrt{\frac{A}{2R}}\left(2Rq^0 + \sqrt{2RA}\, m^0\right) + \frac{q^0R^2}{E_1F} = 0; \tag{12.55}$$

$$\frac{Rq}{AD_{11}}\psi(t) + \frac{1}{AD_{11}}\left(Rq^0 + \sqrt{2RA}\, m^0\right) + \frac{m^0R^2}{E_1J_1} = 0. \tag{12.56}$$

By simultaneous solution of (12.55) and (12.56) we obtain for the unknown quantities

$$m^0 = \frac{-\frac{C_{11} - 0.5C_{12}}{C_{11}} p + \left[2\sqrt{\frac{A}{2R}}\,\theta(t) - \psi(t)\,\gamma_1^0\right] q}{\gamma_1^0\gamma_2^0 - A/R}; \tag{12.57}$$

$$q^0 = \frac{\frac{C_{11} - 0.5C_{12}}{C_{11}} \gamma_2^0 p - \left[2\sqrt{\frac{A}{2R}}\,\gamma_2^0\,\theta(t) - \frac{A}{R}\,\psi(t)\right] q}{\gamma_1^0\gamma_2^0 - A/R}; \tag{12.58}$$

where

$$\gamma_1^0 = 2\sqrt{\frac{A}{2R}} + \frac{\Omega}{C_{11}E_1F}, \quad \gamma_2^0 = 2\sqrt{\frac{A}{2R}} + \frac{D_{11}A}{E_1J_t}R. \tag{12.59}$$

Having the values of m^0 and q^0, the problem may be considered solved.

In those rare cases where the distance between ribs is less than 0.8-0.9 of the edge-effect zone a solution of the problem may be simplified. In this case the contact equations become more complex (for equations with four unknowns) and q (beginning with the third rib) must be taken to be the force defined in formula (12.49). However, these complications introduce no fundamental changes in the presented problem.

The results obtained in this section may also be in reducing the problem of a shell reinforced by lateral ribs to the problem of a laminar anisotropic shell consisting of homogeneous orthotropic layers. As an example of this let us discuss the problem of determining that maximum distance between ribs at which a given shell reinforced by specified lateral ribs may with certain accuracy be regarded as an anisotropic laminar shell consisting of homogeneous orthotropic layers.

For concreteness and simplicity of calculation we discuss an infinitely long shell reinforced by equidistant identical lateral ribs symmetrically arranged relative to the shell itself (Figure 53). It is assumed that the shell is single-layer orthotropic shell with a thickness of $2h_2$. The ribs have rectangular cross-section ($F = a \times H$) and arc spaced at distances of b ($t = b\sqrt{A/2R}$). The radii of curvature of the ribs are the same as the radius of curvature of the shell, that is, $R + h_2 + {}^1/_2 H \approx R$; the shell is loaded by a uniform internal pressure of intensity p.

For the stiffnesses of the presented shell (here regarded as a three-layer shell) we have*

---

*S. G. Lekhnitskiy, Anizotropnyye plastinki, Gostekhizdat, 1957, pp. 258-260.

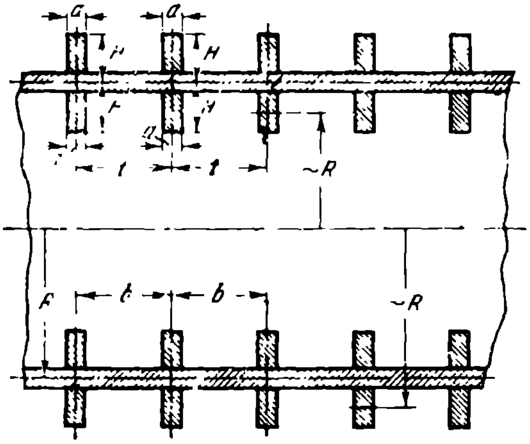

Figure 53.

$$\left.\begin{aligned} C^*_{11} &= 2h_2\frac{E_s}{1-\nu_1\nu_2}; \quad C^*_{12} = 2h_2\frac{\nu_2 E_s}{1-\nu_1\nu_2}; \\ C^*_{22} &= 2h_2\frac{E_\varphi}{1-\nu_1\nu_2} + 2naHE_p; \end{aligned}\right\} \tag{12.60}$$

$$\left.\begin{aligned} D^*_{11} &= \frac{2}{3}h_2^3\frac{E_s}{1-\nu_1\nu_2}; \quad D^*_{12} = \frac{2}{3}h_2^3\frac{\nu_2 E_s}{1-\nu_1\nu_2}; \\ D^*_{22} &= \frac{2}{3}h_2^3\frac{E_\varphi}{1-\nu_1\nu_2} + 2n\left[\frac{aH^3}{3} + aHh_2(H+h_2)\right]E_p, \end{aligned}\right\} \tag{12.61}$$

where E_s and E_φ are the elastic moduli for the principal directions s and φ, respectively; ν_1 and ν_2 are Poisson's ratios; E_p is the elastic modulus of the rib; n is the number of rib pairs (inner and outer) per unit length of shell.

From (12.20) we have for the normal displacement of the presented shell

$$w^* = \frac{C^*_{11} - 0{,}5C^*_{12}}{\Omega^*}R^2p. \tag{12.62}$$

From (12.46) and (12.20) we have for the normal displacement of points under the ribs ($\beta = 0$ или $\beta = t$)

$$w_{(\beta=0)} = \frac{C_{11} - 0{,}5C_{12}}{\Omega}R^2p\left[1 - \frac{\sqrt{\frac{A}{2R}}\,\frac{\sin t + \operatorname{sh} t}{\operatorname{ch} t - \cos t}}{\frac{\Omega}{C_{11}FE_p} + \sqrt{\frac{A}{2R}}\,\frac{\sin t + \operatorname{sh} t}{\operatorname{ch} t - \cos t}}\right]. \tag{12.63}$$

From (12.46) and (12.20) we have for the normal displacement of the mid-points between ribs ($\beta = t/2$) we have

$$w_{(\beta = t/2)} = \frac{C_{11} - 0{,}5C_{12}}{\Omega} R^2 p \left[1 - \frac{2\sqrt{\frac{A}{2R}} \frac{\cos t/2 \operatorname{sh} t/2 + \sin t/2 \operatorname{ch} t/2}{\operatorname{ch} t - \cos t}}{\frac{\Omega}{C_{11}E_pF} + \sqrt{\frac{A}{2R}} \frac{\sin t + \operatorname{sh} t}{\operatorname{ch} t - \cos t}} \right]. \qquad (12.64)$$

It is easily seen that the normal displacement under a rib is less than the corresponding displacement of the given shell, while the normal displacement of the mid-point between ribs, conversely, is greater than the corresponding displacement of the shell, that is, we have

$$\left. \begin{aligned} w^* &= Cw_{(\beta = 0)}, & C &> 1, \\ w^* &= Bw_{(\beta = t/2)}, & B &< 1. \end{aligned} \right\} \qquad (12.65)$$

Inserting the values of displacements from (12.62), (12.63) and (12.64) into formulas (12.65), we obtain after certain transformations

$$\frac{\sin t + \operatorname{sh} t}{\operatorname{ch} t - \cos t} = \sqrt{\frac{2R}{A}} \frac{C\Omega^* - \Omega}{C_{11}FE_p}, \qquad (12.66)$$

$$\frac{\sqrt{A\,2R}}{\operatorname{ch} t - \cos t} \left[\left(\frac{1}{\Omega^*} - \frac{B}{\Omega} \right) (\sin t + \operatorname{sh} t) + \right.$$
$$\left. + \frac{2B}{\Omega} (\cos t/2 \operatorname{sh} t/2 + \sin t/2 \operatorname{ch} t/2) \right] = \frac{B - \frac{\Omega}{\Omega^*}}{C_{11}FE_p} \qquad (12.67)$$

By means of these formulas we may establish those maximum values of parameter t (that is, b) at which, from the standpoint of normal displacement ω and with required accuracy, a shell reinforced by lateral ribs may be regarded as an anisotropic laminar shell consisting of homogeneous layers. Similar formulas may be derived for other design quantities of the shell, particularly for the stresses. In view of the unwieldiness of the derived relationships, we shall not present them here.

Let us examine numerical examples of determination of t by means of formulas (12.66) and (12.67).

Let the shell be isotropic and possess the following elastic characteristics:

$$E_\varphi = E_s = E, \quad \nu_1 = \nu_2 = 0.$$

The shell thickness $h_2 = h$, a $R/h = 25$. For the ribs we assume: $a = h$, $H = 2h$, $E_p = E/m$ where m has the values 1.0 and 0.1.

The values of t are to be found for $C = 1{,}05$, $B = 0{,}95$ and C = 1.1, that is, with the order of 5 and 10%. From (12.66) and (12.67) we have

$$\frac{\sin t + \operatorname{sh} t}{\operatorname{ch} t - \cos t} - \frac{2A}{t} = \frac{(A-1)\,m}{2\sqrt[4]{3}}\sqrt{\frac{R}{h}};$$

$$\frac{1}{\operatorname{ch} t - \cos t}\left[\left(\frac{1}{1+4\sqrt{\frac{h}{R}\,\frac{\sqrt[4]{3}}{tm}}} - B\right)(\sin t + \operatorname{sh} t) + \right.$$

$$\left. + 2B\left(\cos\frac{t}{2}\operatorname{sh}\frac{t}{2} + \sin\frac{t}{2}\operatorname{ch}\frac{t}{2}\right)\right] = \frac{m\sqrt{Rh}}{2\sqrt[4]{3}}\left(B - \frac{1}{1+4\sqrt{\frac{h}{R}\,\frac{\sqrt[4]{3}}{tm}}}\right).$$

Inserting the corresponding values of A and B, we find for parameter t

$m \diagdown C$	1,05	1,10
1,0	2,4	3,0
0,1	1,8	2,2

$m \diagdown B$	0,95	0,90
1,0	2,0	3,0
0,1	0,8	1,1

From these tables we may find those values of t (distance between ribs) at which, from the standpoint of normal displacement, a shell reinforced by ribs may with a given accuracy be regarded as an anisotropic laminar shell consisting of homogeneous layers.

It may be noted that with an increase in the distance between ribs the accuracy of this derivation decreases. The stiffer the rib, the smaller the value of parameter t must be (that is, the closer the required spacing of the ribs) for a given accuracy. For example, with an assigned accuracy of 10% a shell reinforced by ribs for which m = 1.0 may be regarded as a derived shell $t \leqslant 3{,}0$. With a decrease in t the accuracy of the derivation is increased, for example; when $t \leqslant 2{,}0$ the accuracy of the derivation increases to 5%.

CHAPTER V

77806

ANISOTROPIC CYLINDRICAL SHELLS

SECTION 1. BASIC PREMISES. INITIAL RELATIONSHIPS AND EQUATIONS

Let us discuss a circular cylindrical shell consisting of an arbitrary number of homogeneous anisotropic layers. It is assumed that at each point of each layer of the shell there exists only one plane of elastic symmetry parallel to the coordinate surface of the shell. Without disturbing the general validity we shall simplify the expressions for stiffnesses by assuming that the coordinate surface of the shell $(\gamma = 0)$ coincides with the internal boundary surface of the shell (for all points of the shell $\gamma > 0$).

It is also assumed that α and β are orthogonal coordinates coinciding with the lines of principal curvature of the coordinate surface, that is, with the generating lines $(\beta = \text{const})$ and with the direction arcs $(\alpha = \text{const})$ of the cylindrical coordinate surface of the shell.

If α and β are respectively understood to represent the length of the generatrix and the length of the arc of the direction circle (Figure 54), the coefficients of the first quadratic form A and B and the principal radii of curvature R_1 and R_2 of the coordinate surface are defined by the formulas

$$A = 1, \quad B = 1, \quad \frac{1}{R_1} = 0, \quad \frac{1}{R_2} = \frac{1}{R}, \tag{1.1}$$

where R is the radius of curvature of the cylinder.

If we use the dimensionless coordinates s and θ, wherein $s = \alpha/R$ is the dimensionless length of the generating line (that is, the ratio of the true length of the generatrix to the radius R of the shell) and $\theta = \beta/R$ is the dimensionless length of the arc of the direction circle, that is, the central angle of transverse

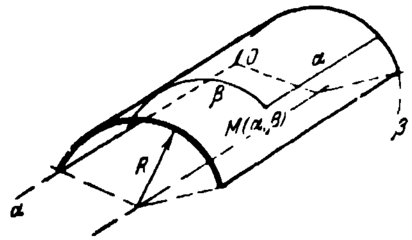

Figure 54

arc reckoned from a certain initial generating line (Figure 55), then for the coefficients of the first quadratic form and the principal radii of curvature of the coordinate surface we shall have

$$A = R, \quad B = R, \quad R_1 = \infty, \quad R_2 = R. \tag{1.2}$$

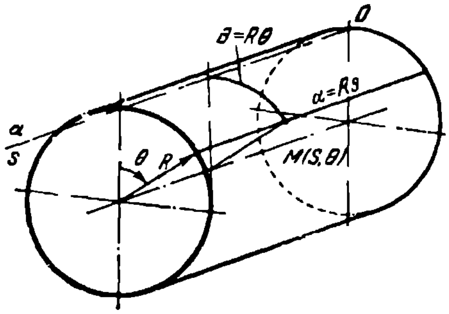

Figure 55

In the subsequent discussion we shall use both representations of coordinates α and β. In this connection, in all the basic formulas and equations coefficients A and B remain unchanged, but, in addition, in virtue of (1.1) and (2.2), we consider that the coefficients of the first quadratic form A and B as well as the radius of curvature $R_2 = R$ are constants.

Thus, for a circular cylindrical shell we have:

the equilibrium equations (Chapter II, Section 7)

$$\frac{1}{A}\frac{\partial T_1}{\partial \alpha} + \frac{1}{B}\frac{\partial S_{21}}{\partial \beta} = -X; \tag{1.3}$$

$$\frac{1}{B}\frac{\partial T_2}{\partial \beta} + \frac{1}{A}\frac{\partial S_{12}}{\partial \alpha} + \frac{1}{R}N_2 = -Y; \tag{1.4}$$

$$\frac{1}{R}T_2 - \frac{1}{A}\frac{\partial N_1}{\partial \alpha} - \frac{1}{B}\frac{\partial N_2}{\partial \beta} = Z; \tag{1.5}$$

$$\frac{1}{A}\frac{\partial M_1}{\partial \alpha} + \frac{1}{B}\frac{\partial H_{21}}{\partial \beta} = N_1; \tag{1.6}$$

$$\frac{1}{B}\frac{\partial M_2}{\partial \beta}+\frac{1}{A}\frac{\partial H_{12}}{\partial \alpha}=N_2; \tag{1.7}$$

$$S_{12}-S_{21}-\frac{1}{R}H_{21}=0; \tag{1.8}$$

Formulas associating the components of deformation with the components of displacement (Chapter II, Section 2)

$$\left.\begin{aligned} \varepsilon_1=\frac{1}{A}\frac{\partial u}{\partial \alpha}; \quad \varepsilon_2=\frac{1}{B}\frac{\partial v}{\partial \beta}+\frac{1}{R}w;\\ \omega=\frac{1}{B}\frac{\partial u}{\partial \beta}+\frac{1}{A}\frac{\partial v}{\partial \alpha};\end{aligned}\right\} \tag{1.9}$$

$$\left.\begin{aligned} \varkappa_1=-\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}; \quad \varkappa_2=-\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}+\frac{1}{R}\frac{1}{B}\frac{\partial v}{\partial \beta};\\ \tau=-\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha\, \partial \beta}+\frac{2}{R}\frac{1}{A}\frac{\partial v}{\partial \alpha};\end{aligned}\right\} \tag{1.10}$$

the elasticity relationships (Chapter II, Section 9)

$$T_1=C_{11}\varepsilon_1+C_{12}\varepsilon_2+C_{16}\omega+K_{11}\varkappa_1+K_{12}\varkappa_2+K_{16}\tau; \tag{1.11}$$

$$T_2=C_{22}\varepsilon_2+C_{12}\varepsilon_1+C_{26}\omega+K_{22}\varkappa_2+K_{12}\varkappa_1+K_{26}\tau; \tag{1.12}$$

$$\begin{aligned} S_{12}= C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2+K_{66}\tau+K_{16}\varkappa_1+K_{26}\varkappa_2+\\ +\frac{1}{R}(K_{66}\omega+K_{16}\varepsilon_1+K_{26}\varepsilon_2+D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2);\end{aligned} \tag{1.13}$$

$$S_{21}=C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2+K_{66}\tau+K_{16}\varkappa_1+K_{26}\varkappa_2; \tag{1.14}$$

$$M_1=D_{11}\varkappa_1+D_{12}\varkappa_2+D_{16}\tau+K_{11}\varepsilon_1+K_{12}\varepsilon_2+K_{16}\omega; \tag{1.15}$$

$$M_2=D_{22}\varkappa_2+D_{12}\varkappa_1+D_{26}\tau+K_{22}\varepsilon_2+K_{12}\varepsilon_1+K_{26}\omega; \tag{1.16}$$

$$\begin{aligned} H_{12}=H_{21}=H=D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2+\\ +K_{66}\omega+K_{16}\varepsilon_1+K_{26}\varepsilon_2,\end{aligned} \tag{1.17}$$

where, by virtue of the fact that the coordinate surface coincides with the inner surface, that is, $\Delta=0$ (Figure 8 and 11), for the stiffnesses we have (Chapter II, Section 8)

$$C_{jk}=\sum_{s=1}^{p} B_{jk}^{s}(\delta_s-\delta_{s-1}); \tag{1.18}$$

$$K_{jk}=\frac{1}{2}\sum_{s=1}^{p} B_{jk}^{s}(\delta_s^2-\delta_{s-1}^2); \tag{1.19}$$

$$D_{jk}=\frac{1}{3}\sum_{s=1}^{p} B_{jk}^{s}(\delta_s^3-\delta_{s-1}^3), \tag{1.20}$$

where $p=m+n$ is the number of layers.

On the basis the stress within the layers (Chapter II, Section 4) of (1.9) and (1.10) will be:

$$\begin{aligned}\sigma'_\alpha = {} & B'_{11}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B'_{12}\frac{1}{B}\frac{\partial v}{\partial\beta} + B'_{16}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) + \\ & + \frac{\gamma}{R}\left(B'_{12}\frac{1}{B}\frac{\partial v}{\partial\beta} + B'_{16}\frac{2}{A}\frac{\partial v}{\partial\alpha}\right) + B'_{12}\frac{1}{R}w - \\ & - \gamma\left(B'_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B'_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B'_{16}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right);\end{aligned} \qquad (1.21)$$

$$\begin{aligned}\sigma'_\beta = {} & B'_{22}\frac{1}{B}\frac{\partial v}{\partial\beta} + B'_{12}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B'_{26}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) + \\ & + \frac{\gamma}{R}\left(B'_{22}\frac{1}{B}\frac{\partial v}{\partial\beta} + B'_{26}\frac{2}{A}\frac{\partial v}{\partial\alpha}\right) + B'_{22}\frac{1}{R}w - \\ & - \gamma\left(B'_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B'_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B'_{26}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right);\end{aligned} \qquad (1.22)$$

$$\begin{aligned}\tau'_{\alpha\beta} = {} & B'_{16}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B'_{26}\frac{1}{B}\frac{\partial v}{\partial\beta} + B'_{66}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) + \\ & + \frac{\gamma}{R}\left(B'_{26}\frac{1}{B}\frac{\partial v}{\partial\beta} + B'_{66}\frac{2}{A}\frac{\partial v}{\partial\alpha}\right) + B'_{26}\frac{1}{R}w - \\ & - \gamma\left(B'_{16}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B'_{26}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B'_{66}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right).\end{aligned} \qquad (1.23)$$

Relationships (1.9)-(1.10) were used in deriving formulas (1.21)-(1.22). In another equation, the equation of compatibility of deformation, which for a cylindrical shell has the following form (Chapter II, Section 3):

$$\frac{\varkappa_1}{R} + \frac{1}{A^2}\frac{\partial^2\varepsilon_2}{\partial\alpha^2} - \frac{1}{AB}\frac{\partial^2\omega}{\partial\alpha\,\partial\beta} + \frac{1}{B^2}\frac{\partial^2\varepsilon_1}{\partial\beta^2} = 0. \qquad (1.24)$$

SECTION 2. SYSTEM OF DIFFERENTIAL EQUATIONS OF SOLUTION IN DISPLACEMENTS

Inserting the values of deformation components ε_1, ε_2, ω, $\varkappa_1$, $\varkappa_2$, and τ from (1.9) and (1.10) into the elasticity relationships (1.11)-(1.17), we obtain expressions for the internal forces and moments represented by means of displacements u, v and w:

$$\begin{aligned}T_1 = {} & \left(C_{11}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left[\left(C_{16} + \frac{2}{R}K_{16}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(C_{12} + \frac{1}{R}K_{12}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left(\frac{1}{R}C_{12} - K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2K_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - K_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w;\end{aligned} \qquad (2.1)$$

$$\begin{aligned}T_2 = {} & \left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left[\left(C_{26} + \frac{2}{R}K_{26}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(C_{22} + \frac{1}{R}K_{22}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left(\frac{1}{R}C_{22} - K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2K_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w;\end{aligned} \qquad (2.2)$$

$$\begin{aligned}S_{12} = {} & \left[\left(C_{16} + \frac{1}{R}K_{16}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \left(C_{66} + \frac{1}{R}K_{66}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]u + \\ & + \left[\left(C_{66} + \frac{3}{R}K_{66} + \frac{2}{R^2}D_{66}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(C_{26} + \frac{2}{R}K_{26} + \frac{1}{R^2}D_{26}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left[\frac{1}{R}\left(C_{26} + \frac{1}{R}K_{26}\right) - \right. \\ & - \left(K_{16} + \frac{1}{R}D_{16}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - 2\left(K_{66} + \frac{1}{R}D_{66}\right)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - \\ & \left. - \left(K_{26} + \frac{1}{R}D_{26}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]w;\end{aligned} \qquad (2.3)$$

$$\begin{aligned}S_{21} = {} & \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left[\left(C_{66} + \frac{2}{R}K_{66}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(C_{26} + \frac{1}{R}K_{26}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left[\frac{1}{R}C_{26} - K_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2K_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - K_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]w;\end{aligned} \qquad (2.4)$$

$$\begin{aligned}M_1 = {} & \left(K_{11}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left[\left(K_{16} + \quad D_{16}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(K_{12} + \frac{1}{R}D_{12}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left(\frac{1}{R}K_{12} - D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2D_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w\end{aligned} \qquad (2.5)$$

$$\begin{aligned}M_2 = {} & \left(K_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left[\left(K_{26} + \frac{2}{R}D_{26}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(K_{22} + \frac{1}{R}D_{22}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left(\frac{1}{R}K_{22} - D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2D_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w.\end{aligned} \qquad (2.6)$$

$$\begin{aligned}H = {} & \left(K_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left[\left(K_{66} + \frac{2}{R}D_{66}\right)\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + \left(K_{26} + \frac{1}{R}D_{26}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v + \left(\frac{1}{R}K_{26} - D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w.\end{aligned} \qquad (2.7)$$

Examining the elasticity relationships (1.13), (1.14), (1.17) and equation (1.8), it is easily seen that the sixth equilibrium equation is identically satisfied. In this connection, here and in subsequent discussion when elasticity relationships (1.11)-(1.17) are taken as the basis of derivation it will be assumed that the sixth equation will not be taken into consideration.

From the equilibrium equations (1.3)-(1.7), excluding the transverse forces, we obtain

$$\frac{1}{A}\frac{\partial T_1}{\partial\alpha} + \frac{1}{B}\frac{\partial S_{21}}{\partial\beta} = -X; \qquad (2.8)$$

$$\frac{1}{B}\frac{\partial T_2}{\partial\beta} + \frac{1}{A}\frac{\partial S_{12}}{\partial\alpha} + \frac{1}{R}\left(\frac{1}{B}\frac{\partial M_2}{\partial\beta} + \frac{1}{A}\frac{\partial H}{\partial\alpha}\right) = -Y; \qquad (2.9)$$

$$\frac{1}{R}T_2 - \frac{1}{A^2}\frac{\partial^2 M_1}{\partial\alpha^2} - \frac{2}{AB}\frac{\partial^2 H}{\partial\alpha\,\partial\beta} - \frac{1}{B^2}\frac{\partial^2 M_2}{\partial\beta^2} = Z. \qquad (2.10)$$

Inserting the values of the internal forces and moments from (2.1)-(2.7) into equilibrium equations (2.8)-(2.10), we obtain the following system of equilibrium equations in displacements:

$$\left.\begin{aligned} L_{11}(C_{jk})u + L_{12}(C_{jk}K_{jk})v + L_{13}(C_{jk},\ K_{jk})w &= X, \\ L_{21}(C_{jk}K_{jk})u + L_{22}(C_{jk}K_{jk}D_{jk})v + L_{23}(C_{jk}K_{jk}D_{jk})w &= Y; \\ L_{31}(C_{jk}K_{jk})u + L_{32}(C_{jk}K_{jk}D_{jk})v + L_{33}(C_{jk}K_{jk}D_{jk})w &= Z, \end{aligned}\right\} \qquad (2.11)$$

where for the linear differential operators we have

$$L_{11}(C_{jk}) = C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2C_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} \qquad (2.12)$$

$$\begin{aligned} L_{22}(C_{jk}K_{jk}D_{jk}) = {} & \left(C_{66} + \frac{4}{R}K_{66} + \frac{4}{R^2}D_{66}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \\ & + \left(2C_{26} + \frac{5}{R}K_{26} + \frac{4}{R^2}D_{26}\right)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + \\ & + \left(C_{22} + \frac{2}{R}K_{22} + \frac{1}{R^2}D_{22}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \end{aligned} \qquad (2.13)$$

$$\begin{aligned} L_{12}(C_{jk}K_{jk}) = L_{21}(C_{jk}K_{jk}) = {} & \left(C_{16} + \frac{2}{R}K_{16}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \\ & + \left[C_{12} + C_{66} + \frac{1}{R}(K_{12} + 2K_{66})\right]\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + \\ & + \left(C_{26} + \frac{1}{R}K_{26}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \end{aligned} \qquad (2.14)$$

$$\begin{aligned} L_{13}(C_{jk}K_{jk}) = L_{31}(C_{jk}K_{jk}) = {} & -\frac{1}{R}\left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right) - \\ & - K_{11}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3} - 3K_{16}\frac{1}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta} - (K_{12} + 2K_{66})\times \\ & \times\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2} - K_{26}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}; \end{aligned} \qquad (2.15)$$

$$\begin{aligned} L_{23}(C_{jk},\ K_{jk},\ D_{jk}) = L_{32}(C_{jk},\ K_{jk}D_{jk}) = {} & \frac{1}{R}\left(C_{22}\frac{1}{B}\frac{\partial}{\partial\beta} + C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right) + \\ & + \frac{1}{R^2}\left(K_{22}\frac{1}{B}\frac{\partial}{\partial\beta} + 2K_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right) - \left(K_{16} + \frac{2}{R}D_{16}\right)\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3} - \\ & - \left[K_{12} + 2K_{66} + \frac{1}{R}(D_{12} + 4D_{66})\right]\frac{1}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta} - \\ & - \left(3K_{26} + \frac{4}{R}D_{26}\right)\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2} - \left(K_{22} + \frac{1}{R}D_{22}\right)\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}; \end{aligned} \qquad (2.16)$$

$$\begin{aligned} L_{33}(C_{jk}K_{jk}D_{jk}) = {} & \frac{1}{R^2}C_{22} - \frac{2}{R}\left(K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \right. \\ & \left. + 2K_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right) + D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4} + \\ & + 4D_{16}\frac{1}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta} + 2(D_{12} + 2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + \\ & + 4D_{26}\frac{1}{B^3A}\frac{\partial^4}{\partial\beta^3\,\partial\alpha} + D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4}. \end{aligned} \qquad (2.17)$$

Thus, solving the system of equations in (2.11) with given boundary conditions we determine the sought functions of u, v and w by means of which, using formulas (2.1)-(2.7) and (1.21)-(1.23), we may determine all the internal forces and fundamental design stresses in the shell layers.

Examining the system of equations in (2.11), we note that it possesses a unique symmetry of construction, namely: the linear operators (L_{12}, L_{21}, L_{13}, L_{23}, L_{32}), symmetrically arranged relative to the principal diagonal operators (L_{11}, L_{22}, L_{33}), have identical values.

The symmetry of the matrix of differential equilibrium equations in displacements in the case of an isotropic shell was first noted by V.Z. Vlasov.*

By means of the operator method the system of differential equations in (2.11) may be reduced to a single first-order differential equation of solution for a certain potential function $\Phi(\alpha, \beta)$, in terms of which all the design quantities of the problem may be represented. However, due to the extreme unwieldiness of the resulting coefficients of the eighth-order equation of solution and of the design formulas we shall not derive them here in the general case of anisotropy of the shell layers.

SECTION 3. CYLINDRICAL SHELLS CONSISTING OF AN ARBITRARY NUMBER OF ORTHOTROPIC LAYERS

Let a cylindrical shell consist of orthotropic layers and let the principal directions of elasticity at each point of each layer coincide with the directions of coordinate lines α, β and γ (i.e., at each point of each layer one of the planes of elastic symmetry is parallel to the coordinate surface of the shell and the other two planes are perpendicular to the corresponding generating and directing arcs of the cylindrical coordinate surface of the shell).

Thus, the discussed laminar shell as a whole is an orthotropic cylindrical shell whose principal directions of elasticity at each point of the shell coincide with the directions of coordinate lines α, β and γ. In this case, for the coefficients $B^i{}_{jk}$ we have (2.12.2), namely:

$$\left.\begin{aligned} &B_{11}^i=\frac{E_1^i}{1-\nu_1^i\nu_2^i}, \quad B_{22}^i=\frac{E_2^i}{1-\nu_1^i\nu_2^i}, \quad B_{66}^i=G_{12}^i, \\ &B_{12}^i=\frac{\nu_2^i E_1^i}{1-\nu_1^i\nu_2^i}=\frac{\nu_1^i E_2^i}{1-\nu_1^i\nu_2^i}, \quad B_{16}^i=B_{26}^i=0. \end{aligned}\right\} \qquad (3.1)$$

---

*V.Z. Vlasov, Obshchaya teoriya obolochek, Gostekhizdat, 1949, pp. 257 and 262.

By virtue of this, from (1.21)-(1.23) we have for the stresses within the layers

$$\sigma_1^i = B_{11}^i \frac{1}{A}\frac{\partial u}{\partial \alpha} + B_{12}^i \frac{1}{B}\frac{\partial v}{\partial \beta} + B_{12}^i \frac{1}{R} w - \\ - \gamma\left(B_{11}^i \frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}^i \frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}\right) + \gamma B_{12}^i \frac{1}{R}\frac{1}{B}\frac{\partial v}{\partial \beta}; \tag{3.2}$$

$$\sigma_2^i = B_{22}^i \frac{1}{B}\frac{\partial v}{\partial \beta} + B_{12}^i \frac{1}{A}\frac{\partial u}{\partial \alpha} + B_{22}^i \frac{1}{R} w - \\ - \gamma\left(B_{22}^i \frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + B_{12}^i \frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}\right) + \gamma B_{22}^i \frac{1}{R}\frac{1}{B}\frac{\partial v}{\partial \beta}; \tag{3.3}$$

$$\tau_{12}^i = B_{66}^i\left(\frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}\right) - \gamma B_{66}^i \frac{2}{AB}\frac{\partial^2 w}{\partial \alpha\, \partial \beta} + \gamma B_{66}^i \frac{2}{R}\frac{1}{A}\frac{\partial v}{\partial \alpha}. \tag{3.4}$$

On the basis of (3.1), from (1.18)-(1.20) we obtain for orthotropic shells in general

$$C_{16} = 0,\quad K_{16} = 0,\quad D_{16} = 0;$$
$$C_{26} = 0,\quad K_{26} = 0,\quad D_{26} = 0;$$

then from (2.1)-(2.7) we have for the internal forces and moments

$$T_1 = C_{11}\frac{1}{A}\frac{\partial u}{\partial \alpha} + \left(C_{12} + \frac{1}{R}K_{12}\right)\frac{1}{B}\frac{\partial v}{\partial \beta} + \\ + \left(\frac{1}{R}C_{12} - K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial \alpha^2} - K_{12}\frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}\right) w; \tag{3.5}$$

$$T_2 = C_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha} + \left(C_{22} + \frac{1}{R}K_{22}\right)\frac{1}{B}\frac{\partial v}{\partial \beta} + \\ + \left(\frac{1}{R}C_{22} - K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial \alpha^2} - K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}\right) w; \tag{3.6}$$

$$S_{12} = \left(C_{66} + \frac{1}{R}K_{66}\right)\frac{1}{B}\frac{\partial u}{\partial \beta} + \left(C_{66} + \frac{3}{R}K_{66} + \\ + \frac{2}{R^2}D_{66}\right)\frac{1}{A}\frac{\partial v}{\partial \alpha} - 2\left(K_{66} + \frac{1}{R}D_{66}\right)\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\, \partial \beta}; \tag{3.7}$$

$$S_{21} = C_{66}\frac{1}{B}\frac{\partial u}{\partial \beta} + \left(C_{66} + \frac{2}{R}K_{66}\right)\frac{1}{A}\frac{\partial v}{\partial \alpha} - 2K_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\, \partial \beta}; \tag{3.8}$$

$$M_1 = K_{11}\frac{1}{A}\frac{\partial u}{\partial \alpha} + \left(K_{12} + \frac{1}{R}D_{12}\right)\frac{1}{B}\frac{\partial v}{\partial \beta} + \\ + \left(\frac{1}{R}K_{12} - D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial \alpha^2} - D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}\right) w; \tag{3.9}$$

$$M_2 = K_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha} + \left(K_{22} + \frac{1}{R}D_{22}\right)\frac{1}{B}\frac{\partial v}{\partial \beta} + \\ + \left(\frac{1}{R}K_{22} - D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial \alpha^2} - D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}\right) w; \tag{3.10}$$

$$H = K_{66}\frac{1}{B}\frac{\partial u}{\partial \beta} + \left(K_{66} + \frac{2}{R}D_{66}\right)\frac{1}{A}\frac{\partial v}{\partial \alpha} - 2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial \alpha\, \partial \beta}. \tag{3.11}$$

Inserting the values of internal forces and moments from (3.5)-(3.11) into equilibrium equations (2.8)-(2.10), we obtain a system of three differential equations for the three unknown functions of u, v and w. Externally it does not differ from the previously derived system in (2.11); however, in the case of an orthotropic shell in linear operators of the system, L_{jk}, have the following form:

$$L_{11}(C_{jk}) = C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{3.12}$$

$$\begin{aligned} L_{22}(C_{jk}, K_{jk}D_{jk}) = {} & \left(C_{66} + \frac{4}{R}K_{66} + \frac{4}{R^2}D_{66}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \\ & + \left(C_{22} + \frac{2}{R}K_{22} + \frac{1}{R^2}D_{22}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \end{aligned} \tag{3.13}$$

$$L_{12}(C_{jk}K_{jk}) = \left[C_{12} + C_{66} + \frac{1}{R}(K_{12} + 2K_{66})\right]\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}; \tag{3.14}$$

$$L_{13}(C_{jk}K_{jk}) = C_{12}\frac{1}{R}\frac{1}{A}\frac{\partial}{\partial\alpha} - K_{11}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3} - (K_{12} + 2K_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}; \tag{3.15}$$

$$\begin{aligned} L_{23}(C_{jk}K_{jk}D_{jk}) = {} & C_{22}\frac{1}{R}\frac{1}{B}\frac{\partial}{\partial\beta} + K_{22}\frac{1}{R^2}\frac{1}{B}\frac{\partial}{\partial\beta} - \\ & - \left[K_{12} + 2K_{66} + \frac{1}{R}(D_{12} + 4D_{66})\right]\frac{1}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta} - \left(K_{22} + \frac{1}{R}D_{22}\right)\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}; \end{aligned} \tag{3.16}$$

$$\begin{aligned} L_{33}(C_{jk}K_{jk}D_{jk}) = {} & \frac{1}{R^2}C_{22} - \frac{2}{R}\left(K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right) + \\ & + D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4} + 2(D_{12} + 2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4}; \end{aligned} \tag{3.17}$$

$$L_{21} = L_{12}, \quad L_{31} = L_{13}, \quad L_{32} = L_{23}. \tag{3.18}$$

As was stated in the preceding section, by means of the operator method the system of differential equations in (2.11) may be reduced to a single eighth-order differential equation of solution for a certain potential function $\Phi(\alpha, \beta)$, which may be used to represent all the design quantities of the problem. The general procedure for such reduction is described in detail in the works by V.Z. Vlasov, A.L. Gol'denveyzer and A.I. Lur'ye. Without going into detail, let us present a few basic premises which are valid also for the general case of anisotropy. The general premises are described in much the same manner as in the cited works.*,**

Let us rewrite the system of equations in (2.11)

---

*A.L. Gol'denveyzer, Teoriya uprugikh tonkikh obolochek, Gostekhizdat, 1953, pp. 208-210.

**A.I. Lur'ye Statika tonkostennykh uprugikh obolochek, Gostekhizdat, 1947, pp. 194-198.

$$\left.\begin{aligned} L_{11}u + L_{12}v + L_{13}w &= -X; \\ L_{21}u + L_{22}v + L_{23}w &= -Y; \\ L_{31}u + L_{32}v + L_{33}w &= Z. \end{aligned}\right\} \tag{3.19}$$

In view of the constancy of the coefficients of equations (3.19), we assume that here we have a complete system of algebraic equations in terms of displacements u, v and w with constant coefficients in the form of operators L_{jk}. Let us obtain the determinate of equations (3.19)

$$d = \begin{vmatrix} L_{11}L_{12}L_{13} \\ L_{21}L_{22}L_{23} \\ L_{31}L_{32}L_{33} \end{vmatrix} \tag{3.20}$$

and calculate its minors

$$\left.\begin{aligned} d_{11} &= L_{22}L_{33} - L_{32}L_{23}; \\ d_{12} &= L_{23}L_{31} - L_{21}L_{33} \end{aligned}\right| \tag{3.21}$$

etc.

We assume

$$u = d_{11}\Phi_1 + d_{12}\Phi_2 + d_{13}\Phi_3; \tag{3.22}$$

$$v = d_{12}\Phi_1 + d_{22}\Phi_2 + d_{23}\Phi_3; \tag{3.23}$$

$$w = d_{31}\Phi_1 + d_{32}\Phi_2 + d_{33}\Phi_3 \tag{3.24}$$

and insert these expressions into equations (2.11) or (3.19); then we obtain three differential equations

$$d\Phi_1 = -X; \quad d\Phi_2 = -Y; \quad d\Phi_3 = Z, \tag{3.25}$$

defining the sought functions $\Phi_1(\alpha, \beta)$, $\Phi_2(\alpha, \beta)$, $\Phi_3(\alpha, \beta)$.

If by the functions Φ_1, Φ_2, and Φ_3 in formulas (3.22)-(3.24) we understand any particular solutions Φ_1^*, Φ_2^*, and Φ_3^* of the equations in (3.25), then the general solution of the system in (3.19) may be obtained by superposition of the particular solution

$$\left.\begin{aligned} u^* &= d_{11}\Phi_1^* + d_{12}\Phi_2^* + d_{13}^*\Phi_3; \\ v^* &= d_{12}\Phi_1^* + d_{22}\Phi_2^* + d_{23}^*\Phi_3; \\ w^* &= d_{31}\Phi_1^* + d_{32}\Phi_2^* + d_{33}\Phi_3^* \end{aligned}\right\} \qquad (3.26)$$

and of the general solution of the homogeneous system corresponding to equations (3.19)

$$\left.\begin{aligned} u^0 &= d_{11}\Phi_1^0 + d_{12}\Phi_2^0 + d_{13}\Phi_3^0; \\ v^0 &= d_{21}\Phi_1^0 + d_{22}\Phi_2^0 + d_{23}\Phi_3^0; \\ w^0 &= d_{31}\Phi_1^0 + d_{32}\Phi_2^0 + d_{33}\Phi_3^0. \end{aligned}\right\} \qquad (3.26')$$

where in determining functions Φ_1^0, Φ_2^0, and Φ_3^0 instead of equations (3.25), we have the following homogeneous equations:

$$d\Phi_1 = 0; \quad d\Phi_2 = 0; \quad d\Phi_3 = 0. \qquad (3.27)$$

Thus, the functions u^0, v^0, and w^0, defined by relationships (3.26') will yield the integral of a homogeneous system corresponding to equations (3.19) at any values of Φ_1^0, Φ_2^0, and Φ_3^0, satisfying homogeneous equations (3.27). In particular, we may assume

$$\Phi_1^0 = 0, \quad \Phi_2^0 = 0, \quad \Phi_3^0 = \Phi. \qquad (3.28)$$

Then to each solution of the homogeneous equation

$$d\Phi = 0 \qquad (3.29)$$

there will correspond the integral of a homogeneous system of equilibrium equations in displacements (3.19) as defined by the formulas

$$u^0 = d_{13}\Phi, \quad v^0 = d_{23}\Phi, \quad w^0 = d_{33}\Phi. \qquad (3.30)$$

For the most widely used forms of external surface load the particular solution for displacements u, v and w may be determined directly from the system in (3.19); then the problem reduces to solution of homogeneous equation (3.29) and to the use of formulas (3.30), (3.2)-(3.11) for determining the displacements, stresses within the layers and the internal forces of the shell.

We shall turn now to the problem of an orthotropic laminar cylindrical shell. By virtue of (3.12)-(3.18) and (3.20), we obtain from (3.29) the following equation of solution of the problem:

$$\begin{aligned} &P_1 \frac{1}{A^8} \frac{\partial^8 \Phi}{\partial \alpha^8} + P_3 \frac{1}{A^6 B^2} \frac{\partial^8 \Phi}{\partial \alpha^6 \partial \beta^2} + P_5 \frac{1}{A^4 B^4} \frac{\partial^8 \Phi}{\partial \alpha^4 \partial \beta^4} + \\ &\quad + P_4 \frac{1}{A^2 B^6} \frac{\partial^8 \Phi}{\partial \alpha^2 \partial \beta^6} + P_2 \frac{1}{B^8} \frac{\partial^8 \Phi}{\partial \beta^8} + Q_1 \frac{1}{A^6} \frac{\partial^6 \Phi}{\partial \alpha^6} + \\ &\quad + Q_3 \frac{1}{A^4 B^2} \frac{\partial^6 \Phi}{\partial \alpha^4 \partial \beta^2} + Q_4 \frac{1}{A^2 B^4} \frac{\partial^6 \Phi}{\partial \alpha^2 \partial \beta^4} + Q_2 \frac{1}{B^6} \frac{\partial^6 \Phi}{\partial \beta^6} + \\ &\quad + R_1 \frac{1}{A^4} \frac{\partial^4 \Phi}{\partial \alpha^4} + R_3 \frac{1}{A^2 B^2} \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2} + R_2 \frac{1}{B^4} \frac{\partial^4 \Phi}{\partial \beta^4} = 0, \end{aligned} \tag{3.31}$$

where the following notation is used:

$$P_1 = G_2(C_{11}D_{11} - K_{11}^2); \quad P_2 = C_{66}(C_{22}D_{22} - K_{22}^2); \tag{3.32}$$

$$\begin{aligned} P_3 = {} & D_{11}(F_2C_{11} + G_2C_{66} - V^2) + 2G_2C_{11}(D_{12} + 2D_{66}) - \\ & - K_{11}[F_2K_{11} + 2G_2(K_{12} + 2K_{66}) - 2VW] - C_{11}W^2; \end{aligned} \tag{3.33}$$

$$\begin{aligned} P_4 = {} & D_{22}(F_2C_{11} + G_2C_{66} - V^2) + 2F_2C_{66}(D_{12} + 2D_{66}) - \\ & - (K_{12} + 2K_{66})\left[F_2(K_{12} + 2K_{66}) - 2V\left(K_{22} + \frac{D_{22}}{R}\right)\right] - \\ & - 2C_{66}\left(K_{22} + \frac{D_{22}}{R}\right)W - C_{11}\left(K_{22} + \frac{D_{22}}{R}\right)^2; \end{aligned} \tag{3.34}$$

$$\begin{aligned} P_5 = {} & D_{11}C_{66}F_2 + D_{22}C_{11}G_2 + 2(D_{12} + 2D_{66})(F_2C_{11} + G_2C_{66} - V^2) - \\ & - K_{11}\left[2F_2(K_{12} + 2K_{66}) - 2\left(K_{22} + \frac{D_{22}}{R}\right)V\right] - \\ & - (K_{12} + 2K_{66})[G_2(K_{12} + 2K_{66}) - 2VW] - \\ & - 2C_{11}\left(K_{22} + \frac{D_{22}}{R}\right)W - C_{66}W^2; \end{aligned} \tag{3.35}$$

$$Q_1 = \frac{2}{R} G_2(C_{12}K_{11} - C_{11}K_{12}); \tag{3.36}$$

$$Q_2 = \frac{2}{R} C_{66}\left[\left(C_{22} + \frac{K_{22}}{R}\right)\left(K_{22} + \frac{D_{22}}{R}\right) - F_2K_{22}\right]; \tag{3.37}$$

$$\begin{aligned} Q_3 = {} & \frac{2}{R}\left\{C_{11}\left(C_{22} + \frac{K_{22}}{R}\right)W + C_{12}[F_2K_{11} + \right. \\ & + G_2(K_{12} + 2K_{66}) - VW] - K_{11}\left(C_{22} + \frac{K_{22}}{R}\right)V - \\ & \left. - K_{12}(F_2C_{11} + G_2C_{66} - V^2) - G_2C_{11}K_{22}\right\}; \end{aligned} \tag{3.38}$$

$$\begin{aligned} Q_4 = {} & \frac{2}{R}\left\{\left(C_{22} + \frac{K_{22}}{R}\right)\left[C_{66}W + C_{11}\left(K_{22} + \frac{D_{22}}{R}\right)\right] + \right. \\ & + C_{12}\left[F_2(K_{12} + 2K_{66}) - \left(K_{22} + \frac{D_{22}}{R}\right)V\right] - K_{22}(F_2C_{11} + G_2C_{66} - V^2) - \\ & \left. - F_2C_{66}K_{12} - (K_{12} + 2K_{66})\left(C_{22} + \frac{K_{22}}{R}\right)V\right\}; \end{aligned} \tag{3.39}$$

$$R_1 = \frac{1}{R^2} G_2 (C_{11} C_{22} - C_{12}^2); \tag{3.40}$$

$$R_2 = \frac{1}{R^4} C_{66} \left[F_2 C_{22} - \left(C_{22} + \frac{K_{22}}{R} \right)^2 \right]; \tag{3.41}$$

$$R_3 = \frac{1}{R^2} \Big\{ (C_{11} C_{22} - C_{12}^2) F_2 + C_{22} (G_2 C_{66} - V^2) + \\ + \left(C_{22} + \frac{K_{22}}{R} \right) \left[2 C_{12} V - C_{11} \left(C_{22} + \frac{K_{22}}{R} \right) \right] \Big\}. \tag{3.42}$$

In these formulas and in subsequent discussion

$$F_2 = C_{22} + \frac{2K_{22}}{R} + \frac{D_{22}}{R^2}; \tag{3.43}$$

$$G_2 = C_{66} + \frac{4K_{66}}{R} + \frac{4D_{66}}{R^2}; \tag{3.44}$$

$$V = C_{12} + C_{66} + \frac{K_{12} + 2K_{66}}{R}; \tag{3.45}$$

$$W = K_{12} + 2K_{66} + \frac{D_{12} + 4D_{66}}{R}. \tag{3.46}$$

In terms of the sought function $\Phi(\alpha, \beta)$ displacements u, v and w are represented by formulas (3.30), which have the following expanded form (in the displacement notation the superscript zeros have been omitted).

$$u = G_2 K_{11} \frac{1}{A^5} \frac{\partial^5 \Phi}{\partial \alpha^5} + [F_2 K_{11} + G_2 (K_{12} + 2K_{66}) - \\ - VW] \frac{1}{A^3 B^2} \frac{\partial^5 \Phi}{\partial \alpha^3 \partial \beta^2} + [F_2 (K_{12} + 2K_{66}) - \\ - \left(K_{22} + \frac{D_{22}}{R} \right) V \Big] \frac{1}{AB^4} \frac{\partial^5 \Phi}{\partial \alpha \partial \beta^4} - \frac{1}{R} G_2 C_{12} \frac{1}{A^3} \frac{\partial^3 \Phi}{\partial \alpha^3} + \\ + \frac{1}{R} \left[\left(C_{22} + \frac{K_{22}}{R} \right) V - C_{12} F_2 \right] \frac{1}{AB^2} \frac{\partial^3 \Phi}{\partial \alpha \partial \beta^2} + u^*; \tag{3.47}$$

$$v = C_{66} \left(K_{22} + \frac{D_{22}}{R} \right) \frac{1}{B^5} \frac{\partial^5 \Phi}{\partial \beta^5} + \Big[C_{66} W + C_{11} \left(K_{22} + \frac{D_{22}}{R} \right) - \\ - (K_{12} + 2K_{66}) V \Big] \frac{1}{A^2 B^3} \frac{\partial^5 \Phi}{\partial \alpha^2 \partial \beta^3} + \\ + \left(C_{11} W - K_{11} V \right) \frac{1}{A^4 B} \frac{\partial^5 \Phi}{\partial \alpha^4 \partial \beta} - \frac{1}{R} C_{66} \left(C_{22} + \frac{K_{22}}{R} \right) \frac{1}{B^3} \frac{\partial^3 \Phi}{\partial \beta^3} + \\ + \frac{1}{R} \left[C_{12} V - C_{11} \left(C_{22} + \frac{K_{22}}{R} \right) \right] \frac{1}{A^2 B} \frac{\partial^3 \Phi}{\partial \alpha^2 \partial \beta} + v^*; \tag{3.48}$$

$$w = G_2 C_{11} \frac{1}{A^4} \frac{\partial^4 \Phi}{\partial \alpha^4} + (F_2 C_{11} + G_2 C_{66} - V^2) \frac{1}{A^2 B^2} \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2} + \\ + F_2 C_{66} \frac{1}{B^4} \frac{\partial^4 \Phi}{\partial \beta^4} + w^*, \tag{3.49}$$

where $u^* = u^*(\alpha, \beta)$, $v^* = v^*(\alpha, \beta)$, and $w^* = w^*(\alpha, \beta)$ are the particular integrals of homogeneous equations (2.11) (with operators (3.12)-(3.18)) defined by the given surface load components $X = X(\alpha, \beta)$, $Y = Y(\alpha, \beta)$, and $Z = Z(\alpha, \beta)$.

Having the values of the displacements, by means of formulas (3.2)-(3.11) we may easily find the values of the design stresses and internal forces.

SECTION 4. ENGINEERING THEORY OF CYLINDRICAL SHELLS CONSISTING OF AN ARBITRARY NUMBER OF ANISOTROPIC LAYERS

Of great practical importance is the so-called engineering theory of cylindrical shells*,**,***,**** which, along with the fundamental hypothesis of nondeformable normals as given for the entire stack of the shell as a whole, is based on the following additional assumptions:

a) In the geometric relationships in (1.10) we may only retain components containing the normal displacement w;

b) We may take the simpler relationships for elasticity in (2.9.1)-(2.9.8), which only approximately satisfy the sixth equilibrium equation (1.8);

c) In the second equilibrium equation (1.4) we may disregard the term N_2/R, which originates in moment;

d) Approximate satisfaction of the second equation of compatibility of deformations (2.3.2) is permissible.

On the basis of the above remarks, for a circular cylindrical shell we obtain:

the equilibrium equations

$$\frac{1}{A}\frac{\partial T_1}{\partial \alpha}+\frac{1}{B}\frac{\partial S}{\partial \beta}=-X; \qquad (4.1)$$

$$\frac{1}{B}\frac{\partial T_2}{\partial \beta}+\frac{1}{A}\frac{\partial S}{\partial \alpha}=-Y; \qquad (4.2)$$

$$\frac{1}{R}T_2-\frac{1}{A}\frac{\partial N_1}{\partial \alpha}-\frac{1}{B}\frac{\partial N_2}{\partial \beta}=Z; \qquad (4.3)$$

$$\frac{1}{A}\frac{\partial M_1}{\partial \alpha}+\frac{1}{B}\frac{\partial H}{\partial \beta}=N_1; \qquad (4.4)$$

$$\frac{1}{B}\frac{\partial M_2}{\partial \beta}+\frac{1}{A}\frac{\partial H}{\partial \alpha}=N_2; \qquad (4.5)$$

the elasticity relationships

---

*V.Z. Vlasov, Fundamental differential equations in the general theory of shells, PMM, Vol. VIII, No. 2, 1944.

**Kh. M. Mushtari, Some generalizations of the theory of thin shells, Izvestiya Fiz.-mat. ob-va pri Kazanskom universitete Vol. XI, series 8, 1938.

***V.V. Novozhilov, Teoriya tonkikh obolochek, Sudpromgiz, 1951, pp. 77-83.

****A.L. Gol'denveyzer, see pp. 250-251 of work cited in footnote *, p.203.

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{16}\omega + K_{11}\varkappa_1 + K_{12}\varkappa_2 + K_{16}\tau; \tag{4.6}$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + C_{26}\omega + K_{22}\varkappa_2 + K_{12}\varkappa_1 + K_{26}\tau; \tag{4.7}$$

$$S = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + K_{66}\tau + K_{16}\varkappa_1 + K_{26}\varkappa_2; \tag{4.8}$$

$$M_1 = D_{11}\varkappa_1 + D_{12}\varkappa_2 + D_{16}\tau + K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{16}\omega; \tag{4.9}$$

$$M_2 = D_{22}\varkappa_2 + D_{12}\varkappa_1 + D_{26}\tau + K_{22}\varepsilon_2 + K_{12}\varepsilon_1 + K_{26}\omega; \tag{4.10}$$

$$H = D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2 + K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\varepsilon_2. \tag{4.11}$$

where, as before, for the stiffnesses we have (1.18)-(1.20);

the geometric relationships

$$\left.\begin{gathered} \varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial\alpha}; \quad \varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{1}{R}w; \\ \omega = \frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}; \end{gathered}\right\} \tag{4.12}$$

$$\left.\begin{gathered} \varkappa_1 = -\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}; \quad \varkappa_2 = -\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}; \\ \tau = -\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}. \end{gathered}\right\} \tag{4.13}$$

Inserting the values of deformation from (4.12) and (4.13) into (2.4.2)-(2.4.4), for the stresses in the shell layers we obtain

$$\begin{aligned} \sigma^i_\alpha = {} & B^i_{11}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B^i_{12}\left(\frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{w}{R}\right) + B^i_{16}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) - \\ & - \gamma\left(B^i_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B^i_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B^i_{16}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right); \end{aligned} \tag{4.14}$$

$$\begin{aligned} \sigma^i_\beta = {} & B^i_{22}\left(\frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{w}{R}\right) + B^i_{12}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B^i_{26}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) - \\ & - \gamma\left(B^i_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B^i_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B^i_{26}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right); \end{aligned} \tag{4.15}$$

$$\begin{aligned} \tau^i_{\alpha\beta} = {} & B^i_{16}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B^i_{26}\left(\frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{w}{R}\right) + B^i_{66}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) - \\ & - \gamma\left(B^i_{16}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B^i_{26}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B^i_{66}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right). \end{aligned} \tag{4.16}$$

Inserting the values of deformation from (4.12) and (4.13) into (4.6)-(4.11), we obtain for the internal forces in displacements

$$\begin{aligned} T_1 = {} & \left(C_{11}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + C_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + \left(\frac{1}{R}C_{12} - K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2K_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - K_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \tag{4.17}$$

$$\begin{aligned} T_2 = {} & \left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + \left(\frac{1}{R}C_{22} - K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2K_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \tag{4.18}$$

$$\begin{aligned} S = {} & \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(C_{66}\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + \left(\frac{1}{R}C_{26} - K_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2K_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - K_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \tag{4.19}$$

$$\begin{aligned} M_1 = {} & \left(K_{11}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(K_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + K_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + \left(\frac{1}{R}K_{12} - D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2D_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \tag{4.20}$$

$$\begin{aligned} M_2 = {} & \left(K_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(K_{26}\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + K_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + \left(\frac{1}{R}K_{22} - D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2D_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \tag{4.21}$$

$$\begin{aligned} H = {} & \left(K_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(K_{66}\frac{1}{A}\frac{\partial}{\partial\alpha} + \right. \\ & \left. + K_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + \left(\frac{1}{R}K_{26} - D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \right. \\ & \left. - 2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} - D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w. \end{aligned} \tag{4.22}$$

Inserting the values of moments, M_1, M_2 and H from (4.20)-(4.22) into the fourth (4.4) and fifth (4.5) equilibrium equations, for the transverse forces in displacements we obtain

$$\begin{aligned} N_1 = {} & \left(K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2K_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + K_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)u + \\ & + \left[K_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + (K_{12}+K_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + K_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]v + \\ & + \frac{1}{R}\left(K_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + K_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)w - E_1(D_{jk})\,w; \end{aligned} \tag{4.23}$$

$$N_2 = \left(K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + 2K_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + K_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)v + \\ + \left[K_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + (K_{12}+K_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + K_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right]u + \\ + \frac{1}{R}\left(K_{22}\frac{1}{B}\frac{\partial}{\partial\beta} + K_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right)w - E_2(D_{jk})\,w. \tag{4.24}$$

where for third-order linear operators we have following notation:

$$E_1(D_{jk}) = D_{11}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3} + 3D_{16}\frac{1}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta} + \\ + (D_{12} + 2D_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2} + D_{26}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}; \tag{4.25}$$

$$E_2(D_{jk}) = D_{22}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3} + 3D_{26}\frac{1}{B^2A}\frac{\partial^3}{\partial\beta^2\,\partial\alpha} + \\ + (D_{12} + 2D_{66})\frac{1}{BA^2}\frac{\partial^3}{\partial\beta\,\partial\alpha^2} + D_{16}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3}. \tag{4.26}$$

Inserting the last derived values of internal forces T_1, T_2, S, N_1 and N_2 into the first three equilibrium equations (4.1)-(4.3), we obtain a symmetrical system of three differential equations in terms of the three sought functions $u(\alpha, \beta)$, $v(\alpha, \beta)$ and $w(\alpha, \beta)$

$$\left.\begin{aligned} &L_{11}(C_{jk})\,u + L_{12}(C_{jk})\,v + L_{13}(C_{jk}K_{jk})\,w = -X; \\ &L_{12}(C_{jk})\,u + L_{22}(C_{jk})\,v + L_{23}(C_{jk}K_{jk})\,w = -Y; \\ &L_{13}(C_{jk}K_{jk})\,u + L_{23}(C_{jk}K_{jk})\,v + L_{33}(C_{jk}K_{jk}D_{jk})\,w = Z, \end{aligned}\right\} \tag{4.27}$$

where the linear operator $L_{11}(C_{jk})$ remains as in (2.12), $L_{13}(C_{jk}K_{jk})$ remains as in (2.15) and $L_{33}(C_{jk}K_{jk}D_{jk})$ remains as in (2.17), while for the other operators we have

$$L_{12}(C_{jk}) = C_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + (C_{12} + C_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + C_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{4.28}$$

$$L_{22}(C_{jk}) = C_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2C_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + C_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{4.29}$$

$$L_{23}(C_{jk}K_{jk}) = \frac{1}{R}\left(C_{22}\frac{1}{B}\frac{\partial}{\partial\beta} + C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right) - E_2(K_{jk}). \tag{4.30}$$

Orthotropic Shell. In the case where the shell consists of an arbitrary number of orthotropic layers such that the principal directions of elasticity at each point of each layer coincide with the directions of the corresponding coordinate lines α, β and γ the design formulas and equations are somewhat simplified (see Section 3).

In this case: for coefficients B^i_{jk} we have (3.1); for the equilibrium equations we have (4.1)-(4.5); for the geometric relationships we have (4.12) and (4.13). In virtue of this, from (4.14)-(4.16) for the stresses: we obtain the following formulas:

$$\sigma^i_\alpha = B^i_{11}\frac{1}{A}\frac{\partial u}{\partial\alpha} + B^i_{12}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{w}{R}\right) - \gamma\left(B^i_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B^i_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}\right); \tag{4.31}$$

$$\sigma^i_\beta = B^i_{22}\left(\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{w}{R}\right) + B^i_{12}\frac{1}{A}\frac{\partial u}{\partial\alpha} - \gamma\left(B^i_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B^i_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}\right); \tag{4.32}$$

$$\tau^i_{\alpha\beta} = B^i_{66}\left(\frac{1}{B}\frac{\partial u}{\partial\beta}+\frac{1}{A}\frac{\partial v}{\partial\alpha}\right) - \gamma B^i_{66}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}. \tag{4.33}$$

From (4.17)-(4.24), in view of the fact that $C_{16} = C_{26} = K_{16} = K_{26} = D_{16} = D_{26} = 0$, for the internal forces and moments we have

$$T_1 = C_{11}\frac{1}{A}\frac{\partial u}{\partial\alpha} + C_{12}\frac{1}{B}\frac{\partial v}{\partial\beta} + \left(\frac{1}{R}C_{12} - K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - K_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w;$$

$$T_2 = C_{12}\frac{1}{A}\frac{\partial u}{\partial\alpha} + C_{22}\frac{1}{B}\frac{\partial v}{\partial\beta} + \left(\frac{1}{R}C_{22} - K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \tag{4.34}$$

$$S = C_{66}\frac{1}{B}\frac{\partial u}{\partial\beta} + C_{66}\frac{1}{A}\frac{\partial v}{\partial\alpha} - K_{66}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}. \tag{4.35}$$

$$M_1 = K_{11}\frac{1}{A}\frac{\partial u}{\partial\alpha} + K_{12}\frac{1}{B}\frac{\partial v}{\partial\beta} + \left(\frac{1}{R}K_{12} - D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \tag{4.36}$$

$$M_2 = K_{12}\frac{1}{A}\frac{\partial u}{\partial\alpha} + K_{22}\frac{1}{B}\frac{\partial v}{\partial\beta} + \left(\frac{1}{R}K_{22} - D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \tag{4.37}$$

(4.38)

$$H = K_{66}\frac{1}{B}\frac{\partial u}{\partial\beta} + K_{66}\frac{1}{A}\frac{\partial v}{\partial\alpha} - D_{66}\frac{2}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta};$$

(4.39)

$$N_1 = \left(K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + K_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)u + (K_{12}+K_{66})\frac{1}{AB}\frac{\partial^2 v}{\partial\alpha\,\partial\beta} + \frac{1}{R}K_{12}\frac{1}{A}\frac{\partial w}{\partial\alpha} - E_1(D_{jk})\,w; \tag{4.40}$$

$$N_2 = \left(K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + K_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)v + (K_{12}+K_{66})\frac{1}{AB}\frac{\partial^2 u}{\partial\alpha\,\partial\beta} + \frac{1}{R}K_{22}\frac{1}{B}\frac{\partial v}{\partial\beta} - E_2(D_{jk})\,w. \tag{4.41}$$

where for the third-order linear operator we have

$$\left.\begin{aligned} E_1(D_{jk}) &= D_{11}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3}+(D_{12}+2D_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2};\\ E_2(D_{jk}) &= D_{22}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}+(D_{12}+2D_{66})\frac{1}{BA^2}\frac{\partial^3}{\partial\alpha^2\,\partial\beta}. \end{aligned}\right\} \qquad (4.42)$$

Inserting the values of internal forces from (5.34)-(5.41) into equilibrium equations (5.1)-(5.3), we obtain a system of equations of solution which coincides externally with the system in (4.27), while for linear operators L_{jk} we have

$$L_{12}(C_{jk}) = (C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}; \qquad (4.43)$$

$$L_{22}(C_{jk}) = C_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+C_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \qquad (4.44)$$

$$L_{23}(C_{jk}K_{jk}) = \frac{1}{R}C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}-E_2(K_{jk}); \qquad (4.45)$$

$L_{11}(C_{jk})$ is the same as in (3.12), $L_{13}(C_{jk}K_{jk})$ is the same as in (3.15) and

$L_{33}(C_{jk}K_{jk}D_{jk})$ is the same as in (3.17).

Proceeding in the same manner as in the previous section, the system of equations in (4.27) may be reduced to a single eighth-order equation of solution for potential function $\Phi(\alpha, \beta)$, in terms of which we may represent all the design quantities of the problem. Proceeding from (3.19)-(3.49) and the basic premises of the engineering theory of cylindrical shells consisting of an arbitrary number of orthotropic layers, we obtain the basic equations and design formulas of the problem.

The eighth-order equation of solution in the case of engineering theory takes the following form:

$$\begin{aligned} &P_1\frac{1}{A^8}\frac{\partial^8\Phi}{\partial\alpha^8}+P_3\frac{1}{A^6B^2}\frac{\partial^8\Phi}{\partial\alpha^6\,\partial\beta^2}+P_5\frac{1}{A^4B^4}\frac{\partial^8\Phi}{\partial\alpha^4\,\partial\beta^4}+\\ &\quad+P_4\frac{1}{A^2B^6}\frac{\partial^8\Phi}{\partial\alpha^2\,\partial\beta^6}+P_2\frac{1}{B^8}\frac{\partial^8\Phi}{\partial\beta^8}+Q_1\frac{1}{A^6}\frac{\partial^6\Phi}{\partial\alpha^6}+\\ &\quad+Q_3\frac{1}{A^4B^2}\frac{\partial^6\Phi}{\partial\alpha^4\,\partial\beta^2}+Q_4\frac{1}{A^2B^4}\frac{\partial^6\Phi}{\partial\alpha^2\,\partial\beta^4}+R_1\frac{1}{A^4}\frac{\partial^4\Phi}{\partial\alpha^4}=0, \end{aligned} \qquad (4.46)$$

where for the coefficients of equation (4.46) we obtain from (3.32)-(3.40);

$$P_1 = C_{66}(C_{11}D_{11}-K_{11}^2), \quad P_2 = C_{66}(C_{22}D_{22}-K_{22}^2); \qquad (4.47)$$

$$\begin{aligned} P_3 = {} & D_{11}(\Omega-2C_{12}C_{66})+2C_{11}C_{66}(D_{12}+2D_{66})-K_{11}[C_{22}K_{11}-\\ & -2C_{12}(K_{12}+2K_{66})]-C_{11}(K_{12}+2K_{66})^2; \end{aligned} \qquad (4.48)$$

$$\begin{aligned} P_4 = {} & D_{22}(\Omega-2C_{12}C_{66})+2C_{22}C_{66}(D_{12}+2D_{66})-\\ & -(K_{12}+2K_{65})[C_{22}(K_{12}+2K_{66})-2C_{12}K_{22}]-C_{11}K_{22}^2; \end{aligned} \qquad (4.49)$$

$$\begin{aligned} P_5 = D_{11}C_{22}C_{66} + D_{22}C_{11}C_{66} + 2(D_{12} + 2D_{66})(\Omega - 2C_{12}C_{66}) - \\ - 2K_{11}[C_{22}(K_{12} + 2K_{66}) - K_{12}(C_{12} + C_{66})] - \\ - 2(K_{12} + 2K_{66})[C_{11}K_{22} - C_{12}(K_{12} + 2K_{66})]; \end{aligned} \tag{4.50}$$

$$Q_1 = \frac{2}{R} C_{66}(C_{12}K_{11} - C_{11}K_{12}); \tag{4.51}$$

$$Q_3 = \frac{2}{R} C_{66}\left[2\left(C_{12}K_{12} + K_{66}\frac{\Omega}{C_{66}}\right) - C_{22}K_{11} - C_{11}K_{22}\right]; \tag{4.52}$$

$$Q_4 = \frac{2}{R} C_{66}(C_{12}K_{22} - C_{22}K_{12}); \tag{4.53}$$

$$R_1 = \frac{1}{R^2} C_{66}(C_{11}C_{22} - C_{12}^2). \tag{4.54}$$

From (3.47)-(3.49), for the displacements, represented in terms of function $\Phi(\alpha, \beta)$, we obtain the following formulas:

$$\begin{aligned} u = C_{66}K_{11}\frac{1}{A^5}\frac{\partial^5\Phi}{\partial\alpha^5} + [C_{22}K_{11} - C_{12}(K_{12} + 2K_{66})]\frac{1}{A^3B^2}\frac{\partial^5\Phi}{\partial\alpha^3\partial\beta^2} + \\ + [C_{22}(K_{12} + 2K_{66}) - K_{22}(C_{12} + C_{66})]\frac{1}{AB^4}\frac{\partial^5\Phi}{\partial\alpha\partial\beta^4} - \\ - \frac{1}{R}C_{12}C_{66}\frac{1}{A^3}\frac{\partial^3\Phi}{\partial\alpha^3} + \frac{1}{R}C_{22}C_{66}\frac{1}{AB^2}\frac{\partial^3\Phi}{\partial\alpha\partial\beta^2} + u^*(\alpha, \beta) \end{aligned} \tag{4.55}$$

$$\begin{aligned} v = C_{66}K_{22}\frac{1}{B^5}\frac{\partial^5\Phi}{\partial\beta^5} + [C_{11}K_{22} - C_{12}(K_{12} + 2K_{66})]\frac{1}{A^2B^3}\frac{\partial^5\Phi}{\partial\alpha^2\partial\beta^3} + \\ + [C_{11}(K_{12} + 2K_{66}) - K_{11}(C_{12} + C_{66})]\frac{1}{A^4B}\frac{\partial^5\Phi}{\partial\alpha^4\partial\beta} - \\ - \frac{1}{R}C_{22}C_{66}\frac{1}{B^3}\frac{\partial^3\Phi}{\partial\beta^3} - \frac{1}{R}(\Omega - C_{12}C_{66})\frac{1}{A^2B}\frac{\partial^3\Phi}{\partial\alpha^2\partial\beta} + v^*(\alpha, \beta); \end{aligned} \tag{4.56}$$

$$\begin{aligned} w = C_{11}C_{66}\frac{1}{A^4}\frac{\partial^4\Phi}{\partial\alpha^4} + C_{22}C_{66}\frac{1}{B^4}\frac{\partial^4\Phi}{\partial\beta^4} + \\ + (\Omega - 2C_{12}C_{66})\frac{1}{A^2B^2}\frac{\partial^4\Phi}{\partial\alpha^2\partial\beta^2} + w^*(\alpha, \beta), \end{aligned} \tag{4.57}$$

where, as usual, $\Omega = C_{11}C_{22} - C_{12}^2$.

Having the values of displacements u, v and w, by means of formulas (4.14)-(4.24) we may easily write the formulas for the internal forces and stresses.

SECTION 5. CONTINUATION OF SECTION 4.

The equations for the engineering theory of anisotropic laminar cylindrical shells may be presented in the form of equations of the combined method in the theory of statically indeterminate systems.*,**,*** Assuming

---

*V.Z. Vlasov, see pp. 301-310 of work cited in footnote *, p. 201

**S.A. Ambartsumyan, Toward calculation of laminar anisotropic shells, Izvestiya AN Arm SSR (FMYeiT nauki), Vol. VI, No. 3, 1953.

***M. Mishonov, Toward a theory of shallow shells, PMM, Vol. XXII, No. 5, 1958.

$$T_1 = \frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - A\int X\,d\alpha, \tag{5.1}$$

$$T_2 = \frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - B\int Y\,d\beta; \tag{5.2}$$

$$S = -\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}, \tag{5.3}$$

where $\varphi = \varphi(\alpha, \beta)$ is the sought function of stresses, we identically satisfy the first two equilibrium equations (4.1) and (4.2), and from the third equation (4.3) we obtain

$$\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \frac{B}{R}\int Y\,d\beta - \frac{1}{A}\frac{\partial N_1}{\partial\alpha} - \frac{1}{B}\frac{\partial N_2}{\partial\beta} = Z. \tag{5.4}$$

Solving the elasticity relationships (4.6)-(4.8) for the deformation components, we obtain

$$\varepsilon_1 = A_{11}T_1 + A_{12}T_2 + A_{16}S - d_{11}\varkappa_1 - d_{12}\varkappa_2 - d_{16}\tau; \tag{5.5}$$

$$\varepsilon_2 = A_{22}T_2 + A_{12}T_1 + A_{26}S - d_{21}\varkappa_1 - d_{22}\varkappa_2 - d_{26}\tau; \tag{5.6}$$

$$\omega = A_{16}T_1 + A_{26}T_2 + A_{66}S - d_{61}\varkappa_1 - d_{62}\varkappa_2 - d_{66}\tau, \tag{5.7}$$

where as coefficients characterizing the elastic properties of the shell we have

$$\left.\begin{aligned}
A_{11} &= \frac{C_{22}C_{66} - C_{26}^2}{\Omega^*}; & A_{22} &= \frac{C_{11}C_{66} - C_{16}^2}{\Omega^*};\\
A_{12} &= \frac{C_{16}C_{26} - C_{12}C_{66}}{\Omega^*}; & A_{66} &= \frac{C_{11}C_{22} - C_{12}^2}{\Omega^*};\\
A_{16} &= \frac{C_{12}C_{26} - C_{16}C_{22}}{\Omega^*}; & A_{26} &= \frac{C_{12}C_{16} - C_{26}C_{11}}{\Omega^*};
\end{aligned}\right| \tag{5.8}$$

$$\Omega^* = (C_{11}C_{22} - C_{12}^2)C_{66} + 2C_{12}C_{16}C_{26} - C_{11}C_{26}^2 - C_{22}C_{16}^2; \tag{5.9}$$

$$d_{11} = A_{11}K_{11} + A_{12}K_{12} + A_{16}K_{16}; \tag{5.10}$$

$$d_{22} = A_{12}K_{12} + A_{22}K_{22} + A_{26}K_{26}; \tag{5.11}$$

$$d_{66} = A_{16}K_{16} + A_{26}K_{26} + A_{66}K_{66}; \tag{5.12}$$

$$d_{12} = A_{11}K_{12} + A_{12}K_{22} + A_{16}K_{26}; \tag{5.13}$$

$$d_{21} = A_{12}K_{11} + A_{22}K_{12} + A_{26}K_{16}; \tag{5.14}$$

$$d_{16} = A_{11}K_{16} + A_{12}K_{26} + A_{16}K_{66}; \tag{5.15}$$

$$d_{61} = A_{16}K_{11} + A_{26}K_{12} + A_{66}K_{16}; \tag{5.16}$$

$$d_{26} = A_{12}K_{16} + A_{22}K_{26} + A_{26}K_{66}; \tag{5.17}$$

$$d_{62} = A_{16}K_{12} + A_{26}K_{22} + A_{66}K_{26}. \tag{5.18}$$

From the elasticity relationships in (4.9)-(4.11), by means of (5.5)-(5.7), eliminating the deformation components ε_1, ε_2, ω, and considering (5.1)-(5.3) and (4.13), we obtain for the moments

$$M_1=\left(\frac{d_{11}}{B^2}\frac{\partial^2}{\partial\beta^2}+\frac{d_{21}}{A^2}\frac{\partial^2}{\partial\alpha^2}-\frac{d_{61}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi-$$

$$-\left(\frac{D_{11}-D_{11}^0}{A^2}\frac{\partial^2}{\partial\alpha^2}+2\frac{D_{16}-D_{16}^0}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+\frac{D_{12}-D_{12}^0}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w+M_1^*; \tag{5.19}$$

$$M_2=\left(\frac{d_{22}}{A^2}\frac{\partial^2}{\partial\alpha^2}+\frac{d_{12}}{B^2}\frac{\partial^2}{\partial\beta^2}-\frac{d_{62}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi-$$

$$-\left(\frac{D_{22}-D_{22}^0}{B^2}\frac{\partial^2}{\partial\beta^2}+2\frac{D_{26}-D_{26}^0}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+\frac{D_{12}-D_{12}^0}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)w+M_2^*; \tag{5.20}$$

$$H=\left(\frac{d_{16}}{B^2}\frac{\partial^2}{\partial\beta^2}+\frac{d_{26}}{A^2}\frac{\partial^2}{\partial\alpha^2}-\frac{d_{66}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi-$$

$$-\left(\frac{D_{16}-D_{16}^0}{A^2}\frac{\partial^2}{\partial\alpha^2}+2\frac{D_{66}-D_{66}^0}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+\frac{D_{26}-D_{26}^0}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w+M_6^*. \tag{5.21}$$

where

$$M_j^*=-\left(d_{1j}A\int X\,d\alpha+d_{2j}B\int Y\,d\beta\right)\qquad (j=1,\ 2,\ 6). \tag{5.22}$$

The formulas for the moments (5.19)-(5.21) contain the derived bending stiffnesses D_{jk}^0, originating from the "tension-compression and shear" stiffness (C_{jk}) and the "interaction" stiffness (K_{jk}), and have the following form:

$$D_{11}^0=K_{11}d_{11}+K_{12}d_{21}+K_{16}d_{61}; \tag{5.23}$$

$$D_{22}^0=K_{12}d_{12}+K_{22}d_{22}+K_{26}d_{62}; \tag{5.24}$$

$$D_{66}^0=K_{16}d_{16}+K_{26}d_{26}+K_{66}d_{66}; \tag{5.25}$$

$$D_{12}^0=K_{11}d_{12}+K_{12}d_{22}+K_{16}d_{62}=K_{12}d_{11}+K_{22}d_{21}+K_{26}d_{61}; \tag{5.26}$$

$$D_{16}^0=K_{11}d_{16}+K_{12}d_{26}+K_{16}d_{66}=K_{16}d_{11}+K_{26}d_{21}+K_{66}d_{61}; \tag{5.27}$$

$$D_{26}^0=K_{12}d_{16}+K_{22}d_{26}+K_{26}d_{66}=K_{16}d_{12}+K_{26}d_{22}+K_{66}d_{62}. \tag{5.28}$$

From the last two equilibrium equations (4.4) and (4.5), considering the values of moments (5.19)-(5.21), we obtain for the transverse forces

$$N_1=\left(\frac{d_{21}}{A^3}\frac{\partial^3}{\partial\alpha^3}+\frac{d_{26}-d_{61}}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta}+\frac{d_{11}-d_{66}}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}+\frac{d_{16}}{B^3}\frac{\partial^3}{\partial\beta^3}\right)\varphi-$$

$$-E_1\left(D_{jk}-D_{jk}^0\right)w+\frac{1}{A}\frac{\partial M_1^*}{\partial\alpha}+\frac{1}{B}\frac{\partial M_6^*}{\partial\beta}; \tag{5.29}$$

$$N_2=\left(\frac{d_{12}}{B^3}\frac{\partial^3}{\partial\beta^3}+\frac{d_{16}-d_{62}}{AB^2}\frac{\partial^2}{\partial\alpha\,\partial\beta^2}+\frac{d_{22}-d_{66}}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta}+\frac{d_{26}}{A^3}\frac{\partial^3}{\partial\alpha^3}\right)\varphi-$$

$$-E_2\left(D_{jk}-D_{jk}^0\right)w+\frac{1}{B}\frac{\partial M_2^*}{\partial\beta}+\frac{1}{A}\frac{\partial M_6^*}{\partial\alpha}. \tag{5.30}$$

where for linear operators $E_j(D^0_{jk} - D_{jk})$ we have (4.25) and (4.26).

Inserting the values of transverse forces N_1 and N_2 from (5.29) and (5.30) into the third equilibrium equation, which after the transformation has the form of (5.4), we obtain

$$L_1(D_{jk} - D^0_{jk})\,w - L_3(d_{jk})\,\varphi + \frac{1}{R}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} =$$
$$= Z + \frac{b}{R}\int Y\,d\beta + \frac{1}{A^2}\frac{\partial^2 M_1^*}{\partial\alpha^2} + \frac{2}{AB}\frac{\partial^2 M_6^*}{\partial\alpha\,\partial\beta} + \frac{1}{B^2}\frac{\partial^2 M_2^*}{\partial\beta^2}, \tag{5.31}$$

where for the linear operators we have

$$L_1(D_{jk} - D^0_{jk}) = \frac{D_{11} - D^0_{11}}{A^4}\frac{\partial^4}{\partial\alpha^4} + 4\,\frac{D_{16} - D^0_{16}}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta} +$$
$$+ 2\,\frac{(D_{12} - D^0_{12}) + 2(D_{66} - D^0_{66})}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} +$$
$$+ 4\,\frac{D_{26} - D^0_{26}}{AB^3}\frac{\partial^4}{\partial\alpha\,\partial\beta^3} + \frac{D_{22} - D^0_{22}}{B^4}\frac{\partial^4}{\partial\beta^4}; \tag{5.32}$$

$$L_3(d_{jk}) = \frac{d_{21}}{A^4}\frac{\partial^4}{\partial\alpha^4} + \frac{2d_{26} - d_{61}}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta} + \frac{d_{11} + d_{22} - 2d_{66}}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} +$$
$$+ \frac{2d_{16} - d_{62}}{AB^3}\frac{\partial^4}{\partial\alpha\,\partial\beta^3} + \frac{d_{12}}{B^4}\frac{\partial^4}{\partial\beta^4}. \tag{5.33}$$

Thus, after certain transformations from the equilibrium equations (4.1)-(4.5) we have arrived at a single equation with two unknown functions $\varphi = \varphi(\alpha, \beta)$ and $w = w(\alpha, \beta)$. The second equation is required in obtaining the complete system of differential equations in terms of the above-mentioned unknown functions and may be obtained on the basis of the third equation of compatibility of deformation (1.24). The first two equations of compatibility of deformation (2.31) and (2.32) are identically satisfied with an accuracy of the additional assumptions of the engineering theory of shells.

Inserting into formulas (5.5)-(5.7) the values of T_1, T_2 and S from (5.1)-(5.3) and $\varkappa_1$, $\varkappa_2$, and τ from (4.13), we obtain for the deformation components ε_1, ε_2, and ω.

$$\varepsilon_1 = \left(\frac{A_{11}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{A_{12}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{A_{16}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi +$$
$$+ \left(\frac{d_{11}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{12}}{B^2}\frac{\partial^2}{\partial\beta^2} + 2\,\frac{d_{16}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)w + \varepsilon_1^*; \tag{5.34}$$

$$\varepsilon_2 = \left(\frac{A_{12}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{A_{22}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{A_{26}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi +$$
$$+ \left(\frac{d_{21}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{22}}{B^2}\frac{\partial^2}{\partial\beta^2} + 2\,\frac{d_{26}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)w + \varepsilon_2^*; \tag{5.35}$$

$$\omega = \left(\frac{A_{16}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{A_{26}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{A_{66}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi + $$
$$+ \left(\frac{d_{61}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{62}}{B^2}\frac{\partial^2}{\partial\beta^2} + 2\frac{d_{66}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)w + \varepsilon_6^*, \qquad (5.36)$$

where

$$\varepsilon_j^* = -\left(A_{1j}A\int X\,d\alpha + A_{2j}B\int Y\,d\beta\right) \qquad (j = 1,\ 2,\ 6). \qquad (5.37)$$

Inserting the values of the deformation components from (4.13) and (5.34)-(5.37) into equation of compatibility (1.24), we obtain the required second equation of the problem.

$$L_2(A_{jk})\varphi + L_3(d_{jk})w - \frac{1}{R}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} =$$
$$= -\frac{1}{A^2}\frac{\partial^2\varepsilon_2^*}{\partial\alpha^2} + \frac{1}{AB}\frac{\partial^2\varepsilon_6^*}{\partial\alpha\,\partial\beta} - \frac{1}{B^2}\frac{\partial^2\varepsilon_1^*}{\partial\beta^2}, \qquad (5.38)$$

where for the new linear operator we have

$$L_2(A_{jk}) = \frac{A_{22}}{A^4}\frac{\partial^4}{\partial\alpha^4} - 2\frac{A_{26}}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta} +$$
$$+ \frac{2A_{12} + A_{66}}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} - 2\frac{A_{16}}{AB^3}\frac{\partial^4}{\partial\alpha\,\partial\beta^3} + \frac{A_{11}}{B^4}\frac{\partial^4}{\partial\beta^4}. \qquad (5.39)$$

Thus, the general problem of the statics of a cylindrical shell consisting of an arbitrary number of anisotropic homogeneous layers in the engineering theory of shells reduces to a system of two differential equations of solution (5.31) and (5.38) in terms of the two sought functions: $\varphi(\alpha, \beta)$ the stress functions, and $w(\alpha, \beta)$ the displacement functions, means of which (using formulas (5.1)-(5.3), (5.19)-(5.20), (5.29), (5.30), (5.34)-(5.36) and (2.4.2)-(2.4.4)) all the design quantities of the problem are determined.

In view of the unwieldiness of the final formulas for the design quantities in the general case of anisotropy we do not derive them here; these formulas are obtained by elementary procedures.

Orthotropic Shell. In the case where the shell consists of an arbitrary number of homogeneous orthotropic layers such that the principal directions of elasticity at each point of each layer coincide with the directions of the corresponding coordinate lines, for the elastic constants of each layer

B^i_{jk} as is known, we have (3.1); moreover, as was stated in Section 3 of this chapter, $C_{16} = C_{26} = K_{16} = K_{26} = D_{16} = D_{26} = 0$. Then, from (5.8-(5.18) and (5.23)-(5.28), for A_{jk}, d_{jk} and D^0_{jk} we obtain (for coefficients differing from zero)

$$\left.\begin{aligned} A_{11} &= \frac{C_{22}}{\Omega}, \quad A_{22} = \frac{C_{11}}{\Omega}, \quad A_{66} = \frac{1}{C_{66}}; \\ A_{12} &= -\frac{C_{12}}{\Omega}, \quad \Omega = C_{11}C_{22} - C_{12}^2; \end{aligned}\right\} \tag{5.40}$$

$$d_{11} = \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}, \quad d_{22} = \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}; \tag{5.41}$$

$$d_{12} = \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}, \quad d_{21} = \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}; \tag{5.42}$$

$$d_{66} = \frac{K_{66}}{C_{66}}; \tag{5.43}$$

$$D_{11}^0 = \frac{K_{11}^2 C_{22} - 2K_{11}K_{12}C_{12} + K_{12}^2 C_{11}}{\Omega}; \tag{5.44}$$

$$D_{12}^0 = \frac{K_{11}K_{12}C_{22} - (K_{11}K_{22} + K_{12}^2)\, C_{12} + K_{22}K_{12}C_{11}}{\Omega}; \tag{5.45}$$

$$D_{22}^0 = \frac{K_{22}^2 C_{11} - 2K_{11}K_{12}C_{12} + K_{12}^2 C_{22}}{\Omega}; \tag{5.46}$$

$$D_{66}^0 = \frac{K_{66}^0}{C_{66}}. \tag{5.47}$$

Then the system of equations of solution (5.31) and (5.38) takes the form

$$\left.\begin{aligned} &L_2(C_{jk})\,\varphi + L_3(d_{jk})\, w - \frac{1}{R}\,\frac{1}{A^2}\,\frac{\partial^2 w}{\partial\alpha^2} = -\frac{1}{A^2}\,\frac{\partial^2 \varepsilon_2^*}{\partial\alpha^2} - \frac{1}{B^2}\,\frac{\partial^2 \varepsilon_1^*}{\partial\beta^2}; \\ &L_1(D_{jk} - D_{jk}^0)\, w - L_3(d_{jk})\,\varphi + \frac{1}{R}\,\frac{1}{A^2}\,\frac{\partial^2\varphi}{\partial\alpha^2} = \\ &\qquad = Z + \frac{B}{R}\int Y\, d\beta + \frac{1}{A^2}\,\frac{\partial^2 M_1^*}{\partial\alpha^2} + \frac{1}{B^2}\,\frac{\partial^2 M_2^*}{\partial\beta^2}, \end{aligned}\right\} \tag{5.48}$$

where for the linear operators and load terms we have

$$\begin{aligned} L_1(D_{jk} - D_{jk}^0) = {} & \frac{D_{11} - D_{11}^0}{A^4}\,\frac{\partial^4}{\partial\alpha^4} + \frac{D_{22} - D_{22}^0}{B^4}\,\frac{\partial^4}{\partial\beta^4} + \\ & + 2\,\frac{(D_{12} - D_{12}^0) + 2\,(D_{66} - D_{66}^0)}{A^2B^2}\,\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}; \end{aligned} \tag{5.49}$$

$$L_2(C_{jk}) = \frac{C_{11}}{\Omega}\,\frac{1}{A^4}\,\frac{\partial^4}{\partial\alpha^4} + \frac{C_{22}}{\Omega}\,\frac{1}{B^4}\,\frac{\partial^4}{\partial\beta^4} + \left(\frac{1}{C_{66}} - 2\,\frac{C_{12}}{\Omega}\right)\frac{1}{A^2B^2}\,\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}; \tag{5.50}$$

$$\begin{aligned} L_3(d_{jk}) = {} & \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\,\frac{1}{A^4}\,\frac{\partial^4}{\partial\alpha^4} + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\,\frac{K_{66}}{C_{66}} + \right. \\ & \left. + \frac{K_{11}C_{22} - K_{12}C_{22}}{\Omega}\right)\frac{1}{A^2B^2}\,\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\,\frac{1}{B^4}\,\frac{\partial^4}{\partial\beta^4}; \end{aligned} \tag{5.51}$$

$$M_1^* = -\frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\, A\int X\, d\alpha - \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\, B\int Y\, d\beta; \tag{5.52}$$

$$M_2^* = -\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}\, B\int Y\, d\beta - \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\, A\int X\, d\alpha; \tag{5.53}$$

$$\varepsilon_1^* = -\frac{C_{22}}{\Omega}\, A\int X\, d\alpha + \frac{C_{12}}{\Omega}\, B\int Y\, d\beta; \tag{5.54}$$

$$\varepsilon_2^* = -\frac{C_{11}}{\Omega} B \int Y\, d\beta + \frac{C_{12}}{\Omega} A \int X\, d\alpha. \tag{5.55}$$

The design formulas for the internal forces and stresses in the layers of the shell in this case are given by the following formulas.

The bending and torsional moments from (5.19)-(5.21)

$$\begin{aligned} M_1 = &-\left(D_{11} - D_{11}^0\right) \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} - \left(D_{12} - D_{12}^0\right) \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} + \\ &+ \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} + M_1^*; \end{aligned} \tag{5.56}$$

$$\begin{aligned} M_2 = &-\left(D_{22} - D_{22}^0\right) \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} - \left(D_{12} - D_{12}^0\right) \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + \\ &+ \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} + \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + M_2^*; \end{aligned} \tag{5.57}$$

$$H = -2\left(D_{66} - D_{66}^0\right) \frac{1}{AB} \frac{\partial^2 w}{\partial \alpha\, \partial \beta} - \frac{K_{66}}{C_{66}} \frac{1}{AB} \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta}. \tag{5.58}$$

The tangential forces T_1, T_2 and S are defined in formulas (5.1)-(5.3).

From (2.4.2)-(2.4.4), in virtue of (3.1), (5.34), (5.36) and (5.40)-(5.43), we obtain for the design stresses in the layers of the shell

$$\begin{aligned} \sigma_\alpha^i = &\left(B_{11}^i \frac{C_{22}}{\Omega} - B_{12}^i \frac{C_{12}}{\Omega}\right) \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} - \left(B_{11}^i \frac{C_{12}}{\Omega} - B_{12}^i \frac{C_{11}}{\Omega}\right) \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + \\ &+ \left(B_{11}^i \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} + B_{12}^i \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^i\right) \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} + \\ &+ \left(B_{11}^i \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} + B_{12}^i \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} - \gamma B_{11}^i\right) \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + \\ &+ B_{11}^i \varepsilon_1^* + B_{12}^i \varepsilon_2^*; \end{aligned} \tag{5.59}$$

$$\begin{aligned} \sigma_\beta^i = &\left(B_{22}^i \frac{C_{11}}{\Omega} - B_{12}^i \frac{C_{12}}{\Omega}\right) \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} - \left(B_{22}^i \frac{C_{12}}{\Omega} - B_{12}^i \frac{C_{22}}{\Omega}\right) \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} + \\ &+ \left(B_{22}^i \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} + B_{12}^i \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^i\right) \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + \\ &+ \left(B_{22}^i \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} + B_{12}^i \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} - \gamma B_{22}^i\right) \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} + \\ &+ B_{22}^i \varepsilon_2^* + B_{12}^i \varepsilon_1^*; \end{aligned} \tag{5.60}$$

$$\tau_{\alpha\beta}^i = -\frac{B_{66}^i}{C_{66}} \frac{1}{AB} \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta} + 2B_{66}^i \left(\frac{K_{66}}{C_{66}} - \gamma\right) \frac{1}{AB} \frac{\partial^2 w}{\partial \alpha\, \partial \beta}. \tag{5.61}$$

Thus, formulas (5.1)-(5.3), (5.56)-(5.58) and (5.59)-(5.61) are used in representing all the design quantities of the problem by means of the unknown functions $w(\alpha, \beta)$ and $\varphi(\alpha, \beta)$, which with given boundary conditions may be determined from the system of differential equations of solution in (5.48)

In the general case of an anisotropic laminar shell all the design quantities of the problem depend both on the stress function $\varphi(\alpha, \beta)$, and on the displacement function $w(\alpha, \beta)$.

The formulas for determining the tangential displacements u and v will not be presented here, since in the case of the combined problem they are of no particular interest; however, they may be defined by means of formulas (4.12).

Of considerable practical interest is the case in which the shell is loaded only by a normally applied surface load, that is, when $X = Y = 0,\ Z \neq 0$. Then the system of equations of solution (5.48) is rewritten as:

$$\left.\begin{aligned} &L_2(C_{jk})\varphi + L_3(d_{jk})w - \frac{1}{R}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} = 0;\\ &L_1\left(D_{jk} - D^0_{jk}\right)w - L_3(d_{jk})\varphi + \frac{1}{R}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} = Z.\end{aligned}\right\} \tag{5.62}$$

The system in (5.62) may be reduced to a single eighth-order differential equation of solution in terms of the potential function $\Psi(\alpha, \beta)$.*

Assuming**

$$\left.\begin{aligned} &w = L_2(C_{jk})\Psi;\\ &\varphi = \frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2} - L_3(d_{jk})\Psi,\end{aligned}\right\} \tag{5.63}$$

we identically satisfy the first equation of the system in (5.62) and from the second equation for the sought function $\Psi(\alpha, \beta)$ we obtain one eighth-order linear differential equation

$$L_0(P_{jk})\Psi - 2F(Q_{jk})\Psi + \frac{1}{R^2}\frac{1}{A^4}\frac{\partial^4\Psi}{\partial\alpha^4} = Z. \tag{5.64}$$

where for the linear operators we have

$$\begin{aligned} L_0(P_{jk}) = {} & P_1\frac{1}{A^8}\frac{\partial^8}{\partial\alpha^8} + P_3\frac{1}{A^6B^2}\frac{\partial^8}{\partial\alpha^6\,\partial\beta^2} + \\ & + P_5\frac{1}{A^4B^4}\frac{\partial^8}{\partial\alpha^4\,\partial\beta^4} + P_4\frac{1}{A^2B^6}\frac{\partial^8}{\partial\alpha^2\,\partial\beta^6} + P_2\frac{1}{B^8}\frac{\partial}{\partial\beta^8}; \end{aligned} \tag{5.65}$$

$$F(Q_{jk}) = Q_1\frac{1}{A^6}\frac{\partial^6}{\partial\alpha^6} + Q_3\frac{1}{A^4B^2}\frac{\partial^6}{\partial\alpha^4\,\partial\beta^2} + Q_4\frac{1}{A^2B^4}\frac{\partial^6}{\partial\alpha^2\,\partial\beta^4} \tag{5.66}$$

---

*V.Z. Vlasov, see p. 319 of work cited in footnote *, on p. 201.
**S.A. Ambartsumyan, see footnote **, on p. 214.

For the coefficients of these operators we have

$$P_1 = (D_{11} - D_{11}^0)\frac{C_{11}}{\Omega} + \left(\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\right)^2;$$

$$P_2 = (D_{22} - D_{22}^0)\frac{C_{22}}{\Omega} + \left(\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\right)^2;$$

$$P_3 = 2[(D_{12} - D_{12}^0) + 2(D_{66} - D_{66}^0)]\frac{C_{11}}{\Omega} + (D_{11} - D_{11}^0)\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) +$$
$$+ 2\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right);$$

$$P_4 = 2[(D_{12} - D_{12}^0) + 2(D_{66} - D_{66}^0)]\frac{C_{22}}{\Omega} + (D_{22} - D_{22}^0)\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) +$$
$$+ 2\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right);$$

$$P_5 = (D_{11} - D_{11}^0)\frac{C_{22}}{\Omega} + 2[(D_{12} - D_{12}^0) + 2(D_{66} - D_{66}^0)]\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) +$$
$$+ (D_{22} - D_{22}^0)\frac{C_{11}}{\Omega} + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right)^2 +$$
$$+ 2\frac{(K_{12}C_{11} - K_{11}C_{12})(K_{12}C_{22} - K_{22}C_{12})}{\Omega^2};$$

$$Q_1 = \frac{1}{R}\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}, \quad Q_4 = \frac{1}{R}\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega};$$

$$Q_3 = \frac{1}{R}\left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right).$$

Representing the design quantities of the problem in terms of the sought function $\Psi(\alpha, \beta)$, we obtain:

For the tangential forces

$$T_1 = \left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\frac{1}{B^2}\frac{\partial^2\Psi}{\partial\beta^2}; \tag{5.67}$$

$$T_2 = \left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2}; \tag{5.68}$$

$$S = -\left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\frac{1}{AB}\frac{\partial^2\Psi}{\partial\alpha\,\partial\beta}; \tag{5.69}$$

for the moments

$$M_1 = -\left[(D_{11} - D_{11}^0)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + (D_{12} - D_{12}^0)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]L_2(C_{jk})\Psi +$$
$$+ \left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\left[\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} +\right.$$
$$\left. + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]\Psi; \tag{5.70}$$

$$M_2 = -\left[(D_{22} - D_{22}^0)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + (D_{12} - D_{12}^0)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right]L_2(C_{jk})\Psi +$$
$$+\left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\left[\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} +\right.$$
$$\left.+\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right]\Psi; \qquad (5.71)$$

$$H = -2(D_{66} - D_{66}^0)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(C_{jk})\Psi -$$
$$-\frac{K_{66}}{C_{66}}\left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}; \qquad (5.72)$$

for the stresses in the shell layers

$$\sigma_\alpha^l = \left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\left[\left(B_{11}^l\frac{C_{22}}{\Omega} - B_{12}^l\frac{C_{12}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} -\right.$$
$$\left.-\left(B_{11}^l\frac{C_{12}}{\Omega} - B_{12}^l\frac{C_{11}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right]\Psi + \left[\left(B_{11}^l\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} +\right.\right.$$
$$\left.+B_{12}^l\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^l\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + \left(B_{11}^l\frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} +\right.$$
$$\left.\left.+B_{12}^l\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} - \gamma B_{11}^l\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right]L_2(C_{jk})\Psi; \qquad (5.73)$$

$$\sigma_\beta^l = \left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\left[\left(B_{22}^l\frac{C_{11}}{\Omega} - B_{12}^l\frac{C_{12}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} -\right.$$
$$\left.-\left(B_{22}^l\frac{C_{12}}{\Omega} - B_{12}^l\frac{C_{22}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]\Psi + \left[\left(B_{22}^l\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} +\right.\right.$$
$$\left.+B_{12}^l\frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^l\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \left(B_{22}^l\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} +\right.$$
$$\left.\left.+B_{12}^l\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} - \gamma B_{22}^l\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right]L_2(C_{jk})\Psi; \qquad (5.74)$$

$$\tau_{\alpha\beta}^l = -\frac{B_{66}^l}{C_{66}}\left[\frac{1}{R}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - L_3(d_{jk})\right]\frac{1}{AB}\frac{\partial^2\Psi}{\partial\alpha\,\partial\beta} +$$
$$+2B_{66}^l\left(\frac{K_{66}}{C_{66}} - \gamma\right)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(C_{jk})\Psi; \qquad (5.75)$$

for the tangential displacements

$$u = \frac{K_{11}}{\Omega}\frac{1}{A^5}\frac{\partial^5\Psi}{\partial\alpha^5} + \frac{1}{C_{66}\Omega}\left[C_{22}K_{11} - C_{12}(K_{12} + 2K_{66})\right]\frac{1}{A^3B^2}\frac{\partial^5\Psi}{\partial\alpha^3\,\partial\beta^2} +$$
$$+\frac{1}{C_{66}\Omega}[C_{22}(K_{12} + 2K_{66}) - K_{22}(C_{12} + C_{66})]\frac{1}{AB^4}\frac{\partial^5\Psi}{\partial\alpha\,\partial\beta^4} -$$
$$-\frac{1}{R}\frac{C_{12}}{\Omega}\frac{1}{A^3}\frac{\partial^3\Psi}{\partial\alpha^3} + \frac{1}{R}\frac{C_{22}}{\Omega}\frac{1}{AB^2}\frac{\partial^3\Psi}{\partial\alpha\,\partial\beta^2}; \qquad (5.76)$$

$$v = \frac{K_{22}}{\Omega}\frac{1}{B^5}\frac{\partial^5\Psi}{\partial\beta^5} + \frac{1}{C_{66}\Omega}[C_{11}K_{22} - C_{12}(K_{12} + 2K_{66})]\frac{1}{A^2B^3}\frac{\partial^5\Psi}{\partial\alpha^2\,\partial\beta^3} +$$
$$+\frac{1}{C_{66}\Omega}[C_{11}(K_{12} + 2K_{66}) - K_{11}(C_{12} + C_{66})]\frac{1}{A^4B}\frac{\partial^5\Psi}{\partial\alpha^4\,\partial\beta} -$$
$$-\frac{1}{R}\frac{C_{22}}{\Omega}\frac{1}{B^3}\frac{\partial^3\Psi}{\partial\beta^3} - \frac{1}{R}\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right)\frac{1}{A^2B}\frac{\partial^3\Psi}{\partial\alpha^2\,\partial\beta}. \qquad (5.77)$$

Comparing equations of solution (4.36) and (5.64) as well as their corresponding design formulas (4.14)-(4.24), (4.55)-(4.57) and (5.67)-(5.63), it is easily seen that they are fundamentally identical and coincide in accuracy if instead of the sought $\Psi(\alpha, \beta)$ we insert $C_{66}\Omega\Phi(\alpha, \beta)$, or, conversely, instead of the sought $\Phi(\alpha, \beta)$ we insert $\Psi(\alpha, \beta)/C_{66}\Omega$.

SECTION 6. CYLINDRICAL SHELLS CONSISTING OF AN ODD NUMBER OF HOMOGENEOUS ANISOTROPIC LAYERS SYMMETRICALLY ARRANGED RELATIVE TO THE MEDIAN SURFACE

In this and the next two sections we discuss shells consisting of an odd number (2m + 1) of layers (Figure 15).

By virtue of symmetry we have for the stiffnesses (Chapter II, Section 13).

$$C_{jk}=2\left[B_{jk}^{m+1}h_{m+1}+\sum_{s=1}^{m}B_{jk}^{s}(h_s-h_{s-1})\right]; \tag{6.1}$$

$$D_{jk}=\frac{2}{3}\left[B_{jk}^{m+1}h_{m+1}^{3}+\sum_{s=1}^{m}B_{jk}^{s}\left(h_s^3-h_{s-1}^3\right)\right]; \tag{6.2}$$

$$K_{jk}=0. \tag{6.3}$$

The elasticity relationships become simplified and take the form

$$\left.\begin{aligned}&T_1=C_{11}\varepsilon_1+C_{12}\varepsilon_2+C_{16}\omega, \quad T_2=C_{22}\varepsilon_2+C_{12}\varepsilon_1+C_{26}\omega;\\&S_{12}=C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2+\frac{1}{R}(D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2);\\&S_{21}=C_{66}\omega+C_{16}\varepsilon_1+C_{26}\varepsilon_2;\end{aligned}\right\} \tag{6.4}$$

$$\left.\begin{aligned}M_1=D_{11}\varkappa_1+D_{12}\varkappa_2+D_{16}\tau; \quad M_2=D_{22}\varkappa_2+D_{12}\varkappa_1+D_{26}\tau\\H=D_{66}\tau+D_{16}\varkappa_1+D_{26}\varkappa_2.\end{aligned}\right\} \tag{6.5}$$

The formulas for determining the stresses (1.21)-(1.23), the equilibrium equations (1.23)-(1.8) and the equation of compatibility of deformation (1.24), as well as the formulas relating the components of deformation and the changes in curvature with the displacement components (1.9)-(1.10) remain unchanged.

From (2.1)-(2.7) for the internal forces and moment represented in terms of displacements u, v, w we obtain

$$\left.\begin{aligned}T_1&=\left(C_{11}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+C_{12}\frac{w}{R};\\T_2&=\left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+C_{22}\frac{w}{R};\\S_{12}&=\left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left[\left(C_{66}+\frac{2}{R^2}D_{66}\right)\frac{1}{A}\frac{\partial}{\partial\alpha}+\right.\end{aligned}\right\} \tag{6.7}$$

$$\begin{aligned}&\left.+\left(C_{26}+\frac{1}{R^2}D_{26}\right)\frac{1}{B}\frac{\partial}{\partial\beta}\right]v+\left(C_{26}-D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-\right.\\&\left.-2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)\frac{w}{R};\end{aligned} \tag{6.8}$$

$$S_{21}=\left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(C_{66}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+C_{26}\frac{w}{R}; \tag{6.9}$$

$$M_1=\left(2D_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+D_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)\frac{v}{R}-\left(D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+\right.$$
$$\left.+2D_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \tag{6.10}$$

$$M_2=\left(2D_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}+D_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)\frac{v}{R}-\left(D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+\right.$$
$$\left.+2D_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \tag{6.11}$$

$$H=\left(2D_{66}\frac{1}{A}\frac{\partial}{\partial\alpha}+D_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)\frac{v}{R}-\left(D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+\right.$$
$$\left.+2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w. \tag{6.12}$$

Then the system of differential equilibrium equations in displacements (2.11) is rewritten as follows:

$$\left.\begin{aligned}
&L_{11}(C_{jk})u+L_{12}(C_{jk})v+L_{13}(C_{jk})w=-X;\\
&L_{21}(C_{jk})u+L_{22}(C_{jk}D_{jk})v+L_{23}(C_{jk}D_{jk})w=-Y;\\
&L_{31}(C_{jk})u+L_{32}(C_{jk}D_{jk})v+L_{33}(C_{jk}D_{jk})w=Z,
\end{aligned}\right\} \tag{6.13}$$

where the linear operators L_{jk} have the following form (for completeness we rewrite L_{11}):

$$L_{11}(C_{jk})=C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+2C_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{6.14}$$

$$L_{22}(C_{jk}D_{jk})=\left(C_{66}+\frac{4}{R^2}D_{66}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+\left(2C_{26}+\frac{4}{R^2}D_{26}\right)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+$$
$$+\left(C_{22}+\frac{1}{R^2}D_{22}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{6.15}$$

$$L_{12}(C_{jk})=C_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+C_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{6.16}$$

$$L_{13}(C_{jk})=\frac{1}{R}\left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right); \tag{6.17}$$

$$L_{23}(C_{jk}D_{jk})=\frac{1}{R}\left(C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}+C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right)-\frac{1}{R}\left[2D_{16}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3}+\right.$$
$$\left.+(D_{12}+4D_{66})\frac{1}{A^2B}\frac{\partial^3}{\partial\alpha^2\,\partial\beta}+4D_{26}\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}+D_{22}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}\right]; \tag{6.18}$$

$$L_{33}(C_{jk}D_{jk})=\frac{1}{R^2}C_{22}+D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4}+4D_{16}\frac{1}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta}+$$
$$+2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+4D_{26}\frac{1}{B^3A}\frac{\partial^4}{\partial\beta^3\,\partial\alpha}+D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4}; \tag{6.19}$$

$$L_{21}=L_{12},\quad L_{31}=L_{13},\quad L_{32}=L_{23}. \tag{6.20}$$

Orthotropic Shell. In the case where the shell consists of orthotropic layers so that the principal directionsof elasticity at each point of each layer coincide with the directions of the corresponding lines α, β, γ (see Section 3), the design formulas and equations, as has already been stated, are considerably simplified. In this case, by virtue of (3.1), for the stresses we obtain formulas (3.2)-(3.4) and for the stiffnesses with subscripts 16 and 26 we evidently have

$$C_{16}=C_{26}=0, \quad D_{16}=D_{26}=0;$$

then from (3.5)-(3.11) for the internal forces and moments we obtain

$$T_1=C_{11}\frac{1}{A}\frac{\partial u}{\partial \alpha}+C_{12}\frac{1}{B}\frac{\partial v}{\partial \beta}+C_{12}\frac{w}{R}; \tag{6.21}$$

$$T_2=C_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha}+C_{22}\frac{1}{B}\frac{\partial v}{\partial \beta}+C_{22}\frac{w}{R}; \tag{6.22}$$

$$S_{12}=C_{66}\frac{1}{B}\frac{\partial u}{\partial \beta}+\left(C_{66}+\frac{2}{R^2}D_{66}\right)\frac{1}{A}\frac{\partial v}{\partial \alpha}-\frac{2}{R}D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}; \tag{6.23}$$

$$S_{21}=C_{66}\frac{1}{B}\frac{\partial u}{\partial \beta}+C_{66}\frac{1}{A}\frac{\partial v}{\partial \alpha}; \tag{6.24}$$

$$M_1=\frac{1}{R}D_{12}\frac{1}{B}\frac{\partial v}{\partial \beta}-D_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}-D_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}; \tag{6.25}$$

$$M_2=\frac{1}{R}D_{22}\frac{1}{B}\frac{\partial v}{\partial \beta}-D_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}-D_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}; \tag{6.26}$$

$$H=\frac{2}{R}D_{66}\frac{1}{A}\frac{\partial v}{\partial \alpha}-2D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}. \tag{6.27}$$

The system of equations of solution does not differ externally from the system in (6.13); however, for the linear operators L_{jk} we have the following values:

$$L_{11}(C_{jk})=C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial \alpha^2}+C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}; \tag{6.28}$$

$$L_{22}(C_{jk}D_{jk})=\left(C_{66}+\frac{4}{R^2}D_{66}\right)\frac{1}{A^2}\frac{\partial^2}{\partial \alpha^2}+\left(C_{22}+\frac{1}{R^2}D_{22}\right)\frac{1}{B^2}\frac{\partial^2}{\partial \beta^2}; \tag{6.29}$$

$$L_{12}(C_{jk})=(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2}{\partial \alpha\,\partial \beta}; \tag{6.30}$$

$$L_{13}(C_{jk})=\frac{1}{R}C_{12}\frac{1}{A}\frac{\partial}{\partial \alpha}; \tag{6.31}$$

$$L_{23}(C_{jk}D_{jk})=\frac{1}{R}C_{22}\frac{1}{B}\frac{\partial}{\partial \beta}-\frac{1}{R}\left[D_{22}\frac{1}{B^3}\frac{\partial^3}{\partial \beta^3}+(D_{12}+4D_{66})\frac{1}{BA^2}\frac{\partial^3}{\partial \beta\,\partial \alpha^2}\right]; \tag{6.32}$$

$$L_{33}(C_{jk}D_{jk})=\frac{1}{R^2}C_{22}+D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial \alpha^4}+2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial \alpha^2\,\partial \beta^2}+D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial \beta^4}, \tag{6.33}$$

$$L_{21}=L_{12}, \quad L_{31}=L_{13}, \quad L_{32}=L_{23}. \tag{6.34}$$

As before, the problem may be reduced to a single eighth-order equation of solution relative to the potential function $\Phi(\alpha, \beta)$. This equation, as was to be expected, has the form of (3.51); however, for coefficients P_j, Q_j and R_j we have the following values:

$$P_1 = G_2C_{11}D_{11}; \qquad P_2 = C_{66}C_{22}D_{22}; \tag{6.35}$$

$$P_3 = D_{11}(F_2C_{11} + G_2C_{66} - V^2) + 2G_2C_{11}(D_{12} + 2D_{66}) - C_{11}W^2; \tag{6.36}$$

$$P_4 = D_{22}(F_2C_{11} + G_2C_{66} - V^2) + 2F_2C_{66}(D_{12} + 2D_{66}) - \\ - \frac{2}{R}C_{66}D_{22}W - \frac{1}{R^2}C_{11}D_{22}^2; \tag{6.37}$$

$$P_5 = F_2C_{66}D_{11} + G_2C_{11}D_{22} + 2(D_{12} + 2D_{66})(F_2C_{11} + G_2C_{66} - V^2) - \\ - C_{66}W^2 - \frac{2}{R}C_{11}D_{22}W; \tag{6.38}$$

$$Q_1 = 0, \qquad Q_2 = \frac{2}{R^2}C_{66}C_{22}D_{22}; \tag{6.39}$$

$$Q_3 = \frac{2}{R}(C_{11}C_{22}W - C_{12}VW); \tag{6.40}$$

$$Q_4 = \frac{2}{R}\left[C_{22}\left(C_{66}W + \frac{1}{R}C_{11}D_{22}\right) - \frac{1}{R}C_{12}D_{22}V\right]; \tag{6.41}$$

$$R_1 = \frac{1}{R^4}G_2(C_{11}C_{22} - C_{12}^2); \tag{6.42}$$

$$R_2 = \frac{1}{R^2}C_{66}C_{22}(F_2 - C_{22}); \tag{6.43}$$

$$R_3 = \frac{1}{R^2}\left[(C_{11}C_{22} - C_{12}^2)F_2 + C_{22}(G_2C_{66} - V^2) + \right. \\ \left. + C_{22}(2C_{12}V - C_{11}C_{22})\right], \tag{6.44}$$

where for F_2, G_2, V and W we have

$$\left.\begin{aligned} F_2 &= C_{22} + \frac{D_{22}}{R^2}, & G_2 &= C_{66} + \frac{4D_{66}}{R^2}; \\ V &= C_{12} + C_{66}, & W &= \frac{D_{12} + 4D_{66}}{R}. \end{aligned}\right\} \tag{6.45}$$

Finally, from (3.47)-(3.49) for the displacements (expressed in terms of function $\Phi(\alpha, \beta)$) we obtain

$$u = -VW\frac{1}{B^2A^3}\frac{\partial^5\Phi}{\partial\alpha^3\partial\beta^2} - \frac{1}{R}D_{22}V\frac{1}{AB^4}\frac{\partial^5\Phi}{\partial\alpha\partial\beta^4} - \\ - \frac{1}{R}G_2C_{12}\frac{1}{A^3}\frac{\partial^3\Phi}{\partial\alpha^3} + \frac{1}{R}(C_{22}V - C_{12}F_2)\frac{1}{AB^2}\frac{\partial^3\Phi}{\partial\alpha\partial\beta^2}; \tag{6.46}$$

$$v = \frac{1}{R}C_{66}D_{22}\frac{1}{B^5}\frac{\partial^5\Phi}{\partial\beta^5} + \left(C_{66}W + \frac{1}{R}C_{11}D_{22}\right)\frac{1}{A^2B^3}\frac{\partial^5\Phi}{\partial\alpha^2\partial\beta^3} + \\ + C_{11}W\frac{1}{A^4B}\frac{\partial^5\Phi}{\partial\alpha^4\partial\beta} - \frac{1}{R}C_{66}C_{22}\frac{1}{B^3}\frac{\partial^3\Phi}{\partial\beta^3} + \\ + \frac{1}{R}(C_{12}V - C_{11}C_{22})\frac{1}{A^2B}\frac{\partial^3\Phi}{\partial\alpha^2\partial\beta}; \tag{6.47}$$

$$w = G_2C_{11}\frac{1}{A^4}\frac{\partial^4\Phi}{\partial\alpha^4} + (F_2C_{11} + G_2C_{66} - V^2)\frac{1}{A^2B^2}\frac{\partial^4\Phi}{\partial\alpha^2\partial\beta^2} + \\ + F_2C_{66}\frac{1}{B^4}\frac{\partial^4\Phi}{\partial\beta^4}. \tag{6.48}$$

Having the values of displacements, by means of formulas (3.2)-(3.4) and (6.14)-(6.20) we may easily write the formulas for the stresses in the shell layers and for the internal forces.

SECTION 7. ENGINEERING THEORY OF CYLINDRICAL SHELLS CONSISTING OF AN ODD NUMBER OF HOMOGENEOUS ANISOTROPIC LAYERS SYMMETRICALLY ARRANGED RELATIVE TO THE MEDIAN SURFACE*

Let us proceed from the general premises of the fourth section of this chapter, assuming the shell consists of an odd number (2m + 1) of layers.

In this case we have for the stiffnesses (6.1)-(6.3); for internal forces T_1, T_2 and moments M_1, M_2 and H we have (6.4) and (6.5). For the internal forces S, N_1 and N_2, from (4.8), (4.23) and (4.24), we obtain

$$S = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2; \tag{7.1}$$

$$N_1 = -E_1(D_{jk})\,w, \quad N_2 = -E_2(D_{jk})\,w. \tag{7.2}$$

where for operators E_j we have (4.25) and (4.26).

Also unchanged in the proposed engineering theory are the formulas for determining the stresses (4.14)-(4.16), the equilibrium equations (4.1)-(4.5), the equation of compatibility of deformation (1.24) and geometric relationships (4.12) and (4.13).

For the internal forces and moments in displacements we obtain

$$\begin{aligned} T_1 = {} & \left(C_{11}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \\ & + \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + C_{12}\frac{w}{R}; \end{aligned} \tag{7.3}$$

$$\begin{aligned} T_2 = {} & \left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \\ & + \left(C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + C_{22}\frac{w}{R}; \end{aligned} \tag{7.4}$$

$$\begin{aligned} S = {} & \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \\ & + \left(C_{66}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + C_{26}\frac{w}{R}; \end{aligned} \tag{7.5}$$

$$M_1 = -\left(D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2D_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \tag{7.6}$$

$$M_2 = -\left(D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + 2D_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)w; \tag{7.7}$$

$$H = -\left(2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w. \tag{7.8}$$

---

*S.A. Ambartsumyan, Calculation of shallow cylindrical shells consisting of anisotropic layers, Izvestiya AN ArmSSR (FMYeiT nauki) Vol. IV, No. 5, 1951.

The system of differential equations of solution in displacements has the following form:

$$\left.\begin{aligned} L_{11}(C_{jk})u+L_{12}(C_{jk})v+L_{13}(C_{jk})w&=-X;\\ L_{12}(C_{jk})u+L_{22}(C_{jk})v+L_{23}(C_{jk})w&=-Y;\\ L_{13}(C_{jk})u+L_{23}(C_{jk})v+L_{33}(C_{jk}D_{jk})w&=Z.\end{aligned}\right\} \tag{7.9}$$

where for the linear operators we have (the values of all operators are rewritten from the previous sections):

$$L_{11}(C_{jk})=C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+2C_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{7.10}$$

$$L_{22}(C_{jk})=C_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+2C_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+C_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{7.11}$$

$$L_{12}(C_{jk})=C_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+C_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{7.12}$$

$$L_{13}(C_{jk})=\frac{1}{R}\left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right); \tag{7.13}$$

$$L_{23}(C_{jk})=\frac{1}{R}\left(C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}+C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right); \tag{7.14}$$

$$\begin{aligned} L_{33}(C_{jk}D_{jk})={}&\frac{1}{R^2}C_{22}+D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4}+4D_{16}\frac{1}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta}+\\ &+2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+4D_{26}\frac{1}{B^3A}\frac{\partial^4}{\partial\alpha\,\partial\beta^3}+\\ &+D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4}.\end{aligned} \tag{7.15}$$

With the given boundary conditions and solving system (7.9), let us determine the unknown functions u, v and w which, by use of formulas (4.14)-(4.16) and (7.2)-(7.8), will permit us to determine all the design stresses and internal forces of the problem.

In the subsequent discussion we shall also be interested in the shearing stresses $\tau^i_{\alpha\gamma}$ and $\tau^i_{\beta\gamma}$, which we may determine by the use of formulas (2.4.7) and (2.4.8). Performing substitutions and applying the conditions at the contact surfaces of adjacent layers (2.5.2) and at the outer surfaces of the shell (2.5.3) and (2.5.4), we obtain for the sought stresses:*

For the layers from the lower first to the middle (m + 1)-th layer, inclusively, that is, for $i=1, 2, 3, \ldots, (m+1)$ (Figure 15)

---

*S.A. Ambartsumyan, Anizotropnyye sloistyye obolochki (dissertation), 1951, pp. 101-105.

$$\tau^i_{\alpha\gamma} = \frac{\gamma^2}{2} E_1(B^i_{jk})w - \gamma\left[L_{11}(B^i_{jk})u + L_{12}(B^i_{jk})v + L_{13}(B^i_{jk})w\right] -$$

$$- \frac{h_0^2}{2} E_1(P^i_{jk})w - h_0\left[L_{11}(R^i_{jk})u + L_{12}(R^i_{jk})v + L_{13}(R^i_{jk})w\right]; \qquad (7.16)$$

$$\tau^i_{\beta\gamma} = \frac{\gamma^2}{2} E_2(B^i_{jk})w - \gamma\left[L_{22}(B^i_{jk})v + L_{12}(B^i_{jk})u + L_{23}(B^i_{jk})w\right] -$$

$$- \frac{h_0^2}{2} E_2(P^i_{jk})w - h_0\left[L_{22}(R^i_{jk})v + L_{12}(R^i_{jk})u + L_{23}(R^i_{jk})w\right]; \qquad (7.17)$$

For the layers from the upper (2m + 1)-th to the middle (m + 1)-th layer, inclusively, that is, for $i = (m+1),\ (m+2),\ \ldots,\ (2m+1)$ (Figure 15).

$$\tau^i_{\alpha\gamma} = \frac{\gamma^2}{2} E_1(B^{2m+2-i}_{jk})w - \gamma\left[L_{11}(B^{2m+2-i}_{jk})u +\right.$$

$$+ L_{12}(B^{2m+2-i}_{jk})v + L_{13}(B^{2m+2-i}_{jk})w\Big] - \frac{h_0^2}{2} E_1(P^{2m+2-i}_{jk})w +$$

$$+ h_0\left[L_{11}(R^{2m+2-i}_{jk})u + L_{12}(R^{2m+2-i}_{jk})v + L_{13}(R^{2m+2-i}_{jk})w\right] + X\ ; \qquad (7.18)$$

$$\tau^i_{\beta\gamma} = \frac{\gamma^2}{2} E_2(B^{2m+2-i}_{jk})w - \gamma\left[L_{22}(B^{2m+2-i}_{jk})v +\right.$$

$$+ L_{12}(B^{2m+2-i}_{jk})u + L_{23}(B^{2m+2-i}_{jk})w\Big] - \frac{h_0^2}{2} E_2(P^{2m+2-i}_{jk})w +$$

$$+ h_0\left[L_{22}(R^{2m+2-i}_{jk})v + L_{12}(R^{2m+2-i}_{jk}) + L_{23}(R^{2m+2-i}_{jk})w\right] + Y\ ; \qquad (7.19)$$

in these formulas

$$R^i_{jk} = \frac{1}{h_0}\left[h_i B^i_{jk} + \sum_{s=1}^{i-1} B^s_{jk}(h_s - h_{s+1})\right], \quad R^1_{jk} = B^1_{jk}; \qquad (7.20)$$

$$P^i_{jk} = \frac{1}{h_0^2}\left[h_i^2 B^i_{jk} + \sum_{s=1}^{i-1} B^s_{jk}(h_s^2 - h_{s+1}^2)\right], \quad P^1_{jk} = B^1_{jk}. \qquad (7.21)$$

In deriving formulas (7.16)-(7.19) it was assumed that the outer tangential forces (X and Y) were applied only to the upper surface ($\gamma = +h$) of the shell.

Having the values of shearing stresses $\tau^i_{\alpha\gamma}$ and $\tau^i_{\beta\gamma}$, we may for the case of the discussed cylindrical shell confirm the result derived in Section 11, Chapter II. From (7.16) and (7.17) at the median surface we obtain the following values for the shearing stresses:

$$\tau^{m+1}_{\alpha\gamma} = -\frac{h^2}{2} E_1(P^{m+1}_{jk})w -$$

$$- h\left[L_{11}(R^{m+1}_{jk})u + L_{12}(R^{m+1}_{jk})v + L_{13}(R^{m+1}_{jk})w\right],$$

$$\tau^{m+1}_{\beta\gamma} = -\frac{h^2}{2} E_2(P^{m+1}_{jk})w -$$

$$- h\left[L_{22}(R^{m+1}_{jk})v + L_{12}(R^{m+1}_{jk})u + L_{23}(R^{m+1}_{jk})w\right].$$

From (7.18) and (7.19), by virtue of (2.7.1) and (2.7.2) we obtain for the same components of stress in the median surface

$$\tau_{\alpha\gamma}^{m+1} = -\frac{h^2}{2} E_1 (P_{jk}^{m+1}) w + \\ + h \left[L_{11}(R_{jk}^{m+1}) u + L_{12}(R_{jk}^{m+1}) v + L_{13}(R_{jk}^{m+1}) w \right];$$

$$\tau_{\beta\gamma}^{m+1} = -\frac{h^2}{2} E_2 (P_{jk}^{m+1}) w + \\ + h \left[L_{22}(R_{jk}^{m+1}) v + L_{12}(R_{jk}^{m+1}) u + L_{23}(R_{jk}^{m+1}) w \right].$$

Comparing the corresponding expressions for one and the same stress and, on the basis of (6.1) and (7.20), considering that $2hR_{jk}^{m+1} = C_{jk}$, we obtain

$$L_{11}(C_{jk}) u + L_{12}(C_{jk}) v + L_{13}(C_{jk}) w = -X;$$
$$L_{12}(C_{jk}) u + L_{22}(C_{jk}) v + L_{23}(C_{jk}) w = -Y.$$

As was to be expected, the derived equations do not differ from the first two equations of the system in (7.9), that is, from the equilibrium equations for the entire shell in displacements of its median surface.

Orthotropic Shell. In the case where the shell consists of homogeneous orthotropic layers so that the principal directions of elasticity at each point of each layer coincide with the direction of the corresponding coordinate lines α, β and γ for coefficients B^i_{jk} we have (3.1); for the equilibrium equations we have (4.1)-(4.5); for the geometric relationships we have (4.12)-(4.13); for the stiffnesses we have (6.1)-(6.3) as well as $C_{16} = C_{26} = D_{16} = D_{26} = 0$.

In this case, for the internal forces and moments we have from (7.2)-(7.8)

$$T_1 = C_{11} \frac{1}{A} \frac{\partial u}{\partial \alpha} + C_{12} \frac{1}{B} \frac{\partial v}{\partial \beta} + C_{12} \frac{w}{R}; \tag{7.22}$$

$$T_2 = C_{12} \frac{1}{A} \frac{\partial u}{\partial \alpha} + C_{22} \frac{1}{B} \frac{\partial v}{\partial \beta} + C_{22} \frac{w}{R}; \tag{7.23}$$

$$S = C_{66} \frac{1}{B} \frac{\partial u}{\partial \beta} + C_{66} \frac{1}{A} \frac{\partial v}{\partial \alpha}; \tag{7.24}$$

$$M_1 = -\left(D_{11} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + D_{12} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} \right) w; \tag{7.25}$$

$$M_2 = -\left(D_{12} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + D_{22} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} \right) w; \tag{7.26}$$

$$H = -D_{66} \frac{2}{AB} \frac{\partial^2 w}{\partial \alpha \, \partial \beta}; \tag{7.27}$$

$$N_1 = -E_1(D_{jk}) w, \quad N_2 = -E_2(D_{jk}) w \tag{7.28}$$

where, as usual in the case of an orthotropic shell, for the linear operators $E_1(D_{jk})$ and $E_2(D_{jk})$ we have (4.42).

The system of differential equations of solution in displacements has the form (6.9), where for the linear operators

$$L_{11}(C_{jk}) = C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{7.29}$$

$$L_{22}(C_{jk}) = C_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + C_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \tag{7.30}$$

$$L_{12}(C_{jk}) = (C_{12} + C_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}; \tag{7.31}$$

$$L_{13}(C_{jk}) = \frac{1}{R}C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha}, \quad L_{23}\left(C_{jk}\right) = \frac{1}{R}C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}; \tag{7.32}$$

$$\begin{aligned} L_{33}(C_{jk}D_{jk}) = {} & \frac{1}{R^2}C_{22} + D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4} + \\ & + 2(D_{12} + 2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4}. \end{aligned} \tag{7.33}$$

Proceeding as in the previous sections, the system of equations in (6.9) may be reduced to a single eighth-order equation of solution in terms of potential function (α, β). For symmetrically assembled anisotropic shells we derive this equation in the next section.

SECTION 8. CONTINUATION OF SECTION 7

The equations for engineering theory of cylindrical shells consisting of an odd number of homogeneous anisotropic layers cylindrically arranged relative to the median surface of the shell may be represented in the form of the equations of the combined method in the theory of statically indeterminate systems.*

In the case of symmetrically assembled laminar shells, as is known, all stiffnesses K_{jk} are equal to (6.3). Then, from (5.10)-(5.18) and (5.23)-(5.28) we have for all coefficients d_{jk} and D^0_{jk}

$$d_{jk} = 0, \quad D^0_{jk} = 0, \tag{8.1}$$

and for the internal forces moments, from (5.1)-(5.3), (5.19)-(5.21) and (5.29)-(5.30), we obtain the following expressions:

---

*S.A. Ambartsumyan, see footnote on p. 229.

$$T_1 = \frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - A\int X\,d\alpha; \tag{8.2}$$

$$T_2 = \frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - B\int Y\,d\beta; \tag{8.3}$$

$$S = -\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}; \tag{8.4}$$

$$M_1 = -D_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} - D_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} - 2D_{16}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}; \tag{8.5}$$

$$M_2 = -D_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} - D_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} - 2D_{26}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}; \tag{8.6}$$

$$H = -2D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - D_{16}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} - D_{26}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}; \tag{8.7}$$

$$N_1 = -E_1(D_{jk})\,w, \quad N_2 = -E_2(D_{jk})\,w. \tag{8.8}$$

From (2.4.2)-(2.4.4), using (5.34)-(5.36), (4.13) and (8.1) we obtain for the stresses in the shell layers

$$\begin{aligned}\sigma_1^i = {} & \left(B_{11}^i A_{12} + B_{12}^i A_{22} + B_{16}^i A_{26}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} + \\ & + \left(B_{11}^i A_{11} + B_{12}^i A_{12} + B_{16}^i A_{16}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - \\ & - \left(B_{11}^i A_{16} + B_{12}^i A_{26} + B_{16}^i A_{56}\right)\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - \\ & - \gamma\left(B_{11}^i\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B_{12}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + 2B_{16}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right) + \\ & + B_{11}^i\overset{*}{\varepsilon}_1 + B_{12}^i\overset{*}{\varepsilon}_2 + B_{16}^i\overset{*}{\varepsilon}_6;\end{aligned} \tag{8.9}$$

$$\begin{aligned}\sigma_2^i = {} & \left(B_{22}^i A_{12} + B_{12}^i A_{11} + B_{26}^i A_{16}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} + \\ & + \left(B_{22}^i A_{22} + B_{12}^i A_{12} + B_{26}^i A_{26}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \\ & - \left(B_{22}^i A_{26} + B_{12}^i A_{16} + B_{26}^i A_{66}\right)\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - \\ & - \gamma\left(D_{22}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + B_{12}^i\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + 2B_{26}^i\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right) + \\ & + B_{22}^i\overset{*}{\varepsilon}_2 + B_{12}^i\overset{*}{\varepsilon}_1 + B_{26}^i\overset{*}{\varepsilon}_6;\end{aligned} \tag{8.10}$$

$$\begin{aligned}\tau_{12}^i = {} & \left(B_{16}^i A_{11} + B_{26}^i A_{12} + B_{66}^i A_{16}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} + \\ & + \left(B_{16}^i A_{12} + B_{26}^i A_{22} + B_{66}^i A_{26}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \\ & - \left(B_{16}^i A_{16} + B_{26}^i A_{26} + B_{66}^i A_{66}\right)\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - \\ & - \gamma\left(B_{16}^i\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B_{26}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + 2B_{66}^i\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right) + \\ & + B_{16}^i\overset{*}{\varepsilon}_1 + B_{26}^i\overset{*}{\varepsilon}_2 + B_{66}^i\overset{*}{\varepsilon}_6.\end{aligned} \tag{8.11}$$

The system of differential equations of solution in (5.31) and (5.38) becomes somewhat simplified and takes the following form:

$$\left.\begin{aligned} L_2(A_{jk})\varphi-\frac{1}{R}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} &= -\frac{1}{A^2}\frac{\partial^2\varepsilon_2^*}{\partial\alpha^2}+\frac{1}{AB}\frac{\partial^2\varepsilon_c^*}{\partial\alpha\,\partial\beta}-\frac{1}{B^2}\frac{\partial^2\varepsilon_1^*}{\partial\beta^2}: \\ L_1(D_{jk})w+\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} &= Z+\frac{B}{R}\int Y\,d\beta, \end{aligned}\right\} \quad (8.12)$$

where for the linear operator L_1 (D_{jk}), evidently,

$$\begin{aligned} L_1(D_{jk}) = {} & \frac{D_{11}}{A^4}\frac{\partial^4}{\partial\alpha^4}+4\frac{D_{16}}{A^3B}\frac{\partial^4}{\partial\alpha^3\,\partial\beta}+ \\ & +2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+4\frac{D_{26}}{AB^3}\frac{\partial^4}{\partial\alpha\,\partial\beta^3}+\frac{D_{22}}{B^4}\frac{\partial^4}{\partial\beta^4}, \end{aligned} \quad (8.13)$$

while for linear operator L_2 (A_{jk}) we have (5.39).

When the shell is loaded only by a normally applied surface load (that is, when X = Y = O, Z ≠ O) the system of equations of solution in (8.12), as with system (5.6), is rewritten as follows:

$$\left.\begin{aligned} L_2(A_{jk})\varphi-\frac{1}{R}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} &= 0; \\ L_1(D_{jk})w+\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} &= Z. \end{aligned}\right\} \quad (8.14)$$

In a corresponding manner we also simplify the design formulas which contain load components X and Y.

As is known, the system of equations (8.14) may be reduced to a single eighth-order differential equation of solution in terms of potential function $\Psi(\alpha, \beta)$.

By analogy with (5.63), assuming

$$w=L_2(A_{jk})\Psi, \quad \varphi=\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2}, \quad (8.15)$$

we identically satisfy the first equation of system (8.14) and from the second equation we obtain

$$L_1(D_{jk})L_2(A_{jk})\Psi+\frac{1}{R^2}\frac{1}{A^4}\frac{\partial^4\Psi}{\partial\alpha^4}=Z, \quad (8.16)$$

which is an eighth-order differential equation of solution in terms of the sought function $\Psi(\alpha, \beta)$.

The principal operator of equation (8.16) has the following expanded form:

$$
\begin{aligned}
L_1(D_{jk})L_2(A_{jk}) = {} & D_{11}A_{22}\frac{1}{A^8}\frac{\partial^8}{\partial\alpha^8} + 2(2A_{22}D_{16} - D_{11}A_{26})\frac{1}{A^7B}\frac{\partial^8}{\partial\alpha^7\partial\beta} + \\
& + [D_{11}(2A_{12} + A_{66}) - 8D_{16}A_{26} + 2(D_{12} + 2D_{66})A_{22}]\frac{1}{A^6B^2}\frac{\partial^8}{\partial\alpha^6\partial\beta^2} + \\
& + [4D_{16}(2A_{12} + A_{66}) + 4D_{26}A_{22} - 2D_{11}A_{16} - 4(D_{12} + 2D_{66})A_{26}] \times \\
& \times \frac{1}{A^5B^3}\frac{\partial^8}{\partial\alpha^5\partial\beta^3} + [D_{11}A_{11} - 8D_{16}A_{16} + 2(D_{12} + 2D_{66})(2A_{12} + A_{66}) - \\
& - 8D_{26}A_{26} + D_{22}A_{22}]\frac{1}{A^4B^4}\frac{\partial^8}{\partial\alpha^4\partial\beta^4} + [4D_{26}(2A_{12} + A_{66}) + \\
& + 4D_{16}A_{11} - 2D_{22}A_{26} - 4(D_{12} + 2D_{66})A_{16}]\frac{1}{A^3B^5}\frac{\partial^8}{\partial\alpha^3\partial\beta^5} + \\
& + [D_{22}(2A_{12} + A_{66}) - 8D_{26}A_{16} + 2(D_{12} + 2D_{66})A_{11}]\frac{1}{A^2B^6}\frac{\partial^8}{\partial\alpha^2\partial\beta^6} + \\
& + 2(2D_{26}A_{11} - D_{22}A_{16})\frac{1}{AB^7}\frac{\partial^8}{\partial\alpha\partial\beta^7} + D_{22}A_{11}\frac{1}{B^8}\frac{\partial^8}{\partial\beta^8}.
\end{aligned}
\tag{8.17}
$$

By virtue of (8.15), from (8.2)-(8.11) we obtain for the internal forces and principle design stresses within the shell layers

$$
\left.
\begin{aligned}
T_1 = \frac{1}{R}\frac{1}{A^2B^2}\frac{\partial^4\Psi}{\partial\alpha^2\partial\beta^2}, \quad T_2 = \frac{1}{R}\frac{1}{A^4}\frac{\partial^4\Psi}{\partial\alpha^4}, \\
S = -\frac{1}{R}\frac{1}{A^3B}\frac{\partial^4\Psi}{\partial\alpha^3\partial\beta};
\end{aligned}
\right\}
\tag{8.18}
$$

$$M_1 = -\left(D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + 2D_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)L_2(A_{jk})\Psi; \tag{8.19}$$

$$M_2 = -\left(D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2D_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)L_2(A_{jk})\Psi; \tag{8.20}$$

$$H = -\left(2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta} + D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)L_2(A_{jk})\Psi; \tag{8.21}$$

$$N_1 = -E_1(D_{jk})L_2(A_{jk})\Psi, \quad N_2 = -E_2(D_{jk})L_2(A_{jk})\Psi; \tag{8.22}$$

$$
\begin{aligned}
\sigma_1^l = {} & \left[\left(B_{11}^lA_{12} + B_{12}^lA_{22} + B_{16}^lA_{26}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \right. \\
& + (B_{11}^lA_{11} + B_{12}^lA_{12} + B_{16}^lA_{16})\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} - \\
& \left. - (B_{11}^lA_{16} + B_{12}^lA_{26} + B_{16}^lA_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right]\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \\
& - \gamma\left(B_{11}^l\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + B_{12}^l\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + 2B_{16}^l\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)L_2(A_{jk})\Psi;
\end{aligned}
\tag{8.23}
$$

$$\begin{aligned}\sigma_\beta^l = \Big[&\left(B_{22}^l A_{12} + B_{12}^l A_{11} + B_{26}^l A_{16}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + \\ &+\left(B_{22}^l A_{22} + B_{12}^l A_{12} + B_{26}^l A_{26}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \\ &-\left(B_{22}^l A_{26} + B_{12}^l A_{16} + B_{26}^l A_{56}\right)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\Big]\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \\ &-\gamma\left(B_{22}^l\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + B_{12}^l\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2B_{26}^l\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)L_2(A_{jk})\Psi;\end{aligned} \qquad (8.24)$$

$$\begin{aligned}\tau_{\alpha\beta}^l = \Big[&\left(B_{16}^l A_{11} + B_{26}^l A_{12} + B_{65}^l A_{16}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + \\ &+\left(B_{16}^l A_{12} + B_{26}^l A_{22} + B_{66}^l A_{26}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} \\ &-\left(B_{16}^l A_{16} + B_{26}^l A_{26} + B_{66}^l A_{66}\right)\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\Big]\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \\ &-\gamma\left(B_{16}^l\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + B_{26}^l\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + 2B_{66}^l\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)L_2(A_{jk})\Psi.\end{aligned} \qquad (8.25)$$

The expressions for shearing stresses $\tau_{\alpha\gamma}^l$ and $\tau_{\beta\gamma}^l$ when X = Y = O are substantially simplified. The second and fourth terms in formulas (7.16)-(7.19), as is seen from Section 7, are proportional to components X and Y of the external surface load; hence, with a zero value for the latter components these terms vanish and the formulas for the shearing stresses $\tau_{\alpha\gamma}^l$, and $\tau_{\beta\gamma}^l$ are represented in the following form:

$$\left.\begin{aligned}\tau_{\alpha\gamma}^l &= -\tau_{\alpha\gamma}^{2m+2-l} = \frac{\gamma^2}{2}E_1\left(B_{jk}^l\right)w - \frac{h_0^2}{2}E_1\left(P_{jk}^l\right)w; \\ \tau_{\beta\gamma}^l &= -\tau_{\beta\gamma}^{2m+2-l} = \frac{\gamma^2}{2}E_2\left(B_{jk}^l\right)w - \frac{h_0^2}{2}E_2\left(P_{jk}^l\right)w,\end{aligned}\right\} \qquad (8.26)$$

or, considering (8.15),

$$\left.\begin{aligned}\tau_{\alpha\gamma}^l &= \left[\frac{\gamma^2}{2}E_1\left(B_{jk}^l\right) - \frac{h_0^2}{2}E_1\left(P_{jk}^l\right)\right]L_2(A_{jk})\Psi; \\ \tau_{\beta\gamma}^l &= \left[\frac{\gamma^2}{2}E_2\left(B_{jk}^l\right) - \frac{h_0^2}{2}E_2\left(P_{jk}^l\right)\right]L_2(A_{jk})\Psi,\end{aligned}\right\} \qquad (8.27)$$

where $l = 1, 2, 3, \ldots, (m+1)$.

Thus, it is easily seen that the formulas for shearing stresses (8.26) and (8.27) do not differ from the corresponding formulas given for an anisotropic laminar plate,* which is easily explained by the presence of the initial relationships in (4.13)

---

*S.G. Lekhnitskiy, Anizotropnyye plastinki, Gostekhizdat, 1957, p. 263.

The displacements of the median surface of the shell determine by means of the following formulas:

$$u = \frac{1}{R}\left(A_{12}\frac{1}{A^3}\frac{\partial^3\Psi}{\partial\alpha^3} + A_{11}\frac{1}{AB^2}\frac{\partial^3\Psi}{\partial\alpha\,\partial\beta^2} - A_{16}\frac{1}{A^2B}\frac{\partial^3\Psi}{\partial\alpha^2\,\partial\beta}\right); \tag{8.28}$$

$$\begin{aligned} v = -\frac{1}{R}\Big[A_{11}\frac{1}{B^3}\frac{\partial^3\Psi}{\partial\beta^3} + (A_{12}+A_{66})\frac{1}{A^2B}\frac{\partial^3\Psi}{\partial\alpha^2\,\partial\beta} - \\ - 2A_{16}\frac{1}{AB^2}\frac{\partial^3\Psi}{\partial\alpha\,\partial\beta^2} - A_{26}\frac{1}{A^3}\frac{\partial^3\Psi}{\partial\alpha^3}\Big], \end{aligned} \tag{8.29}$$

and the normal displacement as determined from (8.15).

By substitution we may confirm that the displacements u, v and w represented in formulas (8.28), (8.29) and (8.15) satisfy the first two equations of system (7.9) when X = Y = 0 and the third equation gives the already known differential equation of solution (8.15).

Examining equation (8.16) and formulas (8.18)-(8.29), we conclude that the problem of calculation of an isotropic laminar cylindrical shell under the action of only a normally applied load reduces to the solution of equation (8.16) with given boundary conditions.

Orthotropic Shell. In the case where the shell consists of homogeneous orthotropic layers such that the principal directions of a elasticity at each point of each layer coincide with the directions of the corresponding coordinate lines the system of differential equations (8.12) takes the following form (we shall not present a detailed enumeration of the basic premises for an orthotropic shell, since they are already known to the reader from the previous sections of this chapter):

$$\left.\begin{aligned} L_2(C_{jk})\varphi - \frac{1}{R}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} &= -\frac{1}{A^2}\frac{\partial^2\varepsilon_2^*}{\partial\alpha^2} - \frac{1}{B^2}\frac{\partial^2\varepsilon_1^*}{\partial\beta^2}. \\ L_1(D_{jk})w - \frac{1}{R}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} &= Z + \frac{B}{R}\int {}'\,d\beta. \end{aligned}\right\} \tag{8.30}$$

where the linear operators L_1 (D_{jk}) and L_2 (C_{jk}) are known

$$\left.\begin{aligned} L_1(D_{jk}) = D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4} + D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4} + \\ + 2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}; \\ L_2(C_{jk}) = \frac{C_{11}}{\Omega}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4} + \frac{C_{22}}{\Omega}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4} + \\ + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}. \end{aligned}\right\} \tag{8.31}$$

Moreover, for ε_1^*, and ε_2^* we here have (5.54) and (5.55).

The design formulas take the following form: for the bending and torsional moments:

$$M_1 = -D_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} - D_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}; \tag{8.32}$$

$$M_2 = -D_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} - D_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}; \tag{8.33}$$

$$H = -2D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}, \tag{8.34}$$

for the transverse forces

$$N_1 = -E_1(D_{jk})\,w; \qquad N_2 = -E_2(D_{jk})\,w, \tag{8.35}$$

where from (4.42) we have for the linear operators E_i

$$E_1(D_{jk}) = \frac{D_{11}}{A^3}\frac{\partial^3}{\partial\alpha^3} + \frac{D_{12}+2D_{66}}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}; \tag{8.36}$$

$$E_2(D_{jk}) = \frac{D_{22}}{B^3}\frac{\partial^3}{\partial\beta^3} + \frac{D_{12}+2D_{66}}{BA^2}\frac{\partial^3}{\partial\beta\,\partial\alpha^2}. \tag{8.37}$$

Tangential forces T_1, T_2 and S are determined from formulas (8.2)-(8.4). The design stresses in the layers

$$\sigma_\alpha^i = \left(B_{11}^i\frac{C_{22}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - \left(B_{11}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{11}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} -$$
$$-\gamma\left(\frac{B_{11}^i}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + \frac{B_{12}^i}{B^2}\frac{\partial^2 w}{\partial\beta^2}\right) + B_{11}^i\varepsilon_1^* + B_{12}^i\varepsilon_2^*; \tag{8.38}$$

$$\sigma_\beta^i = \left(B_{22}^i\frac{C_{11}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \left(B_{22}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{22}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} -$$
$$-\gamma\left(\frac{B_{22}^i}{B^2}\frac{\partial^2 w}{\partial\beta^2} + \frac{B_{12}^i}{A^2}\frac{\partial^2 w}{\partial\alpha^2}\right) + B_{22}^i\varepsilon_2^* + B_{12}^i\varepsilon_1^*; \tag{8.39}$$

$$\tau_{\alpha\beta}^i = -\frac{B_{66}^i}{C_{66}}\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - 2\gamma B_{66}^i\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}. \tag{8.40}$$

In the case where the shell is loaded only by the normally applied surface load, when

$$w = L_2(C_{jk})\Psi, \qquad \varphi = \frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2}, \tag{8.41}$$

the system of equations of solution reduces to a single eighth-order equation of solution in terms of potential function $\Psi(\alpha, \beta)$:

$$P_1\frac{1}{A^8}\frac{\partial^8\Psi}{\partial\alpha^8} + P_3\frac{1}{A^6B^2}\frac{\partial^8\Psi}{\partial\alpha^6\,\partial\beta^2} + P_5\frac{1}{A^4B^4}\frac{\partial^8\Psi}{\partial\alpha^4\,\partial\beta^4} +$$
$$+ P_4\frac{1}{A^2B^6}\frac{\partial^8\Psi}{\partial\alpha^2\,\partial\beta^6} + P_2\frac{1}{B^8}\frac{\partial^8\Psi}{\partial\beta^8} + \frac{1}{R^2}\frac{1}{A^4}\frac{\partial^4\Psi}{\partial\alpha^4} = Z, \tag{8.42}$$

where for coefficients P_j we have

$$\left.\begin{aligned} &P_1 = \frac{D_{11}C_{11}}{\Omega}, \qquad P_3 = 2\frac{C_{11}}{\Omega}(D_{12}+2D_{66}) + D_{11}\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right); \\ &P_2 = \frac{D_{22}C_{22}}{\Omega}, \qquad P_4 = 2\frac{C_{22}}{\Omega}(D_{12}+2D_{66}) + D_{22}\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right); \\ &P_5 = \frac{D_{11}C_{22}}{\Omega} + 2(D_{12}+2D_{66})\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) + \frac{D_{22}C_{11}}{\Omega}. \end{aligned}\right\} \tag{8.43}$$

The design quantities are expressed in terms of function $\Psi = \Psi(\alpha, \beta)$ by the following formulas:

The tangential forces as in (8.18);

The bending and torsional moments

$$M_1 = -\left(D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)L_2(C_{jk})\Psi; \tag{8.44}$$

$$M_2 = -\left(D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)L_2(C_{jk})\Psi; \tag{8.45}$$

$$H = -2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(C_{jk})\Psi; \tag{8.46}$$

the transverse forces

$$N_1 = -\left(\frac{D_{11}}{A^3}\frac{\partial^3}{\partial\alpha^3} + \frac{D_{12}+2D_{66}}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right)L_2(C_{jk})\Psi; \tag{8.47}$$

$$N_2 = -\left(\frac{D_{22}}{B^3}\frac{\partial^3}{\partial\beta^3} + \frac{D_{12}+2D_{66}}{BA^2}\frac{\partial^3}{\partial\beta\,\partial\alpha^2}\right)L_2(C_{jk})\Psi; \tag{8.48}$$

the stresses in the layers

$$\begin{aligned} \sigma_\alpha^i = \Big[&\left(B_{11}^i\frac{C_{22}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} - \\ &-\left(B_{11}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{11}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\Big]\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \\ &-\gamma\left(B_{11}^i\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + B_{12}^i\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)L_2(C_{jk})\Psi, \end{aligned} \tag{8.49}$$

$$\begin{aligned} \sigma_\beta^i = \Big[&\left(B_{22}^i\frac{C_{11}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} - \\ &-\left(B_{22}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{22}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\Big]\frac{1}{R}\frac{1}{A^2}\frac{\partial^2\Psi}{\partial\alpha^2} - \\ &-\gamma\left(B_{22}^i\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + B_{12}^i\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)L_2(C_{jk})\Psi; \end{aligned} \tag{8.50}$$

$$\tau_{\alpha\beta}^i = -\frac{1}{R}\frac{B_{66}^i}{C_{66}}\frac{1}{A^3B}\frac{\partial^4\Psi}{\partial\alpha^3\partial\beta} - 2\gamma B_{66}^i\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(C_{jk})\Psi; \tag{8.51}$$

the displacements of the median surface

$$u = -\frac{1}{R}\frac{C_{12}}{\Omega}\frac{1}{A^3}\frac{\partial^3\Psi}{\partial\alpha^3} + \frac{1}{R}\frac{C_{22}}{\Omega}\frac{1}{AB^2}\frac{\partial^3\Psi}{\partial\alpha\,\partial\beta^2}; \tag{8.52}$$

$$v = -\frac{1}{R}\frac{C_{22}}{\Omega}\frac{1}{B^3}\frac{\partial^3\Psi}{\partial\beta^3} - \frac{1}{R}\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right)\frac{1}{A^2B}\frac{\partial^3\Psi}{\partial\alpha^2\,\partial\beta}, \tag{8.53}$$

and the normal displacement is determined from (8.41).

SECTION 9. INTEGRATION OF THE EQUATIONS OF ENGINEERING THEORY OF A CYLINDRICAL SHELL BY THE METHOD OF DOUBLE TRIGOMETRIC SERIES

The equations of the engineering theory of cylindrical shells, as is known,*,**,***,****,*****,******,******* may be used for the solution of numerous problems of cylindrical shells (both of shallow shells and of steep-curved to closed shells). Herein it must be kept in mind that in the case of a shallow shell the length may be considerably greater; when the shell is steep-curved its length must be limited. As our calculations show****** even with a ratio b/R = 0.5 the engineering theory of cylindrical shells is applicable over a rather wide range of change in λ = a/b (Figure 56); for example, when a cylindrical shell of open profile is freely supported over its entire contour and sustains a distributed sinusoidal load, this range of change in λ lies from 0.1 to 10.

In the case of a cylindrical shell of closed profile applicability of the engineering theory is limited. However, as A.I. Lur'ye's calculations show,*** a horizontal tube partially filled with water may be dealt with in the engineering theory of cylindrical shells even in the case where $l/R = 10$ (see Figure 55). We also point out that with more smoothly varying loads (for example, in the case of an entirely filled tube) this area of applicability of the engineering theory can be expanded.

Without impairing the general validity of the discussion we shall apply the procedure for integration of the equation of solution in the theory of anisotropic laminar shells by the method of double trigonometric series to the example in the engineering theory of shells (sections 4, 5, 7, 8). Where necessary the integration procedure is easily extended to more general equations of solution in the theory of anisotropic linear shells, for example, to equation (3.31).

---

*T.T. Khachaturyan, Shallow cylindrical shells, Soobshcheniya In-ta mat. i mekh. AN ArmSSR, No. 4, 1949.

**V.Z. Vlasov, see pp. 315-397 of work cited in footnote * on p. 201.

***A.I. Lur'ye, see pp. 178-243 of work cited in footnote ** on p. 203.

****A.L. Gol'denveyzer, see pp. 459-460 and 470-541 of work cited in footnote *, on p. 203.

*****V.V. Novozhilov, see footnote *** on p. 208.

******S.A. Ambartsumyan, Toward approximate theories of calculation of shallow cylindrical shells, PMM, Vol. XVIII, No. 3, 1954.

*******Kh.M. Mushtari, Some generalizations in the theory of thin shells with application to solution of stability problems of elastic equilibrium, PMM, Vol. II, No. 4, 1939.

A. Circular Cylindrical Shell of Open Profile. Let us discuss an orthotropic multi-layer cylindrical shell of open circular profile in which the principal directions of elasticity coincide with the directions of the coordinate lines. Let the shell overlap a rectangular plane $(a \times b_1)$ and have the

following dimensions: along generatrix a, along the arc of the transverse circle b, the radius of curvature of the coordinate surface R. The coordinate origin of α and β is located at an angle of the shell, that is, at the point of intersection of two adjacent edges (longitudinal and transverse) of the shell. The coordinate system is so chosen that the coefficients of the first quadratric form of A and B are equal to unity (1.1). Let, moreover, the shell be freely supported over the entire contour (hinged edges free in the tangential direction) and sustain a concentrated force Z = -Q normally applied an arbitrary point $(\alpha = x,\ \beta = y)$ of the surface of the shell (Figure 56).

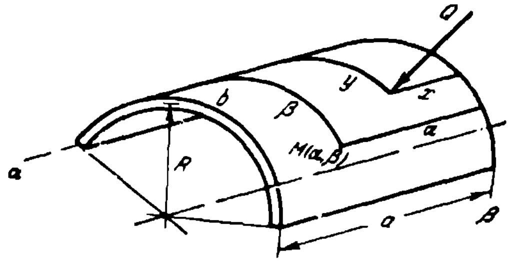

Figure 56

As the equation of solution of the discussed shell we may take (5.64), which in expanded form is rewritten as follows:

$$P_1 \frac{\partial^8 \Psi}{\partial \alpha^8} + P_3 \frac{\partial^8 \Psi}{\partial \alpha^6 \partial \beta^2} + P_5 \frac{\partial^8 \Psi}{\partial \alpha^4 \partial \beta^4} + P_4 \frac{\partial^8 \Psi}{\partial \alpha^2 \partial \beta^6} + P_2 \frac{\partial^8 \Psi}{\partial \beta^8} - {} - 2Q_1 \frac{\partial^6 \Psi}{\partial \alpha^6} - 2Q_3 \frac{\partial^6 \Psi}{\partial \alpha^4 \partial \beta^2} - 2Q_4 \frac{\partial^6 \Psi}{\partial \alpha^2 \partial \beta^4} + \frac{1}{R^2} \frac{\partial^4 \Psi}{\partial \alpha^4} = Z. \tag{9.1}$$

With the given method of fastening the edges of the shell the boundary conditions will have the following form: (2.10.3):

$$\left.\begin{array}{ll} \text{for } \alpha = 0,\ \alpha = a: & v = w = T_1 = M_1 = 0; \\ \text{for } \beta = 0,\ \beta = b: & u = w = T_2 = M_2 = 0. \end{array}\right\} \tag{9.2}$$

Thus, the sought function $\Psi = \Psi(\alpha, \beta)$ must be so defined that with a given load Z the differential equation of solution (9.1) and boundary conditions (9.2) are satisfied.

Examining formulas (5.63), (5.67), (5.68), (5.70), (5.71), (5.76) and (5.77), we easily see that the boundary conditions in (9.2) may be satisfied, seeking the unknown function $\Psi = \Psi(\alpha, \beta)$ in the form of the following double trigonometric series:

$$\Psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}, \tag{9.3}$$

where A_{mn} are the unknown coefficients, m and n are integers (m, n = 1, 2, 3, ...).

In order to determine coefficients A_{mn} let us expand the load Z = Z (α, β) into a double fourier series

$$Z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}, \tag{9.4}$$

where coefficients a_{mn} are determined from the well-known formula

$$a_{mn} = \frac{4}{ab} \int_0^a \int_0^b Z(\alpha, \beta) \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}\, d\alpha\, d\beta. \tag{9.5}$$

Inserting the values of Ψ and Z from (9.3) and (9.4), respectively, into the differential equation of solution (9.1) and comparing the coefficients of the identical sines, for the sought function we obtain

$$\Psi = \frac{a^8}{\pi^8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}, \tag{9.6}$$

where for Δ_{mn} we have

$$\Delta_{mn} = P_1 m^8 + P_3 m^6 n^2 \lambda^2 + P_5 m^4 n^4 \lambda^4 + P_4 m^2 n^6 \lambda^6 + \\ + P_2 n^8 \lambda^8 + 2 \frac{a^2}{\pi^2} (Q_1 m^6 + Q_3 n^4 n^2 \lambda^2 + Q_4 m^2 n^4 \lambda^4) + \frac{1}{R^2} \frac{a^4}{\pi^4} m^4. \tag{9.7}$$

$\lambda = \frac{a}{b}$, P_i, Q_i — как в § 5.
are as in Section 5.

In the discussed case, when force Q is applied at point (x, y), we have for coefficients a_{mn}

$$r_{mn} = -\frac{4Q}{ab}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{9.8}$$

$$(m = 1,\ 2,\ 3\ \ldots,\ n = 1,\ 2,\ 3\ \ldots).$$

Then for the sought $\Psi(\alpha, \beta)$ we obtain from (9.6)

$$\Psi = -\frac{4Qa^8}{\pi^8 ab}\sum_m\sum_n\frac{\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\Delta_{mn}}. \tag{9.9}$$

Inserting the value of function $\Psi(\alpha, \beta)$ into (5.63), (5.67)-(5.75), we obtain:

For the normal displacement

$$w = -\frac{4Qa^4}{\pi^4 ab}\sum_m\sum_n\frac{\Delta'_{mn}}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{9.10}$$

where

$$\Delta'_{mn} = \frac{C_{11}}{\Omega}m^4 + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)m^2n^2\lambda^2 + \frac{C_{22}}{\Omega}n^4\lambda^4; \tag{9.11}$$

For the tangential forces

$$T_1 = -\frac{4Qa^4\lambda^2}{\pi^4 ab}\frac{1}{R}\sum_m\sum_n\left(\frac{\pi^2R}{a^2}\Delta''_{mn} + m^2\right)\times$$
$$\times\frac{n^2}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}; \tag{9.12}$$

$$T_2 = -\frac{4Qa^4}{\pi^4 ab}\frac{1}{R}\sum_m\sum_n\left(\frac{\pi^2R}{a^2}\Delta''_{mn} + m^2\right)\times$$
$$\times\frac{m^2}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}; \tag{9.13}$$

$$S = -\frac{4Qa^4\lambda}{\pi^4 ab}\frac{1}{R}\sum_m\sum_n\left(\frac{\pi^2R}{a^2}\Delta''_{mn} + m^2\right)\times$$
$$\times\frac{mn}{\Delta_{mn}}\cos\frac{m\pi\alpha}{a}\cos\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{9.14}$$

where

$$\Delta''_{mn} = \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}m^4 + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \right.$$
$$\left. + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right)m^2n^2\lambda^2 + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}n^4\lambda^4; \tag{9.15}$$

for the moments

$$M_1 = -\frac{4Qa^2}{\pi^2 ab}\sum_m\sum_n\left\{\left[\left(D_{11} - D^0_{11}\right)m^2 + \left(D_{12} - D^0_{12}\right)n^2\lambda^2\right]\Delta'_{mn} + \right. \tag{9.16}$$

$$+\left[\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega}m^2+\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}n^2\lambda^2\right]\left(\Delta''_{mn}+\frac{a^2m^2}{R\pi^2}\right)\Bigg\}\times$$

$$\times\frac{1}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\tag{9.16}$$

$$M_2=-\frac{4Qa}{\pi^3ab}\sum_m\sum_n\Bigg\{\left[\left(D_{22}-D^0_{22}\right)n^2\lambda^2+\left(D_{12}-D^0_{12}\right)m^2\right]\Delta'_{mn}+$$

$$+\left[\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega}n^2\lambda^2+\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega}m^2\right]\left(\Delta''_{mn}+\frac{a^2m^2}{R\pi^2}\right)\Bigg\}\times$$

$$\times\frac{1}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\tag{9.17}$$

$$H=\frac{4Qa^2\lambda}{\pi^3ab}\sum_m\sum_n\left[2\left(D_{66}-D^0_{66}\right)\Delta'_{mn}-\frac{K_{66}}{C_{66}}\left(\Delta''_{mn}+\frac{a^2m^2}{R\pi^2}\right)\right]\times$$

$$\times\frac{mn}{\Delta_{mn}}\cos\frac{m\pi\alpha}{a}\cos\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\tag{9.18}$$

for the stresses in the shell layers

$$\sigma^i_\alpha=\frac{4Qa^2}{\pi^2ab}\sum_m\sum_n\Bigg\{\left[\left(B^i_{11}\frac{C_{12}}{\Omega}-B^i_{12}\frac{C_{11}}{\Omega}\right)m^2-\right.$$

$$\left.-\left(B^i_{11}\frac{C_{22}}{\Omega}-B^i_{12}\frac{C_{12}}{\Omega}\right)n^2\lambda^2\right]\left(\frac{a^2m^2}{R\pi^2}+\Delta''_{mn}\right)+$$

$$+\left[\left(B^i_{11}\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega}+B^i_{12}\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega}-\gamma B^i_{12}\right)n^2\lambda^2+\right.$$

$$\left.+\left(B^i_{11}\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}+B^i_{12}\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega}-\gamma B^i_{11}\right)m^2\right]\Delta'_{mn}\Bigg\}\times$$

$$\times\frac{1}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\tag{9.19}$$

$$\sigma^i_\beta=\frac{4Qa^2}{\pi^2ab}\sum_m\sum_n\Bigg\{\left[\left(B^i_{22}\frac{C_{12}}{\Omega}-B^i_{12}\frac{C_{22}}{\Omega}\right)n^2\lambda^2-\right.$$

$$\left.-\left(B^i_{22}\frac{C_{11}}{\Omega}-B^i_{12}\frac{C_{12}}{\Omega}\right)m^2\right]\left(\frac{a^2m^2}{R\pi^2}+\Delta''_{mn}\right)+$$

$$+\left[\left(B^i_{22}\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega}+B^i_{12}\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}-\gamma B^i_{12}\right)m^2+\right.$$

$$\left.+\left(B^i_{22}\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega}+B^i_{12}\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega}-\gamma B^i_{22}\right)n^2\lambda^2\right]\Delta'_{mn}\Bigg\}\times$$

$$\times\frac{1}{\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};\tag{9.20}$$

$$\tau^i_{\alpha\beta}=-\frac{4Qa^2\lambda}{\pi^2ab}\sum_m\sum_n\left[\left(\frac{a^2m^2}{R\pi^2}+\Delta''_{mn}\right)\frac{B^i_{66}}{C_{66}}-2B^i_{66}\left(\gamma-\frac{K_{66}}{C_{66}}\right)\Delta'_{mn}\right]\times$$

$$\times\frac{mn}{\Delta_{mn}}\cos\frac{m\pi\alpha}{a}\cos\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.\tag{9.21}$$

In the case where an orthotropic shell consists of an odd number of layers symmetrically arranged relative to the median coordinate surface (sections 7 and 8), the equation of solution has the form (8.42) and the design formulas are (8.18), (8.44)-(8.51). Then, proceeding as in the general case of an arbitrarily laminar orthotropic shell, we obtain:

For the function $\Psi(\alpha, \beta)$ and normal displacement $w(\alpha, \beta)$:

$$\Psi = -\frac{4Qa^6}{\pi^2 ab} \sum_m \sum_n \frac{\sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{\Delta_{mn0}}; \tag{9.22}$$

$$w = -\frac{4Qa^4}{\pi^4 ab} \sum_m \sum_n \frac{\Delta'_{mn}}{\Delta_{mn0}} \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{9.23}$$

where

$$\Delta_{mn_0} = P_1 m^8 + P_3 m^6 n^2 \lambda^2 + P_5 m^4 n^4 \lambda^4 + P_4 m^2 n^6 \lambda^6 + \\ + P_2 n^8 \lambda^8 + \frac{1}{R^2} \frac{a^4}{\pi^4} m^4. \tag{9.24}$$

and for Δ'_{mn} we have (9.11);

For the internal forces and moments

$$T_1 = -\frac{4Qa^4\lambda^2}{\pi^4 abR} \sum_m \sum_n \frac{m^2 n^2}{\Delta_{mn0}} \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}; \tag{9.25}$$

$$T_2 = -\frac{4Qa^4}{\pi^4 abR} \sum_m \sum_n \frac{m^4}{\Delta_{mn0}} \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}; \tag{9.26}$$

$$S = -\frac{4Qa^4\lambda}{\pi^4 abR} \sum_m \sum_n \frac{m^3 n}{\Delta_{mn0}} \cos\frac{m\pi\alpha}{a} \cos\frac{n\pi\beta}{b} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{b}; \tag{9.27}$$

$$M_1 = -\frac{4Qa^2}{\pi^2 ab} \sum_m \sum_n (D_{11} m^2 + D_{12} n^2 \lambda^2) \frac{\Delta'_{mn}}{\Delta_{mn0}} \times \\ \times \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}; \tag{9.28}$$

$$M_2 = -\frac{4Qa^2}{\pi^2 ab} \sum_m \sum_n (D_{22} n^2 \lambda^2 + D_{12} m^2) \frac{\Delta'_{mn}}{\Delta_{mn0}} \times \\ \times \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}; \tag{9.29}$$

$$H = \frac{8Qa^2\lambda}{\pi^2 ab} D_{66} \sum_m \sum_n \frac{mn\Delta'_{mn}}{\Delta_{mn0}} \cos\frac{m\pi\alpha}{a} \cos\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}; \tag{9.30}$$

for the stresses in the shell layers

$$\sigma_\alpha^l = \frac{4Qa^4}{\pi^4 abR} \sum_m \sum_n \left[\left(B_{11}^l \frac{C_{12}}{\Omega} - B_{12}^l \frac{C_{11}}{\Omega} \right) m^2 - \right. \\ \left. - \left(B_{11}^l \frac{C_{22}}{\Omega} - B_{12}^l \frac{C_{12}}{\Omega} \right) n^2 \lambda^2 \right] \frac{m^2}{\Delta_{mn0}} \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} - \\ - \gamma \frac{4Qa^6}{\pi^2 ab} \sum_m \sum_n (B_{11}^l m^2 + B_{12}^l n^2 \lambda^2) \frac{t'_{mn}}{\Delta_{mn0}} \times \\ \times \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{t}; \tag{9.31}$$

$$\sigma_\beta^i = \frac{4Qa^4}{\pi^4 abR} \sum_m \sum_n \left[\left(B_{22}^i \frac{C_{12}}{\Omega} - B_{12}^i \frac{C_{22}}{\Omega} \right) n^2 \lambda^2 - \right.$$

$$\left. - \left(B_{22}^i \frac{C_{11}}{\Omega} - B_{12}^i \frac{C_{12}}{\Omega} \right) m^2 \right] \frac{m^2}{\Delta_{mn0}} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} -$$

$$- \gamma \frac{4Qa^2}{\pi^2 ab} \sum_m \sum_n \left(B_{22}^i n^2 \lambda^2 + B_{12}^i m^2 \right) \frac{\Delta'_{mn}}{\Delta_{mn0}} \times$$

$$\times \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \tag{9.32}$$

$$\tau_{\alpha\beta}^i = - \frac{4Qa^2\lambda}{\pi^2 ab} \sum_m \sum_n \left[\frac{B_{66}^i}{C_{66}} \frac{a^2 m^2}{R\pi^2} - 2\gamma B_{66}^i \Delta'_{mn} \right] \frac{mn}{\Delta_{mn0}} \times$$

$$\times \cos \frac{m\pi\alpha}{a} \cos \frac{n\pi\beta}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{9.33}$$

Design formulas (9.22)-(9.33) may be derived from the corresponding formulas for arbitrarily laminar shells (9.9)-(9.21) if it assumed therein that all K_{jk} are equal to 0.

The design formulas derived in this section are wholly adequate for the solution of freely supported (over the entire contour) (9.2) cylindrical shells of open profile loaded by a normally applied surface load.*,** Let us discuss a few of these problems.

1. A circular cylindrical shell with layers symmetrically arranged relative to the mean coordinate surface is freely supported over its entire contour and sustains a distributed, normally applied load which varies over the surface of the shell as follows:

$$Z = - q_0 \sin \frac{\pi\alpha}{a} \sin \frac{\pi\beta}{b}.$$

In this case the design formulas of the problem are derived if in all formulas (9.23)-(9.33) Q is replaced by $q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \, dx \, dy$ and the resulting expressions are integrated over the entire loaded surface of the shell.

From (9.23), (9.31)-(.33), for the normal displacement and design stresses, after appropriate elementary transformations, we obtain***

---

*S.A. Ambartsumyan, see footnote on p. 229.

**T.T. Khachaturyan, see footnote 1 on p. 240.

***In the ensuing formulas the subscript 0 for Δ_{mn0} is omitted (e.g., $\Delta_{11} = \Delta_{110}$).

$$w = -q_0 \frac{a}{\pi^4} \frac{\Delta_{11}'}{\Delta_{11}} \sin\frac{\pi\alpha}{a} \sin\frac{\pi\beta}{b};$$

$$\sigma_\alpha^i = -q \frac{a^2}{\pi^2}\left\{\frac{a^2}{R\pi^2}\left[\left(B_{11}^i \frac{C_{12}}{\Omega} - B_{12}^i \frac{C_{11}}{\Omega}\right) - \left(B_{11}^i \frac{C_{22}}{\Omega} - B_{12}^i \frac{C_{12}}{\Omega}\right)\lambda^2\right] - \right.$$

$$\left. - \gamma\left(B_{11}^i + B_{12}^i\lambda^2\right)\Delta_{11}'\right\} \frac{\sin\frac{\pi\alpha}{a} \sin\frac{\pi\beta}{b}}{\Delta_{11}};$$

$$\sigma_\beta^i = q_0 \frac{a^2}{\pi^2}\left\{\frac{a^2}{R\pi^2}\left[\left(B_{22}^i \frac{C_{12}}{\Omega} - B_{12}^i \frac{C_{22}}{\Omega}\right)\lambda^2 - \left(B_{22}^i \frac{C_{11}}{\Omega} - B_{12}^i \frac{C_{12}}{\Omega}\right)\right] - \right.$$

$$\left. - \gamma\left(B_{22}^i\lambda^2 + B_{12}^i\right)\Delta_{11}'\right\} \frac{\sin\frac{\pi\alpha}{a} \sin\frac{\pi\beta}{b}}{\Delta_{11}};$$

$$\tau_{\alpha\beta} = -q_0 \frac{a^2\lambda}{\pi^2}\left(\frac{a^2}{R\pi^2}\frac{B_{66}^i}{C_{66}} - 2\gamma B_{66}^i \Delta_{11}'\right) \frac{\cos\frac{\pi\alpha}{a} \cos\frac{\pi\beta}{b}}{\Delta_{11}}.$$

As a numerical example let us discuss a three-layer plywood shell with the following specific dimensions for the median surface: a = b = 50 cm, λ = 1, R = 150 cm. Let the ply of the shell consist of two longitudinal (outer) and one transverse layer. The direction of the grain of the longitudinal plies coincides with the direction of coordinate lines $\beta = \text{const}$. and the direction of the grain of the transverse layer coincides with the direction of coordinate lines $\alpha = \text{const}$. For the elastic constants of each layer of the shell we have:

Layers I and III:

$$E_1^{I} = E_1^{III} = 120\,000 \text{ kg/cm}^2; \qquad E_2^{I} = E_2^{III} = 6000 \text{ kg/cm}^2;$$
$$G_{12}^{I} = G_{12}^{III} = 4800 \text{ kg/cm}^2; \quad \nu_1^{I} = \nu_1^{III} = 0{,}6; \quad \nu_2^{I} = \nu_2^{III} = 0{,}03.$$

Layer II:

$$E_1^{II} = 6000 \text{ kg/cm}^2; \qquad E_2^{II} = 120\,000 \text{ kg/cm}^2;$$
$$G_{12}^{II} = 4800 \text{ kg/cm}^2; \qquad \nu_1^{II} = 0{,}03; \qquad \nu_2^{II} = 0{,}6.$$

The total thickness of the shell $h = 3{,}0$ mm; the thickness of each layer $t_i = 1{,}0$ mm; the design thicknesses: $h_1 = h_3 = 1{,}5$ mm, $h_2 = 0{,}5$ mm (Figure 57).

From (3.1), for coefficients B_{jk}^i we obtain

Layers	B_{11}^i (kg/cm²)	B_{22}^i	B_{12}^i	B_{66}^i
I and II	122 000	6 100	3 660	4 800
II	6 100	122 000	3 660	4 800

By means of formulas (6.1) and (6.2) we obtain for the stiffnesses

$$C_{11} = 25\,010 \text{ kg/см}; \qquad C_{22} = 13\,420 \text{ kg/см};$$
$$C_{66} = 1\,440 \text{ kg/см}; \qquad C_{12} = 1\,100 \text{ kg/см};$$
$$D_{11} = 265 \text{ kg см}; \qquad D_{22} = 23{,}4 \text{ kg см};$$
$$D_{66} = 10{,}8 \text{ kg см}; \qquad D_{12} = 8{,}3 \text{ kg см}.$$

Considering the resulting values of B^{i}_{jk}, C_{jk}, D_{jk}, formulas (9.11), (9.24) and (9.43) and the fact that for the discussed examples $m = 1$, $n = 1$, $\lambda = 1$, we obtain:

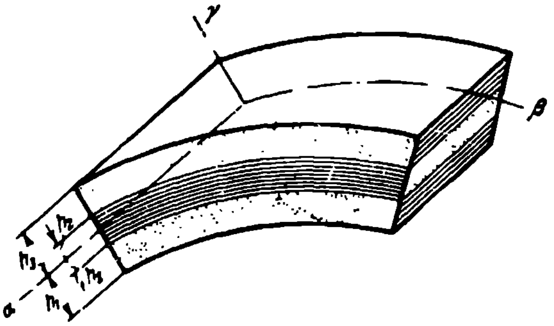

Figure 57

for the flexure of the shell center $\left(\alpha = \frac{a}{2}, \ \beta = \frac{b}{2}\right)$

$$w = -2{,}6q_0 \text{ см};$$

for the normal stresses in each layer of the shell when $\alpha = a/2$ and $\beta = b/2$

Stresses	Layer I		Layer II		Layer III	
	Point 1	Point 2	Point 3	Point 4	Point 5	Point 6
$-\sigma^{i}_{\alpha}$	$520{,}0q_0$	$641{,}0q_0$	$67{,}0q_0$	$77{,}0q_0$	$763{,}0q_0$	$884{,}0q_0$
$-\sigma^{i}_{\beta}$	$70{,}0q_0$	$80{,}0q_0$	$1248{,}0q_0$	$1370{,}0q_0$	$89{,}0q_0$	$99{,}0q_0$

Figure 58 gives the diagrams for the calculated normal stresses.

Also of interest from the standpoint of illustration are the maximum values of shearing stresses $\tau^i_{\alpha\gamma}$ and $\tau^i_{\beta\gamma}$. Inserting the value of w into (8.26) or of Ψ into (8.27), we obtain for the shearing stresses $\tau^i_{\alpha\gamma}$ and $\tau^i_{\beta\gamma}$

$$\tau^i_{\alpha\gamma} = q_0 \frac{a}{\pi} \frac{\Delta'_{11}}{\Delta_{11}} \left\{ \left[\frac{\gamma^2}{2} (B^i_{12} + 2B^i_{66}) - \frac{h_0^2}{2} (P^i_{12} + 2P^i_{66}) \right] \lambda^2 + \right.$$
$$\left. + \left(\frac{\gamma^2}{2} B^i_{11} - \frac{h_0^2}{2} P^i_{11} \right) \right\} \cos \frac{\pi\alpha}{a} \sin \frac{\pi\beta}{b};$$
$$\tau^i_{\beta\gamma} = q_0 \frac{a}{\pi} \frac{\Delta'_{11}}{\Delta_{11}} \left\{ \left[\frac{\gamma^2}{2} (B^i_{12} + 2B^i_{66}) - \frac{h_0^2}{2} (P^i_{12} + 2P^i_{66}) \right] \lambda + \right.$$
$$\left. + \left(\frac{\gamma^2}{2} B^i_{22} - \frac{h_0^2}{2} P^i_{22} \right) \lambda^3 \right\} \sin \frac{\pi\alpha}{a} \cos \frac{\pi\beta}{b}.$$

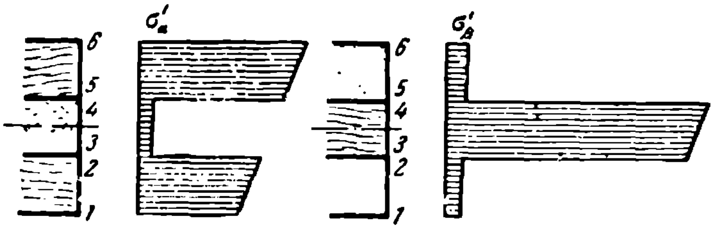

Figure 58

Considering that for the discussed example $h_0 = 1.5$ мм, we obtain from (7.21) for coefficients P^i_{jk}

Layers	P^i_{11} (kg/см²)	P^i_{22}	P^i_{12}	P^i_{66}
I и III	122 000	6 100	3 660	4 800
II	108 500	18 950	3 660	4 800

With consideration of the data listed in this table, for the shearing stress $\tau^i_{\alpha\gamma}$ in each shell layer when $\alpha = 0$, and $\beta = \frac{b}{2}$ and for the shearing stress $\tau^i_{\beta\gamma}$ in each layer when $\alpha = a/2$, and $\beta = 0$ we obtain

Stress	Layer I		Layer II			Layer III	
	Point 1	Point 2	Point 3	Point 0	Point 4	Point 5	Point 6
$\tau_{\alpha\gamma}^i$	0,0	$0{,}84q_0$	$0{,}84q_0$	$0{,}85q_0$	$0{,}84q_0$	$0{,}84q_0$	0,0
$\tau_{\beta\gamma}^i$	0,0	$0{,}12q_0$	$0{,}12q_0$	$0{,}23q_0$	$0{,}12q_0$	$0{,}12q_0$	0,0

Figure 59 gives the diagrams of the calculated shearing stresses. (The scale of the diagram for the shearing stresses differs from that of the diagram for the normal stresses.)

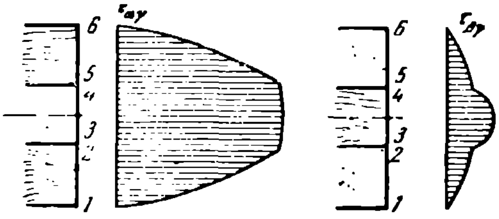

Figure 59

Figures 58 and 59 clearly show the influence of lamination and anisotropicity of the layer material on the distribution of stresses over the thickness of the shell.

2. A circular cylindrical shell with layers symmetrically arranged relative to the median coordinate surface is freely supported over the entire contour (9.2) and sustains a uniformly distributed, normally applied surface load of intensity q. In this case, in formulas (9.23), (9.31) and (9.32), by replacing Q with q dx dy and integrating the resulting expressions over the entire loaded surface of the shell, we obtain for the normal displacement and normal stresses in the shell layers

$$w = -\frac{16qa^4}{\pi^6}\sum_m\sum_n \frac{\Delta'_{mn}}{mn\Delta_{mn}} \sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b};$$

$$\sigma_\alpha^i = \frac{16qa^2}{\pi^4}\sum_m\sum_n\left\{\frac{a^2m^2}{R\pi^2}\left[\left(B_{11}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{11}}{\Omega}\right)m^2 - \right.\right.$$

$$\left.\left. -\left(B_{11}^i\frac{C_{22}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)n^2\lambda^2\right] - \gamma\left(B_{11}^i m^2 + B_{12}^i n^2\lambda^2\right)\Delta'_{mn}\right\}\frac{\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{mn\Delta_{mn}};$$

$$\sigma_\beta^i = \frac{16qa^4}{\pi^4}\sum_m\sum_n\left\{\frac{a^2m^2}{R\pi^2}\left[\left(B_{22}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{22}}{\Omega}\right)n^2\lambda^2 - \right.\right.$$

$$\left.\left. -\left(B_{22}^i\frac{C_{11}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)m^2\right] - \gamma\left(B_{22}^i n^2\lambda^2 + B_{12}^i m^2\right)\Delta'_{mn}\right\}\frac{\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{mn\Delta_{mn}}.$$

In these formulas, for m and n we have

As a numerical example let us discuss a three-ply wooden shell consisting of two layers of plank sheathing separating by an inner row of longitudinal beams. We shall disregard pliancy of the bond between the beam frame and the sheathing. Let the specific dimensions of the shell be: median surface $\alpha = 1150$ cm, $b = 1150$ cm; radius of curvature of median surface $R = 2200$ cm; total thickness of shell $h = 12.0$ cm; thickness of sheathing (i.e., thickness of the first and third layers) $t_1 = t_3 = 2.0$ cm; thickness of middle (second) layer (i.e., height of beam) $t_2 = H = 8.0$ cm; distance between beam axes $s = 12.5$ cm; thickness (width) of each beam $d = 5.0$ cm.

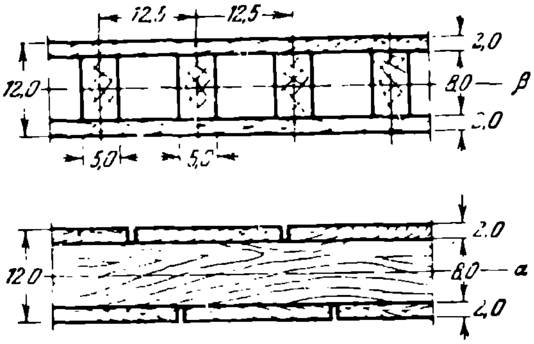

Figure 60

Assuming that the elastic modulus of the shell material (both of the sheathing and of the beams) along the direction of grain is E kg/cm^2, for the elastic moduli of the middle layer (which consists only of longitudinal beams of rectangular cross-section) (Figure 60) we obtain*

$$E_\alpha^{II} = \frac{d}{s} E = 0,4E, \quad E_\beta^{II} = 0,$$

while for the elastic moduli of the transverse sheathing, that is, of the first and third layers:

$$E_\alpha^{I} = E_\alpha^{III} = 0, \quad E_\beta^{I} = E_\beta^{III} = 0.$$

Furthermore, approximately assuming that the shear modulus of the entire shell $G_{12} = 0.1E$ and that the Poisson ratios of all layers of the shell are equal to zero, for the shell stiffnesses, from (6.1) and (6.2), we obtain

---

*S.G. Lekhnitskiy, see pp. 258-260 of work cited in footnote on p.

$$C_{11} = 3{,}2E \text{ kg/см}; \quad C_{22} = 4{,}0E \text{ kg см}; \quad C_{66} = 1{,}2E \text{ kg см};$$
$$D_{11} = 17{,}1E \text{ kg см}; \quad D_{22} = 101{,}3E \text{ kg см}; \quad D_{66} = 14{,}4E \text{ kg см};$$
$$C_{12} = 0; \quad D_{12} = 0; \quad \Omega = 12{,}8E^2 \text{kg}^2/\text{см}^2.$$

Using (3.1), (9.1), (9.4) and (8.43), from the presented formulas for the design quantities and limiting discussion to the first terms of expansion (m = 1, n = 1), we obtain:

for the normal displacement of the shell center

$$w = 10{,}3 \cdot 10^6 \frac{q}{E} \text{ см};$$

for the normal stresses in the sheathing and in the beams, with $\alpha = a/2$ and $\beta = b/2$.

	Layer I		Layer II (beam)		Layer III	
	Point 1	Point 2	Point 3	Point 4	Point 5	Point 6
$-\sigma_\alpha^i$	0,0	0,0	738 q	1353 q	0,0	0,0
$-\sigma_\beta^i$	376 q	530 q	0,0	0,0	1144 q	1298 q

Figure 61 shows the diagrams for the calculated stresses.

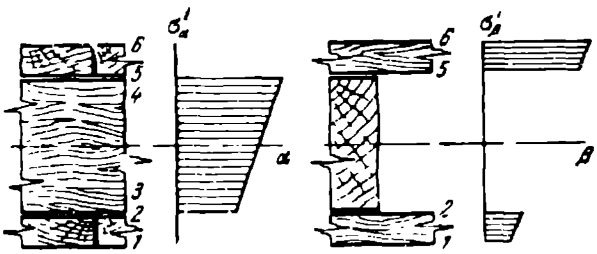

Figure 61

3. A two-layer circular cylindrical shell is freely supported over its entire contour and sustains a uniformly distributed, normally applied surface load $Z = -q$ kg/cm^2. Let the dimensions of the shell be a = b , 50.0 cm; R = = 100.0 cm; total thickness of shell h = 6.0 cm; thickness of first layer $t_1 = 2.0$ cm; thickness of second layer $t_2 = 4.0$ cm. Furthermore, let the elastic constants of the shell layer be: elastic modulus of the first layer

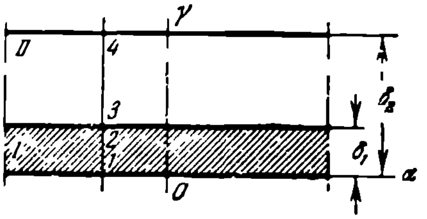

Figure 62

$E'_1 = E'_2 = rE$; the elastic modulus of the second layer $E''_1 = E''_2 = E$; the Poisson ratios of the layers $\nu' = \nu'' = 0$ (Figure 62).

From (3.1) we obtain for coefficients B^i_{jk}

$$B^I_{11} = rE, \quad B^I_{22} = rE, \quad B^I_{12} = 0, \quad B^I_{66} = 0{,}5rE,$$
$$B^{II}_{11} = E, \quad B^{II}_{22} = F, \quad B^{II}_{12} = 0, \quad B^{II}_{66} = 0{,}5E.$$

Considering that the design thicknesses of the shell are $\delta_1 = 2{,}0$ *см*, $\delta_2 = 6{,}0$ *см*, from (1.18)-(1.20) we obtain for the stiffnesses

$$C_{11} = C_{22} = 2E(r+2), \quad C_{66} = E(r+2), \quad C_{12} = 0,$$
$$D_{11} = D_{22} = \frac{8}{3}E(r+26), \quad D_{66} = \frac{4}{3}E(r+26), \quad D_{12} = 0,$$
$$K_{11} = K_{22} = 2E(r+8), \quad K_{66} = E(r+8), \quad K_{12} = 0.$$

On the basis of (5.40)-(5.47), for coefficients $(D_{jk} - D^0_{jk})$ we obtain

$$(D_{11} - D^0_{11}) = (D_{22} - D^0_{22}) = \frac{2}{3}\cdot\frac{r^2 + 64r + 16}{r+2}E;$$
$$(D_{66} - D^0_{66}) = \frac{1}{3}\frac{r^2 + 64r + 16}{(r+2)}E, \quad (D_{12} - D^0_{12}) = 0.$$

On the basis of the formulas presented in Section 5 we obtain for coefficients P_j and Q_j

$$P_1 = P_2 = \frac{1}{3}\frac{r^2 + 64r + 16}{(r+2)^2}, \quad P_3 = P_4 = \frac{4}{3}\frac{r^2 + 64r + 16}{(r+2)^2};$$
$$P_5 = 2\frac{r^2 + 64r + 16}{(r+2)^2}, \quad Q_1 = 0, \quad Q_3 = 0, \quad Q_4 = 0.$$

Now from (9.23), (9.31) and (9.32), after transformation we obtain:

$$w = -\frac{24a^4(r+2)q}{\pi^6 E(r^2+64r+16)} \sum_m \sum_n \frac{(m^2+n^2)^2 \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{a}}{mn\Delta_{mn}};$$

$$\sigma_\alpha^{I,\,II} = -\frac{24a^2(r+2)B_{11}^{I,\,II}q}{\pi^4 E(r^2+64r+16)} \sum_m \sum_n \left[\frac{a^2}{R\pi^2}\frac{mn}{\Delta_{mn}} - \right.$$

$$\left. -\left(\frac{r+8}{r+2} - \gamma\right)\frac{m(m^2+n^2)^2}{n\Delta_{mn}}\right] \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{a};$$

$$\sigma_\beta^{I,\,II} = -\frac{24a^2(r+2)B_{22}^{I,\,II}q}{\pi^4 E(r^2+64r+16)} \sum_m \sum_n \left[\frac{a^2}{R\pi^2}\frac{m^2}{n\Delta_{mn}} - \right.$$

$$\left. -\left(\frac{r+8}{r+2} - \gamma\right)\frac{n(m^2+n^2)^2}{m\Delta_{mn}}\right] \sin\frac{m\pi\alpha}{a} \sin\frac{n\pi\beta}{a}.$$

where

$$\Delta_{mn} = (m^2+n^2)^4 + \frac{1}{R^2}\left(\frac{ma}{\pi}\right)^4 \frac{3(r+2)^2}{r^2+64r+16},$$

while for m and n we have m, n = 1, 3, 5, . . .

By means of the derived formulas we may calculate the values of flexure and stresses in the shell layers with different values of r, that is, with different ratios of the elastic moduli of the layers. Assuming

$$w = -\lambda_r \frac{a^4 q}{\pi^6 E},$$

with different r we obtain for λ_r

r	1,0	2,0	10,0
λ_r	0,1901	0,1393	0,0750

Assuming

$$\sigma_\alpha^{I} = \lambda_{r\alpha}^{I}\frac{a^2 q}{\pi^4}, \quad \sigma_\beta^{I} = \lambda_{r\beta}^{I}\frac{a^2 q}{\pi^4};$$

$$\sigma_\alpha^{II} = \lambda_{r\alpha}^{II}\frac{a^2 q}{\pi^4}, \quad \sigma_\beta^{II} = \lambda_{r\beta}^{II}\frac{a^2 q}{\pi^4},$$

for $\lambda_{r\alpha}$ and $\lambda_{r\beta}$ at different points of the normal passing through point $\alpha = a/2$ and
and $\beta = a/2$ we obtain:

Coefficient	Points	$r=1{,}0$	$r=2{,}0$	$r=10{,}0$
t^{I}_{r1}	1	0,378	0,431	0,493
	2	0,044	—0,058	—0,810
t^{II}_{r1}	3	0,044	—0,029	—0,081
	4	—0,624	—0,518	—0,342
k^{I}_{r1}	1	0,383	0,439	0,515
	2	0,049	—0,050	—0,788
k^{II}_{r1}	3	0,049	—0,025	—0,079
	4	—0,618	—0,514	—0,340

For a graphic illustration of the distribution of normal stresses over the thickness of the shell Figure 63 shows the diagrams of the normal stresses σ'_α for the two values $r=1{,}0$ and $r=10{,}0$.

4. Let us explain the influence of anisotropicity of the shell material on flexure of the center of a circular cylindrical shell for different values

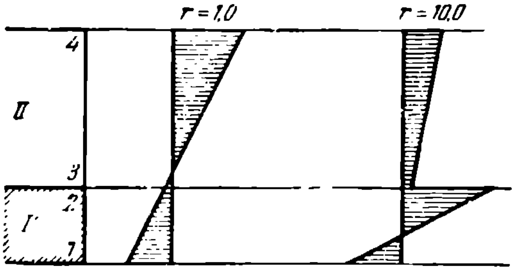

Figure 63

of $\lambda = a/b$. From the preceding problems we have for the normal displacements:

with force Q concentrated at point (x, y)

$$w_Q = \frac{4Qa^2\lambda}{\pi^4}\sum_m\sum_n \frac{\Delta'_{mn}}{\Delta_{mn}} \sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b};$$

with uniformly distributed q

$$w_q = \frac{16qa^4}{\pi^6} \sum_m \sum_n \frac{\Delta'_{mn}}{mn\Delta_{mn}} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b} .$$

For the flexures of an isotropic rectangular $(a \times b)$ plate with the same loads we have

$$\bar{w}_Q = \frac{4Qa^2\lambda}{\pi^4 D} \sum_m \sum_n \frac{\sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{(m^2 + \lambda^2 n^2)^2} ;$$

$$\bar{w}_q = \frac{16qa^4}{\pi^6 D} \sum_m \sum_n \frac{\sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}}{mn(m^2 + \lambda^2 n^2)^2} .$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical stiffness of the plate.

Applying the well-known method of improving the convergence of a series*,** and obtaining from the solution for an anisotropic plate, the solution for a rectangular isotropic plate with sides a, b and c of arbitrary stiffness D, we finally obtain for the displacements of the discussed shells

$$w_Q = \bar{w}_Q - \frac{4Qa^2\lambda}{\pi^4 D} \sum_m \sum_n \frac{\Delta_{mn} - D(m^2 + \lambda^2 n^2)\Delta'_{mn}}{(m^2 + \lambda^2 n^2)^2 \Delta_{mn}} \times$$

$$\times \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} ;$$

$$w_q = \bar{w}_q - \frac{16qa^4}{\pi^6 D} \sum_m \sum_n \frac{\Delta_{mn} - D(m^2 + \lambda^2 n^2)^2 \Delta'_{mn}}{mn(m^2 + \lambda^2 n^2)^2 \Delta_{mn}} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b} .$$

Thus, for the normal displacements we have obtained formulas with two terms, wherein the first components ($\bar{w}_Q$ and $\bar{w}_q$) are the flexures of an arbitrary point (α, β) of a rectangular $(a \times b)$ hinge-supported isotropic plate and the second components are the influence of the elastic characteristics and curvature on the normal displacement of an orthotropic shell.

The values of the first components may be obtained from well-known tables (B.G. Galerkin, S.P. Timoshenko). In the case of the second components, in view of the rapid convergence of the double series, we may limit ourselves to the first terms of expansion. Proceeding in this manner, for the normal displacements of the shell center $(\alpha = a/2,\ \beta = b/2)$ we obtain

---

*T.T. Khachaturyan, see footnote *, on p.240.

**S.A. Ambartsumyan, see footnote on p. 228.

$$w_Q = \frac{Qa^2}{D}\left[\alpha_Q - \frac{4}{\pi^4}\,\frac{\Delta_{11} - D(1+\lambda^2)^2\Delta'_{11}}{(1+\lambda^2)^2\Delta_{11}}\right];$$

$$w_q = \frac{qa^4}{D}\left[\alpha_q - \frac{16}{\pi^6}\,\frac{\Delta_{11} - D(1+\lambda^2)^2\Delta'_{11}}{(1+\lambda^2)^2\Delta_{11}}\right],$$

where α_Q and α_q are the coefficients from the tables. Here, in determining w_Q it is assumed that the force Q is applied to the center of the shell, that is, $x = a/2,\ y = b/2$.

Let us discuss a single-layer orthotropic shell with the dimensions:

$$h = 2\delta_0 = 0{,}1\ \text{м},\quad h/R = 1/50,\quad b/R = 0{,}5.$$

Two cases of combinations of elastic constants are examined:

Case	E_1	E_2	ν_1	ν_2	G_{12}
I	E	$0{,}05E$	0,6	0,03	$0{,}04E$
II	$0{,}05E$	E	0,03	0,6	$0{,}04E$

In these cases we obtain for the stiffnesses:

Case I

$$C_{11} = 10E;\quad C_{22} = 0{,}5E;\quad C_{66} = 0{,}4E;\quad C_{12} = 0{,}3E\ \text{(kg/cм)};$$

$$D_{11} = 83{,}3E;\quad D_{22} = 4{,}17E;\quad D_{66} = 3{,}33E;\quad D_{12} = 2{,}5E\ \text{(kg cм)}.$$

Case II

$$C_{11} = 0{,}5E;\quad C_{22} = 10E;\quad C_{66} = 0{,}4E;\quad C_{12} = 0{,}3E\ \text{(kg/cм)};$$

$$D_{11} = 4{,}17E;\quad D_{22} = 83{,}3E;\quad D_{66} = 3{,}33E;\quad D_{12} = 2{,}5E\ \text{(kg cм)}.$$

In each case we examine three ratios between a and b, namely: $\lambda = a/b = 0{,}5$; $\lambda = 1{,}0$; $\lambda = 1{,}5$.

From the above remarks it is evident that the cylindrical stiffness of an isotropic plate D may be arbitrarily chosen. With this in mind the value of D is sought from the condition that the second components of the formulas w_Q and w_q must vanish, that is,

$$\Delta_{11} - D(1+\lambda^2)^2\Delta'_{11} = 0,$$

whence, for D we obtain

$$D=\frac{\Delta_{11}}{(1+\lambda^2)^2\,\Delta'_{11}},$$

which is the stiffness of an equivalent isotropic plate whose flexures do not differ from the normal displacements of the corresponding points of the anisotropic shell under consideration.

On the basis of (8.40), (9.11), (9.24) and the derived formula we may easily determine the values of cylindrical stiffness of the equivalent isotropic plates for all three values of λ in each case.

Case	$\lambda=0,5$	$\lambda=1,0$	$\lambda=1,5$
I	$58,9E$	$35,4E$	$18,1E$
II	$16,7E$	$35,4E$	$46,4E$

Assuming

$$w_q=\alpha_q^0\frac{b^4q}{E},$$

$$w_Q=\alpha_Q^0\frac{b^2Q}{E}.$$

and using the tables,* for the coefficients α_q and α_Q with different values of λ in each case we obtain

α	Case	$\lambda=0,5$	$\lambda=1,0$	$\lambda=1,5$
α_q^0	I	0,00001	0,00012	0,00016
	II	0,00004	0,00012	0,00015
α_Q^0	I	0,00007	0,00033	0,00086
	II	0,00025	0,00033	0,00034

Figure 64 shows the results of calculation of the flexures of shell centers.

---

*B.G. Galerkin, Uprugiye tonkiye plity (Thin elastic plates), Gosstroyizdat, 1934.

The calculations show that in anisotropic cylindrical shells with small λ $(\lambda < 1)$, (that is, when we have "short" shells) it is more convenient to have large values of stiffness in the longitudinal direction (α) of the shell whereas with large λ $(\lambda > 1)$, (that is, when we have "long" shells) it is more convenient to have large values of stiffness in the transverse direction (β) of the shell.

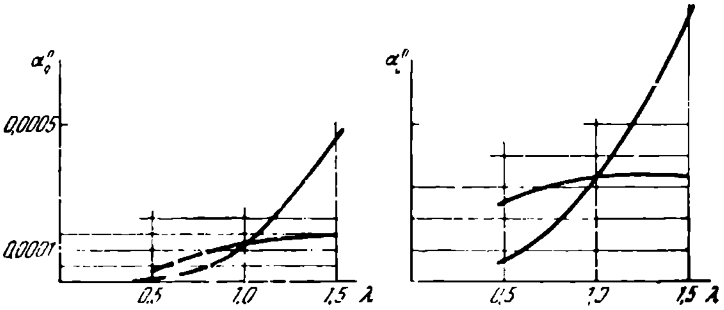

Figure 64

B. Circular Cylindrical Shell of Closed Profile. Let us discuss an orthotropic, multi-layer cylindrical shell of closed circular profile in which the principal directions of elasticity coincide with the directions of the coordinate lines. The system of coordinates is so chosen that A = B = R (1.2) and the origin of coordinate α is in the plane of any end section of the shell. The shell is freely supported at the ends and sustains only a normally applied (radial) surface load $Z = Z(\alpha, \beta)$. The length of the shell is l and the radius of curvature R (Figure 65).

In this case equation of solution (5.64) is rewritten as follows:

$$P_1\frac{\partial^8\Psi}{\partial\alpha^8}+P_3\frac{\partial^8\Psi}{\partial\alpha^6\partial\beta^2}+P_5\frac{\partial^8\Psi}{\partial\alpha^4\partial\beta^4}+P_4\frac{\partial^8\Psi}{\partial\alpha^2\partial\beta^6}+$$
$$+P_2\frac{\partial^8\Psi}{\partial\beta^8}-2R^2Q_1\frac{\partial^6\Psi}{\partial\alpha^6}-2R^2Q_3\frac{\partial^6\Psi}{\partial\alpha^4\partial\beta^2}-$$
$$-2R^2Q_4\frac{\partial^6\Psi}{\partial\alpha^2\partial\beta^4}+R^2\frac{\partial^4\Psi}{\partial\alpha^4}=R^8Z. \tag{9.34}$$

In the mentioned method of fixing the ends of the shell the boundary conditions will have the following form (2.10.3):

$$\left.\begin{array}{lll}\text{when} & \alpha=0: & v=w=T_1=M_1=0;\\ \text{when} & \alpha=\dfrac{l}{R}: & v=w=T_1=M_1=0.\end{array}\right\} \tag{9.35}$$

Thus, the sought function $\Psi=\Psi(\alpha,\beta)$ must be so defined that with a given load Z there are satisfied: equation of solution (9.34), boundary conditions

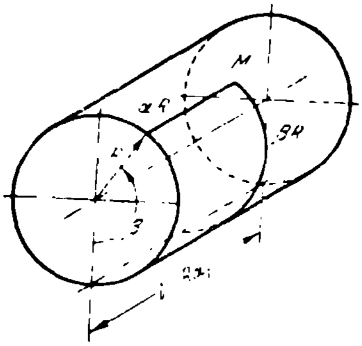

Figure 65

(9.35) and, by virtue of the closed nature of the shell, the conditions of periodicity over coordinate β.

Examining formulas (5.63), (5.67), (5.68), (5.70), (5.71), (5.76) and (5.77), we easily see that the boundary conditions (9.35) and condition of periodicity may be satisfied by finding the function $\Psi = \Psi(\alpha, \beta)$ in the form of a double trigonometric series

$$\Psi = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin \frac{m\pi\alpha}{a_1} \cos n\beta, \tag{9.36}$$

where A_{mn} are the sought coefficients, $\alpha_1 = \frac{l}{R}$ is the relative length of the shell, and m and n are integers $(m = 1, 2, 3, \ldots, n = 0, 1, 2, 3 \ldots)$.

In order to determine coefficients A_{mn} let us expand the load into a double Fourier series

$$Z = \sum_{m=1} \sum_{n=0} a_{mn} \sin \frac{m\pi\alpha}{a_1} \cos n\beta, \tag{9.37}$$

where coefficients a_{mn} are determined from the known formulas

$$\left.\begin{aligned} a_{m0} &= \frac{1}{\pi a_1} \int_{-\pi}^{\pi} \int_{0}^{a_1} Z(\alpha, \beta) \sin \frac{m\pi\alpha}{a_1}\, d\alpha\, d\beta, \\ a_{mn} &= \frac{2}{\pi a_1} \int_{0}^{a_1} \int_{-\pi}^{\pi} Z(\alpha, \beta) \sin \frac{m\pi\alpha}{a_1} \cos n\beta\, d\alpha\, d\beta. \end{aligned}\right\} \tag{9.38}$$

Inserting the values of ψ and Z from (9.36) and (9.37), respectively, into the differential equation of solution (9.34) and comparing the coefficients of the identical trigonometric functions, we obtain

$$\Psi = R^4 \sum_{m=1} \sum_{n=0} \frac{a_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a_1} \cos n\beta, \qquad (9.39)$$

where for Δ_{mn} we have

$$\Delta_{mn} = P_1 \left(\frac{m\pi}{a_1}\right)^8 + P_3 \left(\frac{m\pi}{a_1}\right)^6 n^2 + P_5 \left(\frac{m\pi}{a_1}\right)^4 n^4 + + P_4 \left(\frac{m\pi}{a_1}\right)^2 n^6 + P_2 n^8 + 2R^2 \left[Q_1 \left(\frac{m\pi}{a_1}\right)^6 + + Q_3 \left(\frac{m\pi}{a_1}\right)^4 n^2 + Q_4 \left(\frac{m\pi}{a_1}\right)^2 n^4\right] + R^2 \left(\frac{m\pi}{a_1}\right)^4, \qquad (9.40)$$

P_j and Q_j are as in Section 5.

Inserting the value of function $\Psi = \Psi(\alpha, \beta)$ into (5.63), (5.67)-(5.75), we obtain:

for the normal displacement

$$w = R^4 \sum_{m=1} \sum_{n=0} a_{mn} \frac{\Delta'_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a_1} \cos n\beta, \qquad (9.41)$$

where

$$\Delta'_{mn} = \frac{C_{11}}{\Omega} \left(\frac{m\pi}{a_1}\right)^4 + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) \left(\frac{m\pi}{a_1}\right)^2 n^2 + \frac{C_{22}}{\Omega} n^4; \qquad (9.42)$$

for the tangential forces

$$T_1 = R^2 \sum_{m=1} \sum_{n=0} \left[\left(\frac{m\pi}{a_1}\right)^2 R + \Delta''_{mn}\right] \frac{n^2}{\Delta_{mn}} a_{mn} \sin \frac{m\pi\alpha}{a_1} \cos n\beta; \qquad (9.43)$$

$$T_2 = R^2 \sum_{m=1} \sum_{n=0} \left[\left(\frac{m\pi}{a_1}\right)^2 R + \Delta''_{mn}\right] \left(\frac{m\pi}{a_1}\right)^2 \frac{a_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a_1} \cos n\beta; \qquad (9.44)$$

$$S = -R^2 \sum_{m=1} \sum_{n=0} \left[\left(\frac{m\pi}{a_1}\right)^2 R + \Delta''_{mn}\right] \frac{m\pi}{a_1} \frac{n}{\Delta_{mn}} a_{mn} \cos \frac{m\pi\alpha}{a_1} \sin n\beta, \qquad (9.45)$$

where

$$\Delta''_{mn} = \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} \left(\frac{m\pi}{a_1}\right)^4 + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right) \left(\frac{m\pi}{a_1}\right)^2 n^2 + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} n^4; \qquad (9.46)$$

for the moments

$$M_1 = R^2 \sum_{m=1} \sum_{n=0} \left\{\left[(D_{11} - D_{11}^0) \left(\frac{m\pi}{a_1}\right)^2 + (D_{12} - D_{12}^0) n^2\right] \Delta'_{mn} + + \left[\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} \left(\frac{m\pi}{a_1}\right)^2 + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} n^2\right] \left[\Delta''_{mn} + + \left(\frac{m\pi}{a_1}\right)^2 R\right]\right\} \frac{a_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a_1} \cos n\beta; \qquad (9.47)$$

$$M_2 = R^2 \sum_{m=1} \sum_{n=0} \Big\{ \Big[(D_{22} - D^0_{22}) n^2 + (D_{12} - D^0_{12}) \Big(\frac{m\pi}{a_1}\Big)^2 \Big] \Delta'_{mn} + \\ + \Big[\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} n^2 + \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} \Big(\frac{m\pi}{a_1}\Big)^2 \Big] \Big[\Delta''_{mn} + \\ + \Big(\frac{m\pi}{a_1}\Big)^2 R \Big] \Big\} \frac{a_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a_1} \cos n\beta; \qquad (9.48)$$

$$H = R^2 \sum_{m=1} \sum_{n=0} \Big\{ 2 (D_{66} - D^0_{66}) \Delta_{mn} - \frac{K_{66}}{C_{66}} \Big[\Delta''_{mn} + \\ + \Big(\frac{m\pi}{a_1}\Big)^2 R \Big] \Big\} \frac{m\pi}{a_1} \frac{n}{\Delta_{mn}} \cos \frac{m\pi\alpha}{a_1} \sin n\beta; \qquad (9.49)$$

for the stresses in the shell layers

$$\sigma^i_\alpha = - R^2 \sum_{m=1} \sum_{n=0} \Big\{ \Big[\Big(B^i_{11} \frac{C_{12}}{\Omega} - B^i_{12} \frac{C_{11}}{\Omega} \Big) \Big(\frac{m\pi}{a_1}\Big)^2 - \Big(B^i_{11} \frac{C_{22}}{\Omega} - \\ - B^i_{12} \frac{C_{12}}{\Omega} \Big) n^2 \Big] \Big[\Big(\frac{m\pi}{a_1}\Big)^2 R + \Delta''_{mn} \Big] + \Big[\Big(B^i_{11} \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} + \\ + B^i_{12} \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - \gamma B^i_{12} \Big) n^2 + \Big(B^i_{11} \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} + \\ + B^i_{12} \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} - \gamma B^i_{11} \Big) \Big(\frac{m\pi}{a_1}\Big)^2 \Big] \Delta'_{mn} \Big\} \frac{a_{mn}}{\Delta_{mn}} \times \\ \times \sin \frac{m\pi\alpha}{a_1} \cos n\beta; \qquad (9.50)$$

$$\sigma^i_\beta = - R^2 \sum_{m=1} \sum_{n=0} \Big\{ \Big[\Big(B^i_{22} \frac{C_{12}}{\Omega} - B^i_{12} \frac{C_{22}}{\Omega} \Big) n^2 - \Big(B^i_{22} \frac{C_{11}}{\Omega} - \\ - B^i_{12} \frac{C_{12}}{\Omega} \Big) \Big(\frac{m\pi}{a_1}\Big)^2 \Big] \Big[\Big(\frac{m\pi}{a_1}\Big)^2 R + \Delta''_{mn} \Big] + \Big[\Big(B^i_{22} \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} + \\ + B^i_{12} \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} - \gamma B^i_{12} \Big) \Big(\frac{m\pi}{a_1}\Big)^2 + \Big(B^i_{22} \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} + \\ + B^i_{12} \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} - \gamma B^i_{22} \Big) n^2 \Big] \Delta'_{mn} \Big\} \frac{a_{mn}}{\Delta_{mn}} \sin \frac{m\pi\alpha}{a_1} \cos n\beta; \qquad (9.51)$$

$$\tau^i_{\alpha\beta} = - R^2 \sum_{m=1} \sum_{n=0} \Big\{ \Big[\Big(\frac{m\pi}{a_1}\Big)^2 R + \Delta''_{mn} \Big] \frac{B^i_{66}}{C_{66}} - \\ - 2B^i_{66} \Big(\gamma - \frac{K_{66}}{C_{66}} \Big) \Delta'_{mn} \Big\} \frac{m\pi}{a_1} \frac{n}{\Delta_{mn}} a_{mn} \cos \frac{m\pi\alpha}{a_1} \sin n\beta. \qquad (9.52)$$

Assuming in the derived formulas $K_{jk} = 0$, we obtain the formulas for the design quantities of a circular cylindrical shell of closed profile consisting of an odd number of layers symmetrically arranged relative to the median coordinate surface of the shell (Sections 7 and 8).

As an example let us discuss the problem of a horizontal tube with three symmetrical layers, freely supported at the ends and filled with a liquid with specific weight γ*,**. Disregarding the weight of the material of the tube itself and reckoning angle β from the low point of the tube, for the effective load (the radial pressure of the liquid) we obtain

---

*S.P. Timoshenko, Plastinki i obolochki (Plates and shells), Gostekhizdat, 1948, p. 416.

**A.I. Lur'ye, see pp. 203-205 of work cited in footnote on p. 203.

$$Z = R\gamma(1 - \cos\beta).$$

Inserting the value of z into (9, 38) we obtain for coefficients a_{mn}

$$a_{m0} = \frac{4\gamma R}{m\pi}, \quad a_{m1} = \frac{4\gamma R}{m\pi} \quad (m = 1, 3, 5 \ldots)$$

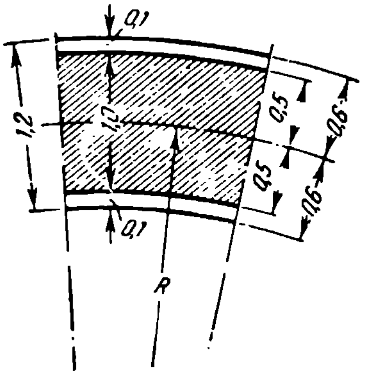

Figure 66

and when $n > 2$, we absolutely find $a_{mn} = 0$.

By virtue of this and (9, 41) we obtain for the normal displacement

$$w = \sum_{m=1} w_{m0} \sin\frac{m\pi\alpha}{a_1} + \cos\beta \sum_{m=1} w_{m1} \sin\frac{m\pi\alpha}{a_1},$$

where

$$w_{m0} = \frac{4\gamma R^3}{\pi} \frac{C_{11}}{\Omega} \frac{R^2}{m\left[P_1\left(\frac{m\pi}{a_1}\right)^4 + R^2\right]};$$

$$w_{m1} = \frac{4\gamma R^3}{\pi} \frac{C_{11}}{\Omega} \frac{R^2\lambda_{m1}}{\Delta_{m1}};$$

$$\lambda_{m1} = \left(\frac{m\pi}{a_1}\right)^4 + \left(\frac{\Omega}{C_{11}C_{66}} - 2\frac{C_{12}}{C_{11}}\right)\left(\frac{m\pi}{a_1}\right)^2 + \frac{C_{22}}{C_{11}};$$

$$\Delta_{m1} = P_1\left(\frac{m\pi}{a_1}\right)^8 + P_3\left(\frac{m\pi}{a_1}\right)^6 + P_5\left(\frac{m\pi}{a_1}\right)^4 + \\ + P_4\left(\frac{m\pi}{a_1}\right)^2 + P_2 + R^2\left(\frac{m\pi}{a_1}\right)^4.$$

Let the specific dimensions of the shell be: $l/R = 0{,}5$; $h/R = 0{,}12$; $h = 1{,}2$ *см*; $t_1 = t_3 = 0{,}1$ *см*; $t_2 = 1{,}0$ *см* (see Figures 65 and 66). Furthermore, assume that we have two variants for the elastic constants of the shell layers:

Variant I

$$E_1^{I} = E_2^{I} = E_1^{III} = E_2^{III} = E,$$
$$G_{12}^{I} = G_{12}^{III} = 0{,}3846E,$$
$$E_1^{II} = 0{,}05E,\ E_2^{II} = 0{,}5E,\ G_{12}^{II} = 0{,}05E,$$
$$\nu_1^{I} = \nu_2^{I} = \nu_1^{III} = \nu_2^{III} = 0{,}3,$$
$$\nu_1^{II} = \nu_2^{II} = 0;$$

Variant II

$$E_1^{I} = E_2^{I} = E_1^{III} = E_2^{III} = E.$$
$$G_{12}^{I} = G_{12}^{III} = 0{,}3846E,$$
$$E_1^{II} = 0{,}5E,\ E_2^{II} = 0{,}05E,\ G_{12}^{II} = 0{,}05E$$
$$\nu_1^{I} = \nu_2^{I} = \nu_1^{III} = \nu_2^{III} = 0{,}3,\ \nu_1^{II} = \nu_2^{II} = 0$$

Using the presented data and formulas (3.1), (5.40), (6.1), (6.2) and (8.43), we obtain for w_{m0} and w_{m1}

m	Variant I		Variant II	
	$\frac{\pi\Omega}{4\gamma R^2 C_{11}} w_{m0}$	$\frac{\pi\Omega}{4\gamma R^2 C_{11}} w_{m1}$	$\frac{\pi\Omega}{4\gamma R^2 C_{11}} w_{m0}$	$\frac{\pi\Omega}{4\gamma R^2 C_{11}} w_{m1}$
1	0,3893	0,3945	0,1351	0,1259
3	$26{,}03 \cdot 10^{-4}$	$25{,}88 \cdot 10^{-4}$	$6{,}42 \cdot 10^{-4}$	$5{,}39 \cdot 10^{-4}$
5	$2{,}038 \cdot 10^{-4}$	$2{,}033 \cdot 10^{-4}$	$0{,}50 \cdot 10^{-4}$	$0{,}498 \cdot 10^{-4}$

Examining these results, it is easily seen that here we have also confirmed the result obtained in the fourth example of a cylindrical shell of open profile, namely: in the case of short shells ($l/R = 0{,}5$) it is invariably better to have larger values of stiffness in the longitudinal direction of the shell. It is also evident from the table that in the discussed case the series entering into the formula for ω converge somewhat rapidly; the remaining design quantities may be found in similar fashion.

SECTION 10. INTEGRATION OF EQUATIONS IN THE ENGINEERING THEORY OF CYLINDRICAL SHELLS BY THE METHOD OF SINGLE TRIGONOMETRIC SERIES

The method of calculation of cylindrical shells by means of single trigonometric series has been widely discussed in the literature.*,**,***,****,***** In this connection we shall here discuss only the fundamental assumptions of the method, which, as it appears to us, are sufficient for a description of the method of calculation of anisotropic laminar cylindrical shells.

The initial premises of the method will be presented in much the same manner as given by A. L. Gol'denveyzer without introducing any substantial changes in his exposition, and without further citation thereof.

We shall deal chiefly with questions of integration of the homogeneous equation of solution of the problem. The particular integrals of inhomogeneous equations, as is known, they may be obtained by any one of a number of methods from inhomogeneous systems (4.27) and (5.48) or, in the special case, from the inhomogeneous equation of solution in (5.64). In addition, in order to determine the internal forces, displacements and stresses in the shell layers corresponding to the particular solution it is necessary to use the corresponding formulas presented in Sections 4 and 5 of this chapter.

In many cases of practical interest, for the ordinarily encountered surface loads the particular integral may be constructed from the membrane theory (see Chapter III).

A. Circular Cylindrical Shell of Open Profile. Let us discuss an orthotropic multi-layer cylindrical shell of open circular directions of the coordinate lines. Let the shell subtend a rectangular plane $(a \times b_1)$ and have

the following dimensions: along the generatrix a, along the arc of transverse circle b, radius of curvature of coordinate surface R.

The coordinate system is so chosen that the coefficients of the first quadratic form A and B are equal to unity (1.1). Moreover, let the shell be freely supported over the lateral curvilinear edges $\alpha = 0$, $\alpha = a$ (see Figure 56), that is, over the curvilinear edges we have the following boundary conditions:

---

*A.I. Lur'ye, see pp. 205-227 of work cited in footnote **, on p. 203.

**V.Z. Vlasov, see pp. 351-398 of work cited in footnote *, on p. 201.

***V.V. Novozhilov, see pp. 179-232 of work cited in footnote ***, on p. 208.

****A.L. Gol'denveyzer, see pp. 211-281 of work cited in footnote *, on p.203.

*****W. Schnell, Krafteinleitung in versteifte Kreiszylinderschalen, Teil I, Die orthotrope Schale. Zeitschrift für Flugwissenschaften, III, No. 12, 1955.

$$\begin{aligned} &\text{when} \quad \alpha = 0: \quad w = 0,\ v = 0,\ T_1 = 0,\ M_1 = 0; \\ &\text{when} \quad \alpha = a: \quad w = 0,\ v = 0,\ T_1 = 0,\ M_1 = 0. \end{aligned} \qquad (10.1)$$

The boundary conditions at the rectilinear edges of the shell ($\beta = \beta_1$, $\beta = \beta_2$). may be arbitrary.

The equation of solution of the discussed shell in the case of the homogeneous problem is described as follows:

$$P_1 \frac{\partial^8 \Psi}{\partial \alpha^8} + P_3 \frac{\partial^8 \Psi}{\partial \alpha^6 \partial \beta^2} + P_5 \frac{\partial^8 \Psi}{\partial \alpha^4 \partial \beta^4} + P_4 \frac{\partial^8 \Psi}{\partial \alpha^2 \partial \beta^6} + P_2 \frac{\partial^8 \Psi}{\partial \beta^8} - -2Q_1 \frac{\partial^6 \Psi}{\partial \alpha^6} - 2Q_3 \frac{\partial^6 \Psi}{\partial \alpha^4 \partial \beta^2} - 2Q_4 \frac{\partial^6 \Psi}{\partial \alpha^2 \partial \beta^4} + \frac{1}{R^2} \frac{\partial^4 \Psi}{\partial \alpha^4} = 0. \qquad (10.2)$$

Let us seek the principal function of equation of solution (10.2) in the form

$$\Psi = \sum_{m=1}^{\infty} \psi_m(\beta) \sin \lambda_m \alpha, \qquad (10.3)$$

where

$$\lambda_m = \frac{m\pi}{a}, \qquad (10.4)$$

m is an integer and $\psi_m(\beta)$ is the new sought function depending only on the single variable β. Inserting the value of $\Psi(\alpha, \beta)$ from (10.3) into equation (10.2), we arrive at relationships of the form

$$\sum_{m=1}^{\infty} d_n(\psi_m) \sin \lambda_m \alpha = 0, \qquad (10.5)$$

which may be fulfilled if it is required that

$$d_m(\psi_m) \equiv P_2 \frac{d^8 \psi_m}{d\beta^8} - P_4 \lambda_m^2 \frac{d^6 \psi_m}{d\beta^6} + (P_5 \lambda_m^4 + 2Q_4 \lambda_m^2) \frac{d^4 \psi_m}{d\beta^4} - -(P_3 \lambda_m^6 + 2Q_3 \lambda_m^4) \frac{d^2 \psi_m}{d\beta^2} + \left(P_1 \lambda_m^8 + 2Q_1 \lambda_m^6 + \frac{1}{R^2} \lambda_m^4\right) \psi_m = 0. \qquad (10.6)$$

The problem of determining the integrals of the homogeneous equation of solution (10.2) also reduces to the integration of such equations with constant coefficients. There will be as many equations of type (10.6) as there are terms in the series (10.3). To each term of the series (10.3) there corresponds a certain stress and deformation of the multi-layer orthotropic shell.

The internal forces, displacements, angles of rotation and stresses within the shell layers may be calculated by means of formulas (5.67)-(5.77). Inserting the value of $\Psi(\alpha, \beta)$ from (10.3) into these formulas, we present each of the design quantities in the form of a trigonometric series

$$
\begin{aligned}
T_1 &= \sum_m T_{1m} \sin \lambda_m \alpha, & M_1 &= \sum_m M_{1m} \sin \lambda_m \alpha, \\
T_2 &= \sum_m T_{2m} \sin \lambda_m \alpha, & M_2 &= \sum_m M_{2m} \sin \lambda_m \alpha, \\
S &= \sum_m S_m \cos \lambda_m \alpha, & H &= \sum_m H_m \cos \lambda_m \alpha, \\
N_1 &= \sum_m N_{1m} \cos \lambda_m \alpha, & N_2 &= \sum_m N_{2m} \sin \lambda_m \alpha, \\
u &= \sum_m U_m \cos \lambda_m \alpha, & v &= \sum_m V_m \sin \lambda_m \alpha, \\
w &= \sum_m W_m \sin \lambda_m \alpha, & \varphi_2 &= \sum_m \varphi_{2m} \sin \lambda_m \alpha,
\end{aligned}
$$

$$\sigma^i_\alpha = \sum_m \sigma^i_{\alpha m} \sin \lambda_m \alpha, \quad \sigma^i_\beta = \sum_m \sigma^i_{\beta m} \sin \lambda_m \alpha$$

$$\tau^i_{\alpha\beta} = \sum_m \tau^i_{\alpha\beta m} \cos \lambda_m \alpha,$$

where all the coefficients of these series depend only on a single variable β and have the following form:

internal forces

$$
\begin{aligned}
T_{1m} = -\Big[& \frac{1}{R} \lambda_m^2 \frac{d^2\psi_m}{d\beta^2} + \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} \lambda_m^4 \frac{d^2\psi_m}{d\beta^2} - \\
& - \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} \right) \lambda_m^2 \frac{d^4\psi_m}{d\beta^4} + \\
& + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} \frac{d^6\psi_m}{d\beta^6} \Big];
\end{aligned}
\tag{10.7}
$$

$$
\begin{aligned}
T_{2m} = & \frac{1}{R} \lambda_m^4 \psi_m + \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} \lambda_m^6 \psi_m - \\
& - \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} \right) \lambda_m^4 \frac{d^2\psi_m}{d\beta^2} + \\
& + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} \lambda_m^2 \frac{d^4\psi_m}{d\beta^4};
\end{aligned}
\tag{10.8}
$$

$$
\begin{aligned}
S_m = & \frac{1}{R} \lambda_m^3 \frac{d\psi_m}{d\beta} + \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} \lambda_m^5 \frac{d\psi_m}{d\beta} - \\
& - \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} \right) \lambda_m^3 \frac{d^3\psi_m}{d\beta^3} + \\
& + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} \lambda_m \frac{d^5\psi_m}{d\beta^5};
\end{aligned}
\tag{10.9}
$$

bending and torsional moments

$$
\begin{aligned}
M_{1m} = & (D_{11} - D_{11}^0) \text{æ}_{1m} + (D_{12} - D_{12}^0) \text{æ}_{2m} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} T_{1m} + \\
& + \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} T_{2m};
\end{aligned}
\tag{10.10}
$$

$$
\begin{aligned}
M_{2m} = & (D_{22} - D_{22}^0) \text{æ}_{2m} + (D_{12} - D_{12}^0) \text{æ}_{1m} + \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} T_{2m} + \\
& + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} T_{1m};
\end{aligned}
\tag{10.11}
$$

$$H_m = 2\left(D_{66} - D_{66}^0\right) G_{12m} + \frac{K_{66}}{C_{66}} S_m, \tag{10.12}$$

where the following additional notation is introduced:

$$G_{1m} = \frac{C_{11}}{\Omega}\lambda_m^6 \psi_m - \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\lambda_m^4 \frac{d^2\psi_m}{d\beta^2} + \frac{C_{22}}{\Omega}\lambda_m^2 \frac{d^4\psi_m}{d\beta^4}; \tag{10.13}$$

$$G_{2m} = -\frac{C_{11}}{\Omega}\lambda_m^4 \frac{d^2\psi_m}{d\beta^2} + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\lambda_m^2 \frac{d^4\psi_m}{d\beta^4} - \frac{C_{22}}{\Omega}\frac{d^6\psi_m}{d\beta^6}; \tag{10.14}$$

$$G_{12m} = -\frac{C_{11}}{\Omega}\lambda_m^5 \frac{d\psi_m}{d\beta} + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\lambda_m^3 \frac{d^3\psi_m}{d\beta^3} - \frac{C_{22}}{\Omega}\lambda_m \frac{d^5\psi_m}{d\beta^5}; \tag{10.15}$$

displacements and angle of rotation

$$W_m = \frac{C_{11}}{\Omega}\lambda_m^4 \psi_m - \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\lambda_m^2 \frac{d^2\psi_m}{d\beta^2} + \frac{C_{22}}{\Omega}\frac{d^4\psi_m}{d\beta^4}; \tag{10.16}$$

$$\begin{aligned} U_m = {} & \frac{K_{11}}{\Omega}\lambda_m^5 \psi_m - \frac{C_{22}K_{11} - C_{12}(K_{12} + 2K_{66})}{C_{66}\Omega}\lambda_m^3 \frac{d^2\psi_m}{d\beta^2} + \\ & + \frac{C_{22}(K_{12} + 2K_{66}) - K_{22}(C_{12} + C_{66})}{C_{66}\Omega}\lambda_m \frac{d^4\psi_m}{d\beta^4} + \\ & + \frac{1}{R}\frac{C_{12}}{\Omega}\lambda_m^3 \psi_m + \frac{1}{R}\frac{C_{22}}{\Omega}\lambda_m \frac{d^2\psi_m}{d\beta^2}; \end{aligned} \tag{10.17}$$

$$\begin{aligned} V_m = {} & \frac{K_{22}}{\Omega}\frac{d^5\psi_m}{d\beta^5} - \frac{C_{11}K_{22} - C_{12}(K_{12} + 2K_{66})}{C_{66}\Omega}\lambda_m^2 \frac{d^3\psi_m}{d\beta^3} + \\ & + \frac{C_{11}(K_{12} + 2K_{66}) - K_{11}(C_{12} + C_{66})}{C_{66}\Omega}\lambda_m^4 \frac{d\psi_m}{d\beta} - \frac{1}{R}\frac{C_{22}}{\Omega}\frac{d^3\psi_m}{d\beta^3} + \\ & + \frac{1}{R}\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right)\lambda_m^2 \frac{d\psi_m}{d\beta}; \end{aligned} \tag{10.18}$$

$$\begin{aligned} \varphi_{2m} = {} & \left(\frac{1}{R}\frac{K_{22}}{\Omega} - \frac{C_{22}}{\Omega}\right)\frac{d^6\psi_m}{d\beta^6} - \left[\frac{1}{R^2}\frac{C_{22}}{\Omega} + \right. \\ & \left. + \frac{C_{11}K_{22} - C_{12}(K_{12} + 2K_{66})}{C_{66}\Omega R}\lambda_m^2 - \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\lambda_m^2\right]\frac{d^4\psi_m}{d\beta^4} + \\ & + \left[\frac{C_{11}(K_{12} + 2K_{66}) - K_{11}(C_{12} + C_{66})}{C_{66}\Omega R}\lambda_m^4 - \frac{C_{11}}{\Omega}\lambda_m^4 + \right. \\ & \left. + \frac{1}{R^2}\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right)\lambda_m^2\right]\frac{d\psi_m}{d\beta}; \end{aligned} \tag{10.19}$$

stresses in the shell layers

$$\begin{aligned} \sigma_{\alpha m}^l = {} & \left(B_{11}^l \frac{C_{22}}{\Omega} - B_{12}^l \frac{C_{12}}{\Omega}\right) T_{1m} - \left(B_{11}^l \frac{C_{12}}{\Omega} - B_{12}^l \frac{C_{11}}{\Omega}\right) T_{2m} - \\ & - \left(B_{11}^l \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} + B_{12}^l \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^l\right) G_{2m} - \\ & - \left(B_{11}^l \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} + B_{12}^l \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} - \gamma B_{11}^l\right) G_{1m}; \end{aligned} \tag{10.20}$$

$$\begin{aligned} \sigma_{\beta m}^l = {} & \left(B_{22}^l \frac{C_{11}}{\Omega} - B_{12}^l \frac{C_{12}}{\Omega}\right) T_{2m} - \left(B_{22}^l \frac{C_{12}}{\Omega} - B_{12}^l \frac{C_{22}}{\Omega}\right) T_{1m} - \\ & - \left(B_{22}^l \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} + B_{12}^l \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^l\right) G_{1m} - \\ & - \left(B_{22}^l \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} + B_{12}^l \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} - \gamma B_{22}^l\right) G_{2m}; \end{aligned} \tag{10.21}$$

$$\tau^{l}_{\alpha\beta m} = \frac{B^{l}_{66}}{C_{66}} S_m - 2B^{l}_{66}\left(\frac{K_{66}}{C_{66}} - \gamma\right) G_{12m}. \tag{10.22}$$

In view of the unwieldiness of the expressions for N_{1m} and N_{2m} they will not be presented here. However, where necessary, having the values of M_{1m}, M_{2m} and H_m, by means of equilibrium equations (4.4) and (4.5) the formulas for N_{1m} and N_{2m} may be given in elementary form.

Let us turn to a discussion of the problems associated with the boundary conditions. Let the particular integral corresponding to the load be obtained and let design quantities corresponding to the partial integral be found and representable by means of trigonometric series in terms of variable α, so that, in particular,

$$\left.\begin{aligned} T_1^0 &= \sum_m T^0_{1m} \sin\lambda_m\alpha, & v^0 &= \sum_m V^0_m \sin\lambda_m\alpha, \\ w^0 &= \sum_m W^0_m \sin\lambda_m\alpha, & M_1^0 &= \sum_m M^0_{1m} \sin\lambda_m\alpha, \end{aligned}\right\} \tag{10.23}$$

where T^0_{1m}, V^0_m, W^0_m, M^0_{1m} are known functions depending only on a single variable β.

Then for the design quantities it is evident that we obtain

$$\left.\begin{aligned} T_1 &= \sum_m T_{1m} \sin\lambda_m\alpha + \sum_m T^0_{1m} \sin\lambda_m\alpha; \\ M_1 &= \sum_m M_{1m} \sin\lambda_m\alpha + \sum_m M^0_{1m} \sin\lambda_m\alpha; \\ v &= \sum_m V_m \sin\lambda_m\alpha + \sum_m V^0_m \sin\lambda_m\alpha; \\ w &= \sum_m W_m \sin\lambda_m\alpha + \sum_m W^0_m \sin\lambda_m\alpha. \end{aligned}\right\} \tag{10.24}$$

Hence, considering (10.4), we note that the boundary conditions (10.1) are satisfied for each term of the expansion individually.

It remains to discuss the boundary conditions at the rectilinear edges of the shell. For the sake of concreteness we shall assume, for example, that with $\beta = \beta_1$ the normal displacement w has the given value $w(\alpha, \beta_1) = w^*(\alpha)$.

Representing $w^*(\alpha)$ in the form of a trigonometric series

$$w^* = \sum_m W^*_m \sin\lambda_m\alpha, \tag{10.25}$$

we must satisfy the quality

$$\sum_m W'_m(\beta_1)\sin\lambda_m\alpha + \sum_m W''_m(\beta_1)\sin\lambda_m\alpha = \sum_m W^*_m \sin\lambda_m\alpha, \tag{10.26}$$

which is equivalent to requiring

$$W_m(\beta_1) + W''_m(\beta_1) = W^*_m, \tag{10.27}$$

that is, also at the rectilinear edges of the shell boundary conditions of the discussed type are satisfied for each term of expansion individually. There are other types of boundary conditions which do not permit term-by-term fulfillment. However, as shall not be interested in these.

Thus, integration of equation of solution (10.2) and imposition of the boundary conditions (both for the rectilinear and for the curvilinear edges of the shell) may be performed for each number m individually. Consequently, for each m we shall have a group of eight boundary conditions (four for each rectilinear edge of the shell) imposed on the extreme values of function ψ_m which satisfied the eighth-order differential equation (10.6).

B. Circular Cylindrical Shell of Closed Profile. Let us discuss an orthotropic multi-layer cylindrical shell of closed circular profile in which the principal directions of elasticity coincide with the directions of the coordinate lines. The coordinate system is so chosen that A = B = R (1.2). Let the length of the shell be l and the radius of curvature R (Figure 65).

The equation of solution of the discussed shell in the case of the homogeneous problem is written as follows:

$$P_1\frac{\partial^8\Psi}{\partial\alpha^8} + P_3\frac{\partial^8\Psi}{\partial\alpha^6\partial\beta^2} + P_5\frac{\partial^8\Psi}{\partial\alpha^4\partial\beta^4} + P_4\frac{\partial^8\Psi}{\partial\alpha^2\partial\beta^6} + P_2\frac{\partial^8\Psi}{\partial\beta^8} - $$
$$- 2R^2\left(Q_1\frac{\partial^6\Psi}{\partial\alpha^6} + Q_3\frac{\partial^6\Psi}{\partial\alpha^4\partial\beta^2} + Q_4\frac{\partial^6\Psi}{\partial\alpha^2\partial\beta^4}\right) + R^2\frac{\partial^4\Psi}{\partial\alpha^4} = 0. \tag{10.28}$$

The principal function of the equation of solution (10.28) will be sought in the form

$$\Psi = \sum_{m=0}^{\infty}(\psi'_m\cos m\beta + \psi''_m\sin m\beta), \tag{10.29}$$

where m is an integer, $\psi'_m(\beta)$ and $\psi''_m(\beta)$ are functions depending only on one variable β. Inserting the value of $\Psi(\alpha, \beta)$ from (10.29) into equation equation (10.28), we arrive at relationships of the form

$$\sum_{m=0}^{\infty}[d_m(\psi'_m)\cos m\beta + d_m(\psi''_m)\sin m\beta] = 0, \tag{10.30}$$

which may be fulfilled if it is required that for each ψ'_m and ψ''_m there exists

$$\begin{aligned} d_m(\psi_m) \equiv P_1 \frac{d^8\psi_m}{d\alpha^8} - (P_3 m^2 + 2R^2 Q_1)\frac{d^6\psi_m}{d\alpha^6} + \\ + (P_5 m^4 + 2R^2 m^2 Q_3 + R^2)\frac{d^4\psi_m}{d\alpha^4} - \\ - (P_4 m^6 + 2R^2 m^4 Q_4)\frac{d^2\psi_m}{d\alpha^2} + P_2 m^8 \psi_m = 0. \end{aligned} \tag{10.31}$$

The problem of determining the integrals of such equations with constant coefficients. To each component of the series (10.29) there corresponds a certain stress and deformation of the anisotropic laminar shell. The internal forces, displacements, angles of rotation and layer stresses corresponding to this stress may be calculated by means of formulas (5.67)-(5.77). For example, assuming

$$\Psi = \psi'_m \cos m\beta, \tag{10.32}$$

we obtain

$$\begin{array}{ll} T_1 = T_{1m}\cos m\beta, & M_1 = M_{1m}\cos m\beta, \\ T_2 = T_{2m}\cos m\beta, & M_2 = M_{2m}\cos m\beta, \\ S = S_m \sin m\beta, & H = H_m \sin m\beta, \\ N_1 = N_{1m}\cos m\beta, & N_2 = N_{2m}\sin m\beta, \\ u = U_m \cos m\beta, & v = V_m \sin m\beta, \\ w = W_m \cos m\beta, & \varphi_1 = \varphi_{1m}\cos m\beta, \\ \sigma^i_\alpha = \sigma^i_{\alpha m}\cos m\beta, & \sigma^i_\beta = \sigma^i_{\beta m}\cos m\beta, \\ \tau^i_{\alpha\beta} = \tau_{\alpha\beta m}\sin m\beta. & \end{array}$$

where all coefficients depend only on a single variable β and have the following form:

$$\begin{aligned} T_{1m} = -\frac{1}{R^5}\Bigg[m^2\frac{d^2\psi'_m}{d\alpha^2} - \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\frac{m^2}{R}\frac{d^4\psi'_m}{d\alpha^4} + \\ + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right)\frac{m^4}{R}\frac{d^2\psi'_m}{d\alpha^2} - \\ - \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\frac{m^6}{R}\psi'_m\Bigg]; \end{aligned} \tag{10.33}$$

$$\begin{aligned} T_{2m} = \frac{1}{R^5}\Bigg[\frac{d^4\psi'_m}{d\alpha^4} - \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\frac{1}{R}\frac{d^6\psi'_m}{d\alpha^6} + \\ + \left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right)\frac{m^2}{R}\frac{d^4\psi'_m}{d\alpha^4} - \\ - \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\frac{m^4}{R}\frac{d^2\psi'_m}{d\alpha^2}\Bigg]; \end{aligned} \tag{10.34}$$

$$S_m = \frac{1}{R^5}\left[m\,\frac{d^3\psi'_m}{da^3} - \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\,\frac{m}{R}\,\frac{d^5\psi'_m}{da^5} + \right.$$
$$+\left(\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - 2\,\frac{K_{66}}{C_{66}} + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\right)\frac{m^3}{R}\,\frac{d^3\psi'_m}{da^3} -$$
$$\left. - \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\,\frac{m^5}{R}\,\frac{d\psi'_m}{da}\right]; \qquad (10.35)$$

bending and torsional moments

$$M_{1m} = (D_{11} - D^0_{11})\,G_{1m} + (D_{12} - D^0_{12})\,G_{2m} +$$
$$+ \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\,T_{1m} + \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\,T_{2m}; \qquad (10.36)$$

$$M_{2m} = (D_{22} - D^0_{22})\,G_{2m} + (D_{12} - D^0_{12})\,G_{1m} +$$
$$+ \frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega}\,T_{2m} + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\,T_{1m}. \qquad (10.37)$$

$$H_m = 2\,(D_{66} - D^0_{66})\,G_{12m} + \frac{K_{66}}{C_{66}}\,S_n, \qquad (10.38)$$

where the following additional notation is introduced:

$$G_{1m} = -\frac{1}{R^6}\left[\frac{C_{11}}{\Omega}\,\frac{d^6\psi'_m}{da^6} - \right.$$
$$\left. - \left(\frac{1}{C_{66}} - 2\,\frac{C_{12}}{\Omega}\right) m^2\,\frac{d^4\psi'_m}{da^4} + \frac{C_{22}}{\Omega}\,m^4\,\frac{d^2\psi'_m}{da^2}\right]; \qquad (10.39)$$

$$G_{2m} = \frac{1}{R^6}\left[\frac{C_{11}}{\Omega}\,m^2\,\frac{d^4\psi'_m}{da^4} - \right.$$
$$\left. - \left(\frac{1}{C_{66}} - 2\,\frac{C_{12}}{\Omega}\right) m^4\,\frac{d^2\psi'_m}{da^2} + \frac{C_{22}}{\Omega}\,m^6\psi'_m\right]; \qquad (10.40)$$

$$G_{12m} = \frac{1}{R^6}\left[\frac{C_{11}}{\Omega}\,m\,\frac{d^5\psi'_m}{da^5} - \right.$$
$$\left. - \left(\frac{1}{C_{66}} - 2\,\frac{C_{12}}{\Omega}\right) m^3\,\frac{d^3\psi'_m}{da^3} + \frac{C_{22}}{\Omega}\,m^5\,\frac{d\psi'_m}{da}\right]; \qquad (10.41)$$

displacements and angle of rotation

$$W_m = \frac{1}{R^4}\left[\frac{C_{11}}{\Omega}\,\frac{d^4\psi_m}{da^4} - \left(\frac{1}{C_{66}} - 2\,\frac{C_{12}}{\Omega}\right) m^2\,\frac{d^2\psi'_m}{da^2} + \frac{C_{22}}{\Omega}\,m^4\psi'_m\right]; \qquad (10.42)$$

$$U_m = \frac{1}{R^5}\left[\frac{K_{11}}{\Omega}\,\frac{d^5\psi'_m}{da^5} - \frac{C_{22}K_{11} - C_{12}(K_{12} + 2K_{66})}{C_{66}\Omega}\,m^2\,\frac{d^3\psi'_m}{da^3} + \right.$$
$$+ \frac{C_{22}(K_{12} + 2K_{66}) - K_{22}(C_{12} + C_{66})}{C_{66}\Omega}\,m^4\,\frac{d\psi'_m}{da} -$$
$$\left. - R\,\frac{C_{12}}{\Omega}\,\frac{d^3\psi'_m}{da^3} - R\,\frac{C_{22}}{\Omega}\,m^2\,\frac{d\psi'_m}{da}\right]; \qquad (10.43)$$

$$V_m = -\frac{1}{R^5}\left[\frac{K_{22}}{\Omega} m^5 \psi'_m - \frac{C_{11}K_{22} - C_{12}(K_{12} + 2K_{66})}{C_{66}\Omega} m^3 \frac{d^2\psi'_m}{d\alpha^2} + \right.$$
$$+ \frac{C_{11}(K_{12} + 2K_{66}) - K_{11}(C_{12} + C_{66})}{C_{66}\Omega} m \frac{d^4\psi'_m}{d\alpha^4} +$$
$$\left. + R\frac{C_{22}}{\Omega} m^3 \psi'_m - R\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right) m \frac{d^2\psi'_m}{d\alpha^2}\right]; \qquad (10.44)$$

$$\varphi_{1m} = -\frac{1}{R^5}\left[\frac{C_{11}}{\Omega}\frac{d^5\psi'_m}{d\alpha^5} - \right.$$
$$\left. - \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) m^2 \frac{d^3\psi'_m}{d\alpha^3} + \frac{C_{22}}{\Omega} m^4 \frac{d\psi'_m}{d\alpha}\right]. \qquad (10.45)$$

Stresses within Shell Layers. Formulas (10.20)-(10.22) remain unchanged, but operators T_{jm} and G_{jm} must be assumed to take their new values (10.33)-(10.35) and (10.39)-(10.41).

Similarly, assuming instead of (10.32)

$$\Psi = \psi''_m \sin m\beta, \qquad (10.46)$$

for the internal forces, displacements, angles of rotation and stresses within the shell layers we obtain formulas analogous to the previous formulas wherein T_1, T_2, M_1, M_2, N_1, u, and ω will contain the factor sin mβ, while S, H, N_2 and v will contain the factor cos mβ. Concerning the operators in (10.33)-(10.44), here ψ'_m is replaced by ψ''_m, and m is replaced by (-m).

Thus, having the integrals of ψ'_m and ψ''_m of equation (10.31), for any m we may construct the stress-strain state of the discussed shell wherein, in accordance with (10.32) and (10.45), this state is analyzed as consisting of a symmetrical state relative to the internal generatrix $\beta = 0$ and of an inversely symmetrical state relative to $\beta = 0$.

Examining the formulas presented here, it is not difficult to see that the conditions of periodicity for variable β are fulfilled in each term of the expansions individually. However, here, in addition to the conditions of periodicity, it is necessary to fulfill the boundary conditions at the edges of the shell. Assuming that the particular integral of the inhomogeneous problem is known, the internal forces, displacements, angles of rotation and stresses in the shell layers corresponding to this particular solution may be represented by means of trigonometric series of the form (10.29) in which, as in (10.23), the coefficients will be known functions. Hence, for concreteness, assuming, for example, that with $\alpha = \alpha_1$ (at one end of the shell) the tangential displacement u has a given value $u(\alpha_1, \beta) = u_*(\beta)$, it must be required that

$$\sum_m [U'_m(\alpha_1)\cos m\beta + U''_m(\alpha_1)\sin m\beta] + \\ + \sum_m [U'_{m0}(\alpha_1)\cos m\beta + U''_{m0}(\alpha_1)\sin m\beta] = \\ = \sum_m [U'_{m_*}\cos m\beta + U''_{m_*}\sin m\beta]. \qquad (10.47)$$

where it is assumed that

$$u_0 = \sum_{m=0}^{\infty} [U'_{m0}(\alpha)\cos m\beta + U''_{m0}(\alpha)\sin m\beta]$$

represents the tangential displacement corresponding to the particular integral, while

$$u_* = \sum_{m=0}^{\infty} [U'_{m_*}\cos m\beta + U''_{m_*}\sin m\beta]$$

is the expansion of the given function $u_*(\beta)$; into a trigonometric series. Hence, as was stated above, coefficients U'_{m_*} and U''_{m_*} are known functions. This equality may be satisfied on the assumption that

$$\left.\begin{aligned} U'_m(\alpha_1) + U'_{m0}(\alpha_1) &= U'_{m_*}, \\ U''_m(\alpha_1) + U''_{m0}(\alpha_1) &= U''_{m_*}, \end{aligned}\right\} \qquad (10.48)$$

that is, the conditions imposed on the coefficients of the symmetrical stress — strain state (the coefficients denoted by a single prime) do not converge with conditions imposed on the coefficients of the inversely symmetrical stress — strain state (the coefficients denoted by a double prime).

There are also other types of boundary conditions which do not permit separate fulfillment of the boundary conditions for the symmetrical and inversely symmetrical stresses. However, they will not be of interest to us.

Thus, integration of the equation of solution (10.28) and imposition of boundary conditions may be performed separately for each number m and separately for the symmetrical and inversely symmetrical stresses of the shell. Then, on each sought function ψ'_m and ψ''_m there will be imposed eight boundary conditions for the definition of the eight arbitrary constants entering into each of these functions.

Without going into detail, we shall state that in the case where the shell consists of an odd number of orthotropic layers symmetrically arranged relative to the median coordinate surface of the shell (Sections 6, 7, 8) all the necessary equations and formulas may be derived by assuming in (10.2)-(10.48) $K_{jk} = 0$.

Examining the problems of orthotropic cylindrical shells of open and closed profile, we note that the first and perhaps the principal stage in calculation of these types of cylindrical shells is the integration of an ordinary eighth-order linear differential equation with constant coefficients (that is, in the case of shells of open profile equation (10.6) and in the case of shells of closed profile equation (10.31)).

As usual, in the case of shells of open profile, assuming

$$\psi_m = C_m e^{s_m 3} \qquad (C_m = \text{const}), \tag{10.49}$$

and in the case of shells of closed profile

$$\psi_m = C_m e^{p_m 2} \qquad (C_m = \text{const}), \tag{10.50}$$

from equations (10.6) and (10.31) we obtain the eighth-order characteristic equations

$$s_m^8 + A_1 s_m^6 + A_2 s_m^4 + A_3 s_m^2 + A_4 = 0; \tag{10.51}$$

$$p_m^8 + B_1 p_m^6 + B_2 p_m^4 + B_3 p_m^2 + B_4 = 0, \tag{10.52}$$

where for the coefficients of the characteristic equations we have:

in the case of shells of open profile

$$\left.\begin{aligned} A_1 &= -\frac{P_4\lambda_m^2}{P_2}, & A_4 &= \frac{P_1\lambda_m^8 + 2Q_1\lambda_m^6 + \frac{1}{R^2}\lambda_m^4}{P_2}, \\ A_3 &= -\frac{P_3\lambda_m^6 + 2Q_3\lambda_m^4}{P_2}, & A_2 &= \frac{P_5\lambda_m^4 + 2Q_4\lambda_m^2}{P_2}; \end{aligned}\right\} \tag{10.53}$$

in the case of shells of closed profile

$$\left.\begin{aligned} B_1 &= -\frac{P_3 m^2 + 2R^2 Q_1}{P_1}, & B_3 &= -\frac{P_4 m^6 + 2R^2 Q_4 m^2}{P_1}, \\ B_4 &= \frac{P_2 m^8}{P_1}, & B_2 &= \frac{P_5 m^4 + 2R^2 Q_3 m^2 + R^2}{P_1}. \end{aligned}\right\} \tag{10.54}$$

It is seen from (10.49) and (10.50) that both in the case of open and in the case of closed shells the form of functions ψ_m depends on the roots of the corresponding characteristic equations (10.51) and (10.52).

For actual problems of anisotropic laminar shells we have a great variety of coefficients A_j and B_j, in virtue of which we will also have various roots for the characteristic equations (10.51) and (10.52). In this connection it is evident that for differential equations (10.6) and (10.31) we shall have solutions of extremely varied types. This variety of solutions of differential equations (10.6) and (10.31) cannot be discussed in detail here. However, in each case it is not difficult to construct the solutions of equations (10.6) and (10.31) and thereby obtain the solutions for the problems presented here. In addition, the entire procedure for calculation of the discussed anisotropic shell is essentially the same as the procedure for calculation for the corresponding isotropic shell.

In this connection it is not considered convenient to introduce numerical examples for they would, in all likelihood, provide no new information concerning the theory of anisotropic laminar shells or the techniques for calculation of shells by the method of single trigonometric series.

22001

CHAPTER VI

SHALLOW ANISOTROPIC SHELLS

SECTION 1. BASIC PREMISES. INITIAL RELATIONSHIPS AND EQUATIONS

We shall discuss a shallow shell consisting of an arbitrary number of homogeneous anisotropic layers. It is assumed that at each point of each layer there exists only one plane of axial symmetry parallel to the coordinate surface of the shell. As in the case of a cylindrical shell, without disturbing the general validity we assume that the coordinate surface of the shell $(\gamma = 0)$ coincides with the internal boundary surface of the shell (see Figures 8 and 11).

Along with the fundamental hypothesis of nondeformable normals as given for the entire stack of the shell as a whole, the theory proposed here is also based on the following additional assumptions*, **:

a) In the first two equilibrium equations we may disregard the terms k_1N_1 and k_2N_2;

b) In the expressions relating the components of bending deformation $(\varkappa_1, \varkappa_2, \tau)$ to displacements we may retain only those terms which contain the normal displacement w;

c) The first two equations of continuity and the sixth equilibrium equation are identically satisfied (in fact, they are satisfied only approximately);

---

*V. Z. Vlasov, Obshchaya teoriya obolochek, Gostekhizdat, 1949, pp. 301-304.

**A. L. Gol'denveyzer, Teoriya uprugikh tonkikh obolochek, Gostekhizdat, 1953, pp. 365-367, 454-459.

d) The system of curvilinear orthogonal coordinates α and β is so chosen so that it satisfies the emphatic inequality $\frac{AB}{R_1 R_2} \ll 1$.

In virtue of these assumptions for the discussed shell we obtain the following initial relationships and equations:

equilibrium equations

$$\frac{\partial BT_1}{\partial \alpha} - \frac{\partial B}{\partial \alpha} T_2 + \frac{\partial AS}{\partial \beta} + \frac{\partial A}{\partial \beta} S = -ABX; \qquad (1.1)$$

$$\frac{\partial AT_2}{\partial \beta} - \frac{\partial A}{\partial \beta} T_1 + \frac{\partial BS}{\partial \alpha} + \frac{\partial B}{\partial \alpha} S = -ABY \cdot \qquad (1.2)$$

$$-(k_1 T_1 + k_2 T_2) + \frac{1}{AB}\left(\frac{\partial BN_1}{\partial \alpha} + \frac{\partial AN_2}{\partial \beta}\right) = -Z; \qquad (1.3)$$

$$\frac{\partial BM_1}{\partial \alpha} + \frac{\partial AH}{\partial \beta} + \frac{\partial A}{\partial \beta} H - \frac{\partial B}{\partial \alpha} M_2 = ABN_1; \qquad (1.4)$$

$$\frac{\partial AM_2}{\partial \beta} + \frac{\partial BH}{\partial \alpha} + \frac{\partial B}{\partial \alpha} H - \frac{\partial A}{\partial \beta} M_1 = ABN_2; \qquad (1.5)$$

elasticity relationships

$$T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{16}\omega + K_{11}\varkappa_1 + K_{12}\varkappa_2 + K_{16}\tau; \qquad (1.6)$$

$$T_2 = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + C_{26}\omega + K_{22}\varkappa_2 + K_{12}\varkappa_1 + K_{26}\tau; \qquad (1.7)$$

$$S = C_{66}\omega + C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + K_{66}\tau + K_{16}\varkappa_1 + K_{26}\varkappa_2; \qquad (1.8)$$

$$M_1 = D_{11}\varkappa_1 + D_{12}\varkappa_2 + D_{16}\tau + K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + K_{16}\omega; \qquad (1.9)$$

$$M_2 = D_{22}\varkappa_2 + D_{12}\varkappa_1 + D_{26}\tau + K_{22}\varepsilon_2 + K_{12}\varepsilon_1 + K_{26}\omega; \qquad (1.10)$$

$$H = D_{66}\tau + D_{16}\varkappa_1 + D_{26}\varkappa_2 + K_{66}\omega + K_{16}\varepsilon_1 + K_{26}\omega. \qquad (1.11)$$

where, as usual, for the stiffnesses we have (Chapter II, Section 8) (see Figures 8 and 11)

$$C_{jk} = \sum_{s=1}^{p} B_{jk}^{s}(\delta_s - \delta_{s-1}); \qquad (1.12)$$

$$K_{jk} = \frac{1}{2}\sum_{s=1}^{p} B_{jk}^{s}(\delta_s^2 - \delta_{s-1}^2); \qquad (1.13)$$

$$D_{jk} = \frac{1}{3}\sum_{s=1}^{p} B_{jk}^{s}(\delta_s^3 - \delta_{s-1}^3); \qquad (1.14)$$

geometric relationships

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_1 w; \qquad (1.15)$$

$$\varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + k_2 w; \qquad (1.16)$$

$$\omega = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right); \qquad (1.17)$$

$$\varkappa_1 = -\frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta}; \qquad (1.18)$$

$$\varkappa_2 = -\frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) - \frac{1}{BA^2}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha}; \qquad (1.19)$$

$$\tau = -\frac{2}{AB}\left(\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right); \qquad (1.20)$$

equation of compatibility of deformation

$$\begin{aligned} k_2\varkappa_1 + k_1\varkappa_2 + \frac{1}{AB}\Big\{\frac{\partial}{\partial\alpha}\frac{1}{A}\Big[B\frac{\partial\varepsilon_2}{\partial\alpha} + \frac{\partial B}{\partial\alpha}(\varepsilon_2 - \varepsilon_1) - \\ - \frac{A}{2}\frac{\partial\omega}{\partial\beta} - \frac{\partial A}{\partial\beta}\omega\Big] + \frac{\partial}{\partial\beta}\frac{1}{B}\Big[A\frac{\partial\varepsilon_1}{\partial\beta} + \frac{\partial A}{\partial\beta}(\varepsilon_1 - \varepsilon_2) - \\ - \frac{B}{2}\frac{\partial\omega}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\omega\Big]\Big\} = 0. \qquad (1.21) \end{aligned}$$

We have listed only those relationships and equations which will be used. The stresses in the layers will be determined by means of the usual formulas (2.4.2)—(2.4.9).

An approximate theory of shells, based on the presented initial relationships and equations (1.1) -(1.21) in the case of an isotropic shell, has been expounded by A. L. Gol'denveyzer* as a theory for the calculation of shells with large variability index. However, as has been correctly pointed out by A.L. Gol'denveyzer, this theory may also be used in solution of problems which would appear to have little in common with the problem of calculating stresses with large variability index (e.g., in investigating problems of shallow shells and shells with zero Gaussian curvature, in discussing problems of representation of a simple edge effect).

The initial relationships and equations (1.1) —(1.21) have been used by many authors, who have found for them completely different applications. L. Donnell**, Kh. M. Mushtari*** and U.N. Rabotnov**** have used them for investigation of local stability. S.M. Feynberg*****, V.V. Novozhilov******, A.I. Lur'ye******* and T.T. Khachaturyan******** have used these relationships in constructing a theory of cylindrical shells. In the solution of problems

---

*A.L. Gol'denveyzer, see pp. 358-368 and 454-460 of work cited in footnote ** on p. 277.

**L. Donnell, Stability of Thin Walled Tubes under Torsion. NACA, Rep. No. 479, 1933.

***Kh.M.Mushtari, Some generalizations of the theory of thin shells, Izvestiya fiz.-mat. ob-va pri Kazanskom un-te, Vol. XI, series 8, 1938.

****Yu.N. Rabotnov, Local stability of shells, DAN SSSR, LII, No. 2, 1946.

*****S.M. Feynberg, Toward the problem of a moment theory for cylindrical shells, Proyekt i standart, No. 12, 1936.

******V.V. Novozhilov, Teoriya tonkikh obolochek, Oborongiz, 1941. (Cited in same author's monograph of same title, 1951).

*******A.I. Lur'ye, Statika tonkostennykh uprugikh obolochek, Gostekhizdat, 1947, pp. 177-243.

********T.T. Khachaturyan, Shallow cylindrical shells, Soobshcheniya In-ta mat. i mekh. AN ArmSSR, No. 4, 1949.

of shallow shells these relationships have been used by V. Z. Vlasov*, S. A. Ambartsumyan** and N. A. Alumyae***. A. L. Gol'denveyzer**** and Yu. N. Rabotnov***** have used these relationships in constructing equations for the edge effect.

Here we have cited only a small part of the vast number of works devoted to the discussed problem. However, the citations listed here suffice to confirm that the approximate theory of shells based on the initial relationships and equations in (1.1) -(1.21) may be used for investigation of numerous, completely different problems in the theory of shells. This is to say that although interpretation of the discussed approximate theory of shells as a theory of shells with large variability index or as a theory of shallow shells is perfectly accurate, such interpretation does not reflect the wide possibilities of this theory. From this point of view the title of the present chapter, "shallow anisotropic shells," is doubly formal in nature.

SECTION 2. EQUATIONS OF SOLUTION AND DESIGN FORMULAS

Let us limit ourselves to investigation of the case in which a shell is loaded by a normally applied surface load $Z \neq 0$. The remaining components of the surface load X and Y are equal to 0. Assuming******, *******

$$T_1 = \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial\varphi}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial\varphi}{\partial\alpha}; \qquad (2.1)$$

$$T_2 = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial\varphi}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial\varphi}{\partial\beta}; \qquad (2.2)$$

$$S = -\frac{1}{AB}\left(\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\varphi}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial\varphi}{\partial\alpha}\right), \qquad (2.3)$$

where $\varphi = \varphi(\alpha, \beta)$ is the sought function of stress, we identically (with the accuracy of the given assumptions) satisfy the first two equilibrium equations (1.1) and (1.2) ($X = 0$, $Y = 0$), and from the third equation (1.3) we obtain

$$\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}k_2\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}k_1\frac{\partial}{\partial\beta}\right)\right]\varphi - \\ - \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BN_1) + \frac{\partial}{\partial\beta}(AN_2)\right] = Z. \qquad (2.4)$$

---

*V.Z. Vlasov, Basic differential equations of the general theory of shells, PMM, Vol. VIII, No. 2, 1944.

**S.A. Ambartsumyan, Calculation of cylindrical shells with curvilinear generatrix, 1946 (dissertation) .

***N.A. Alumyae, Differential equilibrium equations of thin-walled elastic shells in the post-critical stage, PMM, Vol. XIII, No. 1, 1949.

****A.L. Gol'denveyzer, Qualitative investigation of stress of a thin shell, PMM, Vol. XI, No. 6, 1945.

*****Yu. N. Rabotnov, Boundary-zone equation in the theory of shells, DAN SSSR, XLVII, No. 5, 1945.

******V.Z. Vlasov, see footnote * on p. 277.

*******A.L. Gol'denveyzer, see footnote ** on p. 277.

Solving the elasticity relationships (1.9) -(1.11) for the deformation components, we obtain

$$\left.\begin{aligned} \varepsilon_1 &= A_{11}T_1 + A_{12}T_2 + A_{16}S - d_{11}\varkappa_1 - d_{12}\varkappa_2 - d_{16}\tau; \\ \varepsilon_2 &= A_{12}T_1 + A_{22}T_2 + A_{26}S - d_{21}\varkappa_1 - d_{22}\varkappa_2 - d_{26}\tau; \\ \omega &= A_{16}T_1 + A_{26}T_2 + A_{66}S - d_{61}\varkappa_1 - d_{62}\varkappa_2 - d_{66}\tau, \end{aligned}\right\} \qquad (2.5)$$

where for elastic coefficients A_{jk} and d_{jk} we have (5.5.8) -(5.5.18) .

From elasticity relationships (1.9) -(1.11) , by means of (2.5) , eliminating the deformation components ε_1, ε_2 and ω, then considering (2.1) -(2.3) and (1.18) -(1.20) , we obtain for the moments

$$M_1 = I_1(d_{11})\varphi - I_2(D_{11} - D^0_{11})w; \qquad (2.6)$$

$$M_2 = I_1(d_{22})\varphi - I_2(D_{22} - D^0_{22})w; \qquad (2.7)$$

$$H = I_1(d_{66})\varphi - I_2(D_{66} - D^0_{66})w, \qquad (2.8)$$

where for linear operators I_j we have

$$\begin{aligned} I_1(d_{jk}) = {} & \frac{d_{1k}}{B}\left[\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + \frac{1}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha}\right] - \\ & - \frac{d_{6k}}{AB}\left[\frac{\partial^2}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\alpha}\right] + \\ & + \frac{d_{2k}}{A}\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\right) + \frac{1}{B^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}\right]; \end{aligned} \qquad (2.9)$$

$$\begin{aligned} I_2(D_{jk} - D^0_{jk}) = {} & \frac{(D_{2k} - D^0_{2k})}{B}\left[\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + \frac{1}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha}\right] + \\ & + 2\frac{(D_{6k} - D^0_{6k})}{AB}\left[\frac{\partial^2}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\alpha}\right] + \\ & + \frac{(D_{1k} - D^0_{1k})}{A}\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\right) + \frac{1}{B^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}\right]. \end{aligned} \qquad (2.10)$$

The formulas for moments (2.6) -(2.8) contain the bending stiffnesses D^0_{jk} , which have the form (5.5.23) -(5.5.28) . From the last two equilibrium equations (1.4) -(1.5) , considering the values of moments (2.6) -(2.8) , for the transverse forces we obtain

$$\begin{aligned} N_1 = {} & \frac{1}{AB}\left\{\frac{\partial}{\partial\beta}[AI_1(d_{66})] + \frac{\partial A}{\partial\beta}I_1(d_{66}) + \right. \\ & \left. + \frac{\partial}{\partial\alpha}[BI_1(d_{11})] - \frac{\partial B}{\partial\alpha}I_1(d_{22})\right\}\varphi - \\ & - \frac{1}{AB}\left\{\frac{\partial}{\partial\beta}[AI_2(D_{66} - D^0_{66})] + \frac{\partial A}{\partial\beta}I_2(D_{66} - D^0_{66}) + \right. \\ & \left. + \frac{\partial}{\partial\alpha}[BI_2(D_{11} - D^0_{11})] - \frac{\partial B}{\partial\alpha}I_2(D_{22} - D^0_{22})\right\}w; \end{aligned} \qquad (2.11)$$

$$\begin{aligned} N_2 = \frac{1}{AB}\Big\{ & \frac{\partial}{\partial\alpha}[B l_1(d_{66})] + \frac{\partial B}{\partial\alpha} l_1(d_{66}) + \\ & + \frac{\partial}{\partial\beta}[A l_1(d_{22})] - \frac{\partial A}{\partial\beta} l_1(d_{11})\Big\}\varphi - \\ & - \frac{1}{AB}\Big\{\frac{\partial}{\partial\alpha}[B l_2(D_{66}-D^0_{66})] + \frac{\partial B}{\partial\alpha} l_2(D_{66}-D^0_{66}) + \\ & + \frac{\partial}{\partial\beta}[A l_2(D_{22}-D^0_{22})] - \frac{\partial A}{\partial\beta} l_2(D_{11}-D^0_{11})\Big\} w. \end{aligned} \qquad (2.12)$$

From (2.5) , considering (2.1) -(2.3) and (1.18) -(1.20) , we obtain for the relative deformations and shear

$$\varepsilon_1 = l_1(A_{11})\varphi + l_2^*(d_{11}) w; \qquad (2.13)$$

$$\varepsilon_2 = l_1(A_{22})\varphi + l_2^*(d_{22}) w; \qquad (2.14)$$

$$\omega = l_1(A_{66})\varphi + l_2^*(d_{66}) w, \qquad (2.15)$$

where for the linear operators l_j^* we have

$$\begin{aligned} l_2^*(d_{kj}) = \frac{d_{k2}}{B}\Big[& \frac{\partial}{\partial\beta}\Big(\frac{1}{B}\frac{\partial}{\partial\beta}\Big) + \frac{1}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha}\Big] + \\ & + 2\frac{d_{k6}}{AB}\Big[\frac{\partial^2}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\alpha}\Big] + \\ & + \frac{d_{k1}}{A}\Big[\frac{\partial}{\partial\alpha}\Big(\frac{1}{A}\frac{\partial}{\partial\alpha}\Big) + \frac{1}{B^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}\Big]. \end{aligned} \qquad (2.16)$$

Inserting the values of transverse forces from (2.11) and (2.12) into equilibrium equation (2.4) as well as the values of deformation and of the components of change in curvature from (2.13) -(2.15) , (1.18) and (1.19) into the equation of compatibility of deformation (1.21) , we obtain the following system of differential equations of solution of the problem:

$$\left.\begin{aligned} & L_1(D_{jk}-D^0_{jk})w - L_3(d_{jk})\varphi + \nabla_r\varphi = Z; \\ & L_2(A_{jk})\varphi + L_4(d_{jk})w - \nabla_r w = 0. \end{aligned}\right\} \qquad (2.17)$$

where for the linear operators L_j and ∇_r we have

$$\begin{aligned} L_1(D_{jk}-D^0_{jk}) = \frac{1}{AB}\frac{\partial}{\partial\alpha}\frac{1}{A}\Big\{ & \frac{\partial}{\partial\beta}[A l_2(D_{66}-D^0_{66})] + \\ & + \frac{\partial A}{\partial\beta} l_2(D_{66}-D^0_{66}) + \frac{\partial}{\partial\alpha}[B l_2(D_{11}-D^0_{11})] - \\ & - \frac{\partial B}{\partial\alpha} l_2(D_{22}-D^0_{22})\Big\} + \frac{1}{AB}\frac{\partial}{\partial\beta}\frac{1}{B}\Big\{\frac{\partial}{\partial\alpha}[B l_2(D_{66}-D^0_{66})] + \\ & + \frac{\partial B}{\partial\alpha} l_2(D_{66}-D^0_{66}) + \frac{\partial}{\partial\beta}[A l_2(D_{22}-D^0_{22})] - \frac{\partial A}{\partial\beta} l_2(D_{11}-D^0_{11})\Big\}; \end{aligned} \qquad (2.18)$$

$$\begin{aligned} L_2(A_{jk}) = \frac{1}{AB}\frac{\partial}{\partial\alpha}\frac{1}{A}\Big\{ & B\frac{\partial}{\partial\alpha} l_1(A_{22}) + \frac{\partial B}{\partial\alpha}[l_1(A_{22}) - l_1(A_{11})] - \\ & - \frac{A}{2}\frac{\partial}{\partial\beta} l_1(A_{66}) - \frac{\partial A}{\partial\beta} l_1(A_{66})\Big\} + \frac{1}{AB}\frac{\partial}{\partial\beta}\frac{1}{B}\Big\{A\frac{\partial}{\partial\beta} l_1(A_{11}) + \\ & + \frac{\partial A}{\partial\beta}[l_1(A_{11}) - l_1(A_{22})] - \frac{B}{2}\frac{\partial}{\partial\alpha} l_1(A_{66}) - \frac{\partial B}{\partial\alpha} l_1(A_{66})\Big\}; \end{aligned} \qquad (2.19)$$

$$L_3(d_{jk}) = \frac{1}{AB}\frac{\partial}{\partial\alpha}\frac{1}{A}\left\{\frac{\partial}{\partial\beta}[AI_1(d_{66})] + \frac{\partial A}{\partial\beta} I_1(d_{66}) + \frac{\partial}{\partial\alpha}[BI_1(d_{11})] - \right.$$
$$\left. - \frac{\partial B}{\partial\alpha} I_1(d_{22})\right\} + \frac{1}{AB}\frac{\partial}{\partial\beta}\frac{1}{B}\left\{\frac{\partial}{\partial\alpha}[BI_1(d_{66})] + \frac{\partial B}{\partial\alpha} I_1(d_{66}) + \right.$$
$$\left. + \frac{\partial}{\partial\beta}[AI_1(d_{22})] - \frac{\partial A}{\partial\beta} I_1(d_{11})\right\}; \qquad (2.20)$$

$$L_4(d_{jk}) = \frac{1}{AB}\frac{\partial}{\partial\alpha}\frac{1}{A}\left\{B\frac{\partial}{\partial\alpha} I_2^*(d_{22}) + \frac{\partial B}{\partial\alpha}[I_2^*(d_{22}) - I_2^*(d_{11})] - \right.$$
$$\left. - \frac{A}{2}\frac{\partial}{\partial\beta} I_2^*(d_{66}) - \frac{\partial A}{\partial\beta} I_2^*(d_{66})\right\} + \frac{1}{AB}\frac{\partial}{\partial\beta}\frac{1}{B}\left\{A\frac{\partial}{\partial\beta} I_2^*(d_{11}) + \right.$$
$$\left. + \frac{\partial A}{\partial\beta}[I_2^*(d_{11}) - I_2^*(d_{22})] - \frac{B}{2}\frac{\partial}{\partial\alpha} I_2^*(d_{66}) - \frac{\partial B}{\partial\alpha} I_2^*(d_{66})\right\}; \qquad (2.21)$$

$$\nabla_r = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A} k_2 \frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B} k_1 \frac{\partial}{\partial\beta}\right)\right]. \qquad (2.22)$$

Thus, the problem of equilibrium of a "shallow" multi-layer anisotropic shell subtending an arbitrary surface reduces to a system of two differential equations of solution (2.17) in terms of two unknown functions: the stress function: $\varphi(\alpha, \beta)$ and the displacement function $w(\alpha, \beta)$, in terms of which the design quantities of the problem are defined.

It is interesting to note that in the special case of a multi-layer anisotropic plate ($k_1 = 0$, $k_2 = 0$) system (2.17) is written as follows:

$$\left.\begin{aligned} &L_1(D_{jk} - D_{jk}^0) w - L_3(d_{jk}) \varphi = Z; \\ &L_2(A_{jk}) \varphi + L_4(d_{jk}) w = 0, \end{aligned}\right\} \qquad (2.23)$$

that is, in distinction from single-layer or symmetrical multi-layer plates, it does not break down into two independent equations*, **.

Orthotropic Shell. In the special case where the shell consists of an arbitrary number of homogeneous orthotropic layers so that the principal directions of elasticity at each point of each layer coincide with the directions of the corresponding coordinate lines, for coefficients B_{jk} we have (2.12.2), that is:

$$\left.\begin{aligned} &B_{11}^i = \frac{E_1^i}{1 - \nu_1^i\nu_2^i}, \quad B_{22}^i \frac{E_2^i}{1 - \nu_1^i\nu_2^i}, \quad B_{66}^i = G_{12}^i, \\ &B_{12}^i = \frac{\nu_2^i E_1^i}{1 - \nu_1^i\nu_2^i} = \frac{\nu_1^i E_2^i}{1 - \nu_1^i\nu_2^i}, \quad B_{16}^i = B_{26}^i = 0. \end{aligned}\right\} \qquad (2.24)$$

---

*S.A. Ambartsumyan, Some basic equations in the theory of a thin laminar shell, DAN ArmSSR, VIII, No. 5, 1948.

**S.A. Ambartsumyan, Toward the problem of calculation of laminar anisotropic shells, Izvestiya An ArmSSR (FMYeiT nauki), VI, No. 3, 1953.

By virtue of (2.24) all quantities with subscripts 16, 61, 26, and 62 become zero, whereas in the case of a cylindrical shell the quantities A_{jk}, d_{jk} and D^0_{jk} will be determined by means of formulas (5.5.40) -(5.5.47) . In this case the system of equations of solution (2.17) remains unchanged externally and operators L_j (2.18) -(2.22) also remain unchanged while for operators I_j we have from (2.9) , (2.10) and (2.16)

$$I_1(d_{11}) = d_{11}\nabla_{p\beta} + d_{21}\nabla_{p\alpha};$$
$$I_1(d_{22}) = d_{22}\nabla_{p\alpha} + d_{12}\nabla_{p\beta};$$
$$I_1(d_{66}) = -d_{66}\nabla_{h\alpha\beta};$$
$$I_1(A_{11}) = \frac{C_{22}}{\Omega}\nabla_{p\beta} - \frac{C_{12}}{\Omega}\nabla_{p\alpha};$$
$$I_1(A_{22}) = \frac{C_{11}}{\Omega}\nabla_{p\alpha} - \frac{C_{12}}{\Omega}\nabla_{p\beta};$$
$$I_1(A_{66}) = -\frac{1}{C_{66}}\nabla_{h\alpha\beta};$$
$$I_2(D_{11} - D^0_{11}) = (D_{11} - D^0_{11})\nabla_{p\alpha} + (D_{21} - D_{21})\nabla_{p\beta};$$
$$I_2(D_{22} - D^0_{22}) = (D_{22} - D^0_{22})\nabla_{p\beta} + (D_{12} - D^0_{12})\nabla_{p\alpha};$$
$$I_2(D_{66} - D^0_{66}) = 2(D_{66} - D^0_{66})\nabla_{h\alpha\beta};$$
$$I_2^*(d_{11}) = d_{11}\nabla_{p\alpha} + d_{12}\nabla_{p\beta};$$
$$I_2^*(d_{22}) = d_{22}\nabla_{p\beta} + d_{21}\nabla_{p\alpha};$$
$$I_2^*(d_{66}) = 2d_{66}\nabla_{h\alpha\beta}.$$

where for the hyperbolic and parabolic operators we have

$$\nabla_{h\alpha\beta} = \frac{1}{AB}\left(\frac{\partial^2}{\partial\alpha\,\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\beta}\right); \qquad (2.25)$$

$$\nabla_{p\alpha} = \frac{1}{A}\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\right) + \frac{1}{B^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}\right]; \qquad (2.26)$$

$$\nabla_{p\beta} = \frac{1}{B}\left[\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial}{\partial\beta}\right) + \frac{1}{A^2}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha}\right]. \qquad (2.27)$$

For the internal tangential forces T_1, T_2 and S we have (2.1) -(2.3) and from (2.6) -(2.8) , considering the values of operators I_j, we obtain for the moments

$$M_1 = -(D_{11} - D^0_{11})\nabla_{p\alpha}w - (D_{12} - D^0_{12})\nabla_{p\beta}w + \\ + \frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega}\nabla_{p\beta}\varphi + \frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega}\nabla_{p\alpha}\varphi; \qquad (2.28)$$

$$M_2 = -(D_{22} - D^0_{22})\nabla_{p\beta}w - (D_{12} - D^0_{12})\nabla_{p\alpha}w + \\ + \frac{K_{22}C_{11} - C_{12}K_{12}}{\Omega}\nabla_{p\alpha}\varphi + \frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega}\nabla_{p\beta}\varphi; \qquad (2.29)$$

$$H = -\gamma\left(D_{66} - D_{66}^{0}\right)\nabla_{h\alpha\beta}w - \frac{K_{66}}{C_{66}}\nabla_{h\alpha\beta}\varphi. \qquad (2.30)$$

For the design stresses in the shell layers, proceeding from the well-known, often cited formulas and relationships we obtain

$$\begin{aligned}\sigma_{\alpha}^{i} = {} & \left(B_{11}^{i}\frac{C_{22}}{\Omega} - B_{12}^{i}\frac{C_{12}}{\Omega}\right)\nabla_{p\beta}\varphi - \left(B_{11}^{i}\frac{C_{12}}{\Omega} - B_{12}^{i}\frac{C_{11}}{\Omega}\right)\nabla_{p\alpha}\varphi + \\ & + \left(B_{11}^{i}\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} + B_{12}^{i}\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^{i}\right)\nabla_{p\beta}w + \\ & + \left(B_{11}^{i}\frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} + B_{12}^{i}\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} - \gamma B_{11}^{i}\right)\nabla_{p\alpha}w;\end{aligned} \qquad (2.31)$$

$$\begin{aligned}\sigma_{\beta}^{i} = {} & \left(B_{22}^{i}\frac{C_{11}}{\Omega} - B_{12}^{i}\frac{C_{12}}{\Omega}\right)\nabla_{p\alpha}\varphi - \left(B_{22}^{i}\frac{C_{12}}{\Omega} - B_{12}^{i}\frac{C_{22}}{\Omega}\right)\nabla_{p\beta}\varphi + \\ & + \left(B_{22}^{i}\frac{K_{12}C_{11} - K_{11}C_{12}}{\Omega} + B_{12}^{i}\frac{K_{11}C_{22} - K_{12}C_{12}}{\Omega} - \gamma B_{12}^{i}\right)\nabla_{p\alpha}w + \\ & + \left(B_{22}^{i}\frac{K_{22}C_{11} - K_{12}C_{12}}{\Omega} + B_{12}^{i}\frac{K_{12}C_{22} - K_{22}C_{12}}{\Omega} - \gamma B_{22}^{i}\right)\nabla_{p\beta}w;\end{aligned} \qquad (2.32)$$

$$\tau_{\alpha\beta}^{i} = -\frac{B_{66}^{i}}{C_{66}}\nabla_{h\alpha\beta}\varphi + 2B_{66}^{i}\left(\frac{K_{66}}{C_{66}} - \gamma\right)\nabla_{h\alpha\beta}w. \qquad (2.33)$$

Thus, all the design quantities of the problem are determined through formulas (2.1) , (2.3) , (2.28) -(2.33) by means of the sought functions $w(\alpha, \beta)$ and $\varphi(\alpha, \beta)$, which under the boundary conditions may be derived from the system of equations in (2.17) .

The equations of solution in the approximate theory of shells as based on the initial relationships and equations in (1.1) -(1.21) may also be written in displacements (in a manner similar to that in the case of cylindrical shells) ; however, in view of the unwieldiness of these equations we shall not present them here.

SECTION 3. SHELLS CONSISTING OF AN ODD NUMBER OF HOMOGENEOUS ANISOTROPIC LAYERS SYMMETRICALLY ARRANGED RELATIVE TO THE MEDIAN SURFACE

We shall discuss an anisotropic laminar shell consisting of an odd number (2m + 1) of homogeneous anisotropic layers symmetrically arranged relative to the coordinate (median) surface of the shell (see Figure 15) .

In virtue of the symmetrical structure of the shell, for the stiffnesses we have (Chapter II, Section 13)

$$C_{jk} = 2\left[B_{jk}^{m+1}h_{m+1} + \sum_{s=1}^{m} B_{jk}^{s}(h_{s} - h_{s-1})\right]; \qquad (3.1)$$

$$D_{jk}=\frac{2}{3}\left[B_{jk}^{m+1}h_{m+1}^{3}+\sum_{s=1}^{n}B_{jk}^{s}(h_{s}^{3}-h_{s-1}^{3})\right]. \quad (3.2)$$

$$K_{jk}=0. \quad (3.3)$$

by virtue of which, from (5.5.10) -(5.5.18) and (5.5.23) -(5.5.28) , for coefficients d_{jk} and D_{jk}^{0} we have

$$d_{jk}=0, \quad D_{jk}^{0}=0. \quad (3.4)$$

Then from (2.6) -(2.8) , (2.11) and (2.12) , we obtain for the moments and transverse forces

$$M_{1}=-I_{2}(D_{11}-D_{11}^{0})w; \quad (3.5)$$

$$M_{2}=-I_{2}(D_{22}-D_{22}^{0})w; \quad (3.6)$$

$$H=-I_{2}(D_{66}-D_{66}^{0})w; \quad (3.7)$$

$$N_{1}=-\frac{1}{AB}\left\{\frac{\partial}{\partial\beta}[AI_{2}(D_{66}-D_{66}^{0})]+\frac{\partial A}{\partial\beta}I_{2}(D_{66}-D_{66}^{0})+\right.$$
$$\left.+\frac{\partial}{\partial\alpha}[BI_{2}(D_{11}-D_{11}^{0})]-\frac{\partial B}{\partial\alpha}I_{2}(D_{22}-D_{22}^{0})\right\}w; \quad (3.8)$$

$$N_{2}=-\frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}[BI_{2}(D_{66}-D_{66}^{0})]+\frac{\partial B}{\partial\alpha}I_{2}(D_{66}-D_{66}^{0})+\right.$$
$$\left.+\frac{\partial}{\partial\beta}[AI_{2}(D_{22}-D_{22}^{0})]-\frac{\partial A}{\partial\beta}I_{2}(D_{11}-D_{11}^{0})\right\}w. \quad (3.9)$$

while for the internal tangential forces T_1, T_2 and S, as usual, we have (2.1) - (2.3) .

Finally, from (2.4.2) -(2.4.4) and by virture of (1.18) -(1.20) , (2.1) -(2.3) , (2.13) -(2.15) and (2.25) -(2.26) , for the stresses in the shell layers we obtain

$$\sigma_{\alpha}^{i}=(B_{11}^{i}A_{12}+B_{12}^{i}A_{22}+B_{16}^{i}A_{26})\nabla_{p\alpha}\varphi+$$
$$+(B_{11}^{i}A_{11}+B_{12}^{i}A_{12}+B_{16}^{i}A_{16})\nabla_{p\beta}\varphi-(B_{11}^{i}A_{16}+B_{12}^{i}A_{26}+B_{16}^{i}A_{66})\nabla_{h\alpha\beta}\varphi-$$
$$-\gamma(B_{11}^{i}\nabla_{p\alpha}w+B_{12}^{i}\nabla_{p\beta}w+2B_{16}^{i}\nabla_{h\alpha\beta}w); \quad (3.10)$$

$$\sigma_{\beta}^{i}=(B_{22}^{i}A_{12}+B_{12}^{i}A_{11}+B_{26}^{i}A_{16})\nabla_{p\beta}\varphi+$$
$$+(B_{22}^{i}A_{22}+B_{12}^{i}A_{12}+B_{26}^{i}A_{26})\nabla_{p\alpha}\varphi-$$
$$-(B_{22}^{i}A_{26}+B_{12}^{i}A_{16}+B_{26}^{i}A_{66})\nabla_{h\alpha\beta}\varphi-$$
$$-\gamma(B_{22}^{i}\nabla_{p\beta}w+B_{12}^{i}\nabla_{p\alpha}w+2B_{26}^{i}\nabla_{h\alpha\beta}w); \quad (3.11)$$

$$\tau_{\alpha\beta}^{i}=(B_{16}^{i}A_{11}+B_{26}^{i}A_{12}+B_{66}^{i}A_{16})\nabla_{p\beta}\varphi+$$
$$+(B_{16}^{i}A_{12}+B_{26}^{i}A_{22}+B_{66}^{i}A_{26})\nabla_{p\alpha}\varphi-$$
$$-(B_{16}^{i}A_{16}+B_{26}^{i}A_{26}+B_{66}^{i}A_{66})\nabla_{h\alpha\beta}\varphi-$$
$$-\gamma(B_{16}^{i}\nabla_{p\alpha}w+B_{26}^{i}\nabla_{p\beta}w+2B_{66}^{i}\nabla_{h\alpha\beta}w). \quad (3.12)$$

The system of differential equations of solution (2.17) is somewhat simplified and takes the following form:

$$\left.\begin{aligned} L_1(D_{jk})\,w+\nabla_r\varphi &= Z;\\ L_2(A_{jk})\,\varphi-\nabla_r w &= 0, \end{aligned}\right\} \qquad (3.13)$$

where for linear operator $L_1(D_{jk})$ we have from (2.18)

$$\begin{aligned} L_1(D_{jk}) = {} & \frac{1}{AB}\frac{\partial}{\partial\alpha}\frac{1}{A}\left\{\frac{\partial}{\partial\beta}[Al_2(D_{66})]+\right.\\ & \left.+\frac{\partial A}{\partial\beta}l_2(D_{66})+\frac{\partial}{\partial\alpha}[Bl_2(D_{11})]-\frac{\partial B}{\partial\alpha}l_2(D_{22})\right\}+\\ & +\frac{1}{AB}\frac{\partial}{\partial\beta}\frac{1}{B}\left\{\frac{\partial}{\partial\alpha}[Bl_2(D_{66})]+\frac{\partial B}{\partial\alpha}l_2(D_{66})+\right.\\ & \left.+\frac{\partial}{\partial\beta}[Al_2(D_{22})]-\frac{\partial A}{\partial\beta}l_2(D_{11})\right\}, \end{aligned} \qquad (3.14)$$

while linear operators $L_2(A_{jk})$ and ∇_r remain unchanged, that is, they have the values (2.19) and (2.22) .

Thus, in the case of an anisotropic laminar shell with symmetrically arranged layers the equilibrium problem reduces to a simpler system (simpler than in (2.17)) of two differential equations (3.13) in terms of two unknown functions: $\varphi(\alpha,\beta)$ and $w(\alpha,\beta)$. We note that in the special case of a multi-layer anisotropic plate with symmetrically arranged layers ($k_1 = 0$, $k_2 = 0$) the system breaks down into two independent differential equations

$$L_1(D_{jk})\,w = Z; \qquad (3.15)$$

$$L_2(A_{jk})\,\varphi = 0. \qquad (3.16)$$

The first of these quations (3.15) is the equation for transverse bending of a symmetrically assembled anisotropic plate in orthogonal curvilinear coordinates α, β.

Orthotropic Shell. In the case of an orthotropic shell where the shell is symmetrically assembled so that the principal directions of elasticity at each point of each orthotropic layer coincide with the directions of the coordinate lines, for coefficients B^i_{jk} we have (2.24) .

Let us derive the final results. The system of equations of solution (3.13) remains externally unchanged. Operators $L_1(D_{jk})$ in (3.14) , $L_2(A_{jk})$ in (2.19) and ∇_r in (2.22) also remain unchanged, while for the operators l_j entering into $L_1(D_{jk})$ and $L_2(A_{jk})$ we have

$$\left.\begin{aligned}
I_1(A_{11}) &= \frac{C_{22}}{\Omega}\nabla_{p\beta} - \frac{C_{12}}{\Omega}\nabla_{p\alpha};\\
I_1(A_{22}) &= \frac{C_{11}}{\Omega}\nabla_{p\alpha} - \frac{C_{12}}{\Omega}\nabla_{p\beta};\\
I_1(A_{66}) &= -\frac{1}{C_{66}}\nabla_{h\alpha\beta};\\
I_2(D_{11}) &= D_{11}\nabla_{p\alpha} + D_{12}\nabla_{p\beta};\\
I_2(D_{22}) &= D_{22}\nabla_{p\beta} + D_{12}\nabla_{p\alpha};\\
I_2(D_{66}) &= 2D_{66}\nabla_{h\alpha\beta}.
\end{aligned}\right\} \qquad (3.17)$$

The formulas for internal tangential forces T_1, T_2 and S (2.1) -(2.3) remain unchanged.

From (2.28) -(2.30) or from (3.5) -(3.7) , we have for the moments

$$M_1 = -D_{11}\nabla_{p\alpha}w - D_{12}\nabla_{p\beta}w; \qquad (3.18)$$
$$M_2 = -D_{22}\nabla_{p\beta}w - D_{12}\nabla_{p\alpha}w; \qquad (3.19)$$
$$H = -2D_{66}\nabla_{h\alpha\beta}w. \qquad (3.20)$$

From (2.31) -(2.33) or from (3.10) -(3.12) for the design stresses in the shell layers we obtain

$$\begin{aligned}
\sigma_\alpha = \left(B^i_{11}\frac{C_{22}}{\Omega} - B^i_{12}\frac{C_{12}}{\Omega}\right)\nabla_{p\beta}\varphi - {}\\
-\left(B^i_{11}\frac{C_{12}}{\Omega} - B^i_{12}\frac{C_{11}}{\Omega}\right)\nabla_{p\alpha}\varphi - \gamma\left(B^i_{11}\nabla_{p\alpha}w + B^i_{12}\nabla_{p\beta}w\right);
\end{aligned} \qquad (3.21)$$

$$\begin{aligned}
\sigma^i_\beta = \left(B^i_{22}\frac{C_{11}}{\Omega} - B^i_{12}\frac{C_{12}}{\Omega}\right)\nabla_{p\alpha}\varphi - \left(B^i_{22}\frac{C_{12}}{\Omega} - B^i_{12}\frac{C_{22}}{\Omega}\right)\nabla_{p\beta}\varphi - {}\\
-\gamma\left(B^i_{22}\nabla_{p\beta}w + B^i_{12}\nabla_{p\alpha}w\right);
\end{aligned} \qquad (3.22)$$

$$\tau^i_{\alpha\beta} = -\frac{B^i_{66}}{C_{66}}\nabla_{h\alpha\beta}\varphi - 2\gamma B^i_{66}\nabla_{h\alpha\beta}w. \qquad (3.23)$$

Thus, in the case of an orthotropic shell the problem also reduces to the solution of the system of differential equations in (3.3) with the given boundary conditions, since by means of formulas (2.1) -(2.3) , (3.18) -(3.23) all the design quantities of the problem are represented in terms of the sought functions $w(\alpha, \beta)$ and $\varphi(\alpha, \beta)$.

SECTION 4. EXTREMELY SHALLOW SHELLS. BASIC PREMISES, INITIAL RELATIONSHIPS AND EQUATIONS

The title of the present section, "extremely shallow shells," as that of this chapter, is conditional in nature, since the approximate theory of shells discussed herein is applicable not only for calculation of extremely shallow shells[*], [**], [***] but also for calculation of shells with large variability indexes[****], for representation of a simple edge effect[*****], [******], etc. Thus, in treating the theory proposed herein as a theory for extremely shallow shells, we will not forget that it is also quite valid for investigation of other problems in the theory of shells.

We shall discuss an extremely shallow, anisotropic, laminar shell inscribed over a certain portion of an arbitrary surface. It is assumed that the rise of the shell $f \lesssim \frac{1}{5} l$, where l is the smallest specific dimension of the shell at the coordinate surface. It is approximately assumed that the internal geometry of the coordinate surface ($\gamma = 0$) does not differ from the ordinary geometry at a plane. (For shells with zero Gaussian curvature this assumption is omitted.)

We shall assume that in the plan view the discussed shell has the shape of a rectangle with sides a and b. Let x and y be the Cartesian coordinates of a point of the plane of the shell (Figure 67) . Then the square of a linear element on plane xOy is given as follows:

$$ds_n^2 = dx^2 + dy^2, \tag{4.1}$$

that is, the coefficients of the first quadratic form A and B in the chosen coordinate system are equal to unity:

$$A = B = 1. \tag{4.2}$$

For the discussed extremely shallow shell, represented in the curvilinear orthogonal system of coordinates α, β, γ, the first quadratic form of the coordinate surface of the shell is represented with sufficient accuracy in the form (4.1) , that is,

$$ds^2 = d\alpha^2 + d\beta^2, \tag{4.3}$$

which is equivalent to the assumption in (4.2) , that is,

---

[*] S.A. Ambartsumyan, Toward calculation of shallow shells, PMM, Vol. XI, No. 5, 1947.

[**] S.A. Ambartsumyan, Approximate method of calculation of thin, shallow shells, DAN ArmSSR, VI, No. 3, 1947.

[***] V.Z. Vlasov, see pp. 436-453 of work cited in footnote * on p. 277.

[****] A.L. Gol'denveyzer, see footnote on p. 279.

[*****] A.L. Gol'denveyzer, see footnote **** on p. 280.

[******] Yu.N. Rabotnov, see footnote ***** on p. 280.

$$A \approx 1, \quad B \approx 1. \tag{4.4}$$

In the case where coordinates α and β are dimensionless, instead of (4.4) it is assumed that the coefficients of the first quadratic form $A = A(\alpha, \beta)$ and $B = B(\alpha, \beta)$ upon differentiation behave as constants. With the same accuracy for an extremely shallow shell it is assumed that the principal curvatures of the coordinate surface $k_1 = k_1(\alpha, \beta)$ and $k_2 = k_2(\alpha, \beta)$ also behave as constants.

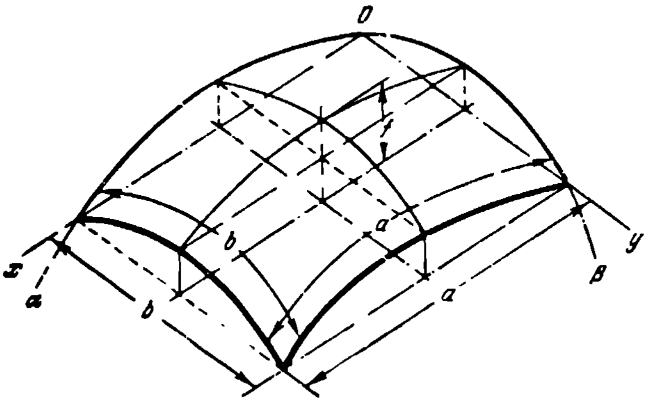

Figure 67

The purely geometric definitions presented here are hardly sufficient for the construction of a theory of extremely shallow shells. To these assumptions (A = const, B = const, k_1 = const, k_2 = const) we must add all the initial assumptions from the theory of shallow shells in general (see Section of this chapter).

With these remarks in mind, for an extremely shallow shell we obtain the following initial relationships and equations:

equilibrium equations

$$\frac{1}{A}\frac{\partial T_1}{\partial \alpha}+\frac{1}{B}\frac{\partial S}{\partial \beta}=-X; \tag{4.5}$$

$$\frac{1}{B}\frac{\partial T_2}{\partial \beta}+\frac{1}{A}\frac{\partial S}{\partial \alpha}=-Y; \tag{4.6}$$

$$-(k_1T_1+k_2T_2)+\frac{1}{A}\frac{\partial N_1}{\partial \alpha}+\frac{1}{B}\frac{\partial N_2}{\partial \beta}=-Z; \tag{4.7}$$

$$\frac{1}{A}\frac{\partial M_1}{\partial \alpha}+\frac{1}{B}\frac{\partial H}{\partial \beta}=N_1; \tag{4.8}$$

$$\frac{1}{B}\frac{\partial M_2}{\partial \beta}+\frac{1}{A}\frac{\partial H}{\partial \alpha}=N_2; \tag{4.9}$$

elasticity relationships (1.6)-(1.11) and formulas for stiffness (1.12)-(1.14) remain unchanged (and it is assumed that the coordinate surface $\gamma = 0$ coincides with the lower surface of ths shell) ;

the geometric relationships

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + k_1 w; \quad \varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + k_2 w;$$
$$\omega = \frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}; \tag{4.10}$$
$$\varkappa_1 = -\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}; \quad \varkappa_2 = -\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}; \quad \tau = -\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta};$$

the equation of compatibility of deformation

$$k_2\varkappa_1 + k_1\varkappa_2 + \frac{1}{A^2}\frac{\partial^2 \varepsilon_2}{\partial \alpha^2} - \frac{1}{AB}\frac{\partial^2 \omega}{\partial \alpha\,\partial \beta} + \frac{1}{B^2}\frac{\partial^2 \varepsilon_1}{\partial \beta^2} = 0; \tag{4.11}$$

stresses in the layers are determined by means of the usual formulas (2.4.2) -(2.4.4)

$$\sigma_\alpha^i = B_{11}^i\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{w}{R_1}\right) + B_{12}^i\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{w}{R_2}\right) + B_{16}^i\left(\frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}\right) -$$
$$- \gamma\left(B_{11}^i \frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + B_{16}^i\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}\right); \tag{4.12}$$

$$\sigma_\beta^i = B_{22}^i\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{w}{R_2}\right) + B_{12}^i\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{w}{R_1}\right) + B_{26}^i\left(\frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}\right) -$$
$$- \gamma\left(B_{22}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + B_{12}^i\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + B_{26}^i\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}\right); \tag{4.13}$$

$$\tau_{\alpha\beta}^i = B_{16}^i\left(\frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{w}{R_1}\right) + B_{26}^i\left(\frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{w}{R_2}\right) + B_{66}^i\left(\frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}\right) -$$
$$- \gamma\left(B_{16}^i\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + B_{26}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + B_{66}^i\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}\right). \tag{4.14}$$

The initial relationships and equations in (4.5) -(4.14) in the the theory of extremely shallow shells may be derived on the assumption that the unknown quantities are substantially increased by differentiation.*,**,***,****

---

*Kh.M. Mushtari, see footnote *** on p. 279.

**S.M. Feynberg, Toward an approximate moment theory of thin-walled shells of arbitrary profile, "Issledovaniya po teorii sooruzheniy," sbornik No. 3, 1939.

***A.L. Gol'denveyzer, see p. 457 of work cited in footnote ** on p. 277.

****A.L. Gol'denveyzer, see pp. 366-367 and 469 of work cited in footnote** on p. 277.

SECTION 5. EQUATIONS OF SOLUTION AND DESIGN FORMULAS IN THE THEORY OF EXTREMELY SHALLOW SHELLS CONSISTING OF AN ARBITRARY NUMBER OF ANISOTROPIC LAYERS

From (1.6) -(1.11) , by virtue of (4.10), we obtain for the internal forces and moments

$$\begin{aligned} T_1 = &\left(C_{11}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+\\ &+\left(\frac{1}{R_1}C_{11}+\frac{1}{R_2}C_{12}-K_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-2K_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-K_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \qquad (5.1)$$

$$\begin{aligned} T_2 = &\left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+\\ &+\left(\frac{1}{R_2}C_{22}+\frac{1}{R_1}C_{12}-K_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-2K_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-K_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \qquad (5.2)$$

$$\begin{aligned} S = &\left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(C_{66}\frac{1}{A}\frac{\partial}{\partial\alpha}+C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+\\ &+\left(\frac{1}{R_1}C_{16}+\frac{1}{R_2}C_{26}-K_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-2K_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-K_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \qquad (5.3)$$

$$\begin{aligned} M_1 = &\left(K_{11}\frac{1}{A}\frac{\partial}{\partial\alpha}+K_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(K_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+K_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+\\ &+\left(\frac{1}{R_1}K_{11}+\frac{1}{R_2}K_{12}-D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-2D_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \qquad (5.4)$$

$$\begin{aligned} M_2 = &\left(K_{12}\frac{1}{A}\frac{\partial}{\partial\alpha}+K_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(K_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}+K_{22}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+\\ &+\left(\frac{1}{R_2}K_{22}+\frac{1}{R_1}K_{12}-D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-2D_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w; \end{aligned} \qquad (5.5)$$

$$\begin{aligned} H = &\left(K_{16}\frac{1}{A}\frac{\partial}{\partial\alpha}+K_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u+\left(K_{66}\frac{1}{A}\frac{\partial}{\partial\alpha}+K_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v+\\ &+\left(\frac{1}{R_1}K_{16}+\frac{1}{R_2}K_{26}-D_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}-2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}-D_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w. \end{aligned} \qquad (5.6)$$

Inserting the values of internal forces into the first three equilibrium equations (4.5) -(4.7) and with consideration of equations (4.8) and (4.9) , we obtain a symmetrical system of three differential equations in terms of the three sought displacements $u(\alpha, \beta)$, $v(\alpha, \beta)$ and $w(\alpha, \beta)$

$$\left.\begin{aligned} &L_{11}(C_{jk})u+L_{12}(C_{jk})v+L_{13}(C_{jk}K_{jk})w=-X;\\ &L_{12}(C_{jk})u+L_{22}(C_{jk})v+L_{23}(C_{jk}K_{jk})w=-Y;\\ &L_{13}(C_{jk}K_{jk})u+L_{23}(C_{jk},K_{jk})v+L_{33}(C_{jk}K_{jk}D_{jk})w=Z. \end{aligned}\right| \qquad (5.7)$$

where for the linear operators we have

$$L_{11}(C_{jk})=C_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}+2C_{16}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}+C_{66}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \qquad (5.8)$$

$$L_{22}(C_{jk}) = C_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + 2C_{26}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + C_{66}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}; \qquad (5.9)$$

$$L_{12}(C_{jk}) = C_{16}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + (C_{12} + C_{66})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + C_{26}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}; \qquad (5.10)$$

$$L_{13}(C_{jk}K_{jk}) = (k_1C_{11} + k_2C_{12})\frac{1}{A}\frac{\partial}{\partial\alpha} + (k_1C_{16} + k_2C_{26})\frac{1}{B}\frac{\partial}{\partial\beta} - E_1(K_{jk}); \qquad (5.11)$$

$$L_{23}(C_{jk}K_{jk}) = (k_2C_{22} + k_1C_{12})\frac{1}{B}\frac{\partial}{\partial\beta} + (k_2C_{26} + k_1C_{16})\frac{1}{A}\frac{\partial}{\partial\alpha} - E_2(K_{jk}); \qquad (5.12)$$

$$\begin{aligned} L_{33}(C_{jk}K_{jk}D_{jk}) = -2\Big[&(k_1K_{11} + k_2K_{12})\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \\ &+ 2(k_1K_{16} + k_2K_{26})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta} + (k_2K_{22} + k_1K_{12})\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\Big] + \\ &+ (k_1^2C_{11} + 2k_1k_2C_{12} + k_2^2C_{22}) + L_1(D_{jk}). \end{aligned} \qquad (5.13)$$

As usual, for the linear operators $E_1(K_{jk})$, $E_2(K_{jk})$ and $L_1(D_{jk})$ we have, respectively, (5.4.25) , (5.4.26) , and (5.5.32) or (5.8.13) .

Thus, with the assigned boundary conditions, by solving the system of differential equations in (5.7) we determine the unknown displacements u, v, and w in terms of which all the design quantities of the problem are represented by means of formulas (4.12) -(4.14) and (5.1) -(5.6) .

The equations fo the theory of extremely shallow anisotropic laminar shells may also be represented in the form of equations of the combined method. Let us construct the equations of the combined method on the basis of the results obtained in Section 2 of this chapter, that is, for the case of practical importance wherein X = 0 and Y = 0*. For an extremely shallow shell, assuming

$$A = \text{const}, \quad B = \text{const}, \qquad (5.14)$$

we obtain from (2.17) the following system of equations of solution:

$$\left.\begin{aligned} L_1(D_{jk} - D_{jk}^0)w - L_3(d_{jk})\varphi + \nabla_r\varphi = Z; \\ L_2(A_{jk})\varphi + L_3(d_{jk})w - \nabla_r w = 0. \end{aligned}\right\} \qquad (5.15)$$

where for the linear operators we have

---

*The equations of the theory of extremely shallow anisotropic laminar shells may also be constructed for the general case in which X ≠ 0, Y ≠ 0 and Z ≠ 0 (see M. Mishonov, Toward a theory of shallow shells, PMM, Vol. XXII, No. 5, 1958 and Section 5, Chapter V of this work) .

$$L_1(D_{jk}-D^0_{jk}) = \frac{D_{11}-D^0_{11}}{A^4}\frac{\partial^4}{\partial\alpha^4} + 4\frac{D_{16}-D^0_{16}}{A^3B}\frac{\partial^4}{\partial\alpha^3\partial\beta} + 2\frac{(D_{12}-D^0_{12})+2(D_{66}-D^0_{66})}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + 4\frac{D_{26}-D^0_{26}}{AB^3}\frac{\partial^4}{\partial\alpha\partial\beta^3} + \frac{D_{22}-D^0_{22}}{B^4}\frac{\partial^4}{\partial\beta^4}; \qquad (5.16)$$

$$L_2(A_{jk}) = \frac{A_{22}}{A^4}\frac{\partial^4}{\partial\alpha^4} - 2\frac{A_{26}}{A^3B}\frac{\partial^4}{\partial\alpha^3\partial\beta} + \frac{A_{66}+2A_{12}}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\partial\beta^2} - 2\frac{A_{16}}{AB^3}\frac{\partial^4}{\partial\alpha\partial\beta^3} + \frac{A_{11}}{B^4}\frac{\partial^4}{\partial\beta^4}; \qquad (5.17)$$

$$L_3(d_{jk}) = \frac{d_{21}}{A^4}\frac{\partial^4}{\partial\alpha^4} + \frac{2d_{26}-d_{61}}{A^3B}\frac{\partial^4}{\partial\alpha^3\partial\beta} + \frac{d_{11}+d_{22}-2d_{66}}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{2d_{16}-d_{62}}{AB^3}\frac{\partial^4}{\partial\alpha\partial\beta^3} + \frac{d_{12}}{B^4}\frac{\partial^4}{\partial\beta^4}; \qquad (5.18)$$

$$\nabla_r = k_2\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + k_1\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}. \qquad (5.19)$$

For the internal forces and moments, from (2.1) -(2.3) and (2.6) - (2.8) , we obtain

$$T_1 = \frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2}, \quad T_2 = \frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2}, \quad S = -\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\partial\beta}; \qquad (5.20)$$

$$M_1 = \left(\frac{d_{11}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{d_{21}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{d_{61}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)\varphi - \left(\frac{D_{11}-D^0_{11}}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2\frac{D_{16}-D^0_{16}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{D_{12}-D^0_{12}}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w, \qquad (5.21)$$

$$M_2 = \left(\frac{d_{22}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{12}}{B^2}\frac{\partial^2}{\partial\beta^2} - \frac{d_{62}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)\varphi - \left(\frac{D_{22}-D^0_{22}}{B^2}\frac{\partial^2}{\partial\beta^2} + 2\frac{D_{26}-D^0_{26}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{D_{12}-D^0_{12}}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)w; \qquad (5.22)$$

$$H = \left(\frac{d_{16}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{d_{26}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{d_{66}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)\varphi - \left(\frac{D_{16}-D^0_{16}}{A^2}\frac{\partial^2}{\partial\alpha^2} + 2\frac{D_{66}-D^0_{66}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{D_{26}-D^0_{26}}{B^2}\frac{\partial^2}{\partial\beta^2}\right)w. \qquad (5.23)$$

The stresses in the layers may be determined on the basis of the general formulas (2.4.2) -(2.4.4) wherein, along with (4.10), it is necessary to consider that

$$\varepsilon_1 = \left(\frac{A_{11}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{A_{12}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{A_{16}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)\varphi + \left(\frac{d_{11}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{12}}{B^2}\frac{\partial^2}{\partial\beta^2} + 2\frac{d_{16}}{AB}\frac{\partial^2}{\partial\alpha\partial\beta}\right)w; \qquad (5.24)$$

$$\varepsilon_2 = \left(\frac{A_{12}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{A_{22}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{A_{26}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi + \\ + \left(\frac{d_{21}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{22}}{B^2}\frac{\partial^2}{\partial\beta^2} + 2\frac{d_{26}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)w; \qquad (5.25)$$

$$\omega = \left(\frac{A_{16}}{B^2}\frac{\partial^2}{\partial\beta^2} + \frac{A_{26}}{A^2}\frac{\partial^2}{\partial\alpha^2} - \frac{A_{66}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\varphi + \\ + \left(\frac{d_{61}}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{d_{62}}{B_2}\frac{\partial^2}{\partial\beta^2} + 2\frac{d_{66}}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)w. \qquad (5.26)$$

The system of equations of solution of the combined method in (5.15) may be reduced to an equivalent single eighth-order differential equation of solution. Assuming

$$w = L_2(A_{jk})\Psi; \qquad (5.27)$$

$$\varphi = \nabla_r\Psi - L_3(d_{jk})\Psi. \qquad (5.28)$$

where $\Psi = \Psi(\alpha, \beta)$ is a certain unknown potential function of coordinates and , we identically satisfy the second equation of the system in (5.15) and from the first equation we obtain

$$L_0(P_{jk})\Psi - 2F(Q_{jk})\Psi + \nabla_r^2\Psi = Z, \qquad (5.29)$$

where $L_0(P_{jk})$ is an eighth-order linear operator, $F(Q_{jk})$ is a sixth-order linear operator, ∇_r^2 is a fourth-order linear operator; these operators have the following form:

$$\left.\begin{aligned} L_0(P_{jk}) &= L_1(D_{jk} - D_{jk}^0)L_2(A_{jk}) + [L_3(d_{jk})]^2; \\ F(Q_{jk}) &= L_3(d_{jk})\nabla_r; \quad \nabla_r^2 = \nabla_r\nabla_r. \end{aligned}\right\} \qquad (5.30)$$

These operators (except ∇_r^2) in the general case of anisotropy are extremely unwieldy, hence we shall not present them in expanded form.

Thus, the eight-order linear differential equation (5.29) is an equation of solution in the theory of extremely shallow anisotropic laminar shells. With given boundary conditions, having the solution of equation (5.28) [that is, $\Psi = \Psi(\alpha, \beta)$], by means of formulas (5.27), (5.28), (5.20) -(5.26) we may find the values of all design quantities of the problem.

Orthotropic Shell*. In the case where the shell consists of an arbitrary number of homogeneous orthotropic layers such that the principal directions of elasticity at each point of each layer coincide with the directions of the corresponding coordinate lines all quantities with subscripts 16, 61, 26, and 62

---

*S.A. Ambartsumyan, see footnote ** on p. 283.

become zero. Then for constants B^i_{jk}, A_{jk}, d_{jk}, D^0_{jk} we will have (2.24) , (5.5.40) -(5.5.47) .

In this case all internal forces, moments and stresses in the layers will be represented by means of the following formulas:

internal forces

$$T_1 = C_{11}\frac{1}{A}\frac{\partial u}{\partial \alpha} + C_{12}\frac{1}{B}\frac{\partial v}{\partial \beta} + C_{11}k_1 w + {} + C_{12}k_2 w - K_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} - K_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}; \quad (5.31)$$

$$T_2 = C_{22}\frac{1}{B}\frac{\partial v}{\partial \beta} + C_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha} + C_{22}k_2 w + {} + C_{12}k_1 w - K_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} - K_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}; \quad (5.32)$$

$$S = C_{66}\frac{1}{A}\frac{\partial v}{\partial \alpha} + C_{66}\frac{1}{B}\frac{\partial u}{\partial \beta} - 2K_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}; \quad (5.33)$$

the bending and torsional moments

$$M_1 = -D_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} - D_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + K_{11}k_1 w + {} + K_{12}k_2 w + K_{11}\frac{1}{A}\frac{\partial u}{\partial \alpha} + K_{12}\frac{1}{B}\frac{\partial v}{\partial \beta}; \quad (5.34)$$

$$M_2 = -D_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} - D_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + K_{22}k_2 w + {} + K_{12}k_1 w + K_{22}\frac{1}{B}\frac{\partial v}{\partial \beta} + K_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha}; \quad (5.35)$$

$$H = -2D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta} + K_{66}\left(\frac{1}{A}\frac{\partial v}{\partial \alpha} + \frac{1}{B}\frac{\partial u}{\partial \beta}\right); \quad (5.36)$$

stresses in the layers

$$\sigma^i_\alpha = B^i_{11}\frac{1}{A}\frac{\partial u}{\partial \alpha} + B^i_{12}\frac{1}{B}\frac{\partial v}{\partial \beta} + B^i_{11}k_1 w + {} + B^i_{12}k_2 w - \gamma\left(B^i_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + B^i_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}\right); \quad (5.37)$$

$$\sigma^i_\beta = B^i_{22}\frac{1}{B}\frac{\partial v}{\partial \beta} + B^i_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha} + B^i_{22}k_2 w + {} + B^i_{12}k_1 w - \gamma\left(B^i_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + B^i_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}\right); \quad (5.38)$$

$$\tau^i_{\alpha\beta} = B^i_{66}\left(\frac{1}{A}\frac{\partial v}{\partial \alpha} + \frac{1}{B}\frac{\partial u}{\partial \beta}\right) - 2\gamma B^i_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\,\partial \beta}. \quad (5.39)$$

Assuming that the unknowns are displacements $u(\alpha, \beta)$, $v(\alpha, \beta)$ and $w(\alpha, \beta)$, in order to determine them we obtain the following symmetrical system of three differential equations in terms of three unknowns;

$$C_{11}\frac{1}{A^2}\frac{\partial^2 u}{\partial \alpha^2}+C_{66}\frac{1}{B^2}\frac{\partial^2 u}{\partial \beta^2}+(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2 v}{\partial \alpha\,\partial \beta}+$$

$$+(k_1C_{11}+k_2C_{12})\frac{1}{A}\frac{\partial w}{\partial \alpha}-E_1(K_{jk})w=-X; \qquad (5.40)$$

$$C_{22}\frac{1}{B^2}\frac{\partial^2 v}{\partial \beta^2}+C_{66}\frac{1}{A^2}\frac{\partial^2 v}{\partial \alpha^2}+(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2 u}{\partial \alpha\,\partial \beta}+$$

$$+(k_2C_{22}+k_1C_{12})\frac{1}{B}\frac{\partial w}{\partial \beta}-E_2(K_{jk})w=-Y; \qquad (5.41)$$

$$(k_1C_{11}+k_2C_{12})\frac{1}{A}\frac{\partial u}{\partial \alpha}-E_1(K_{jk})u+$$

$$+(k_2C_{22}+k_1C_{12})\frac{1}{B}\frac{\partial v}{\partial \beta}-E_2(K_{jk})v+$$

$$+(k_1^2C_{11}+2k_1k_2C_{12}+k_2^2C_{22})w-2(k_1K_{11}+$$

$$+k_2K_{12})\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2}-2(k_2K_{22}+k_1K_{12})\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2}+L_1(D_{jk})w=Z, \qquad (5.42)$$

where

$$E_1(K_{jk})=K_{11}\frac{1}{A^3}\frac{\partial^3}{\partial \alpha^3}+(K_{12}+2K_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial \alpha\,\partial \beta^2}; \qquad (5.43)$$

$$E_2(K_{jk})=K_{22}\frac{1}{B^3}\frac{\partial^3}{\partial \beta^3}+(K_{12}+2K_{66})\frac{1}{BA^2}\frac{\partial^3}{\partial \beta\,\partial \alpha^2}; \qquad (5.44)$$

$$L_1(D_{jk})=D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial \alpha^4}+D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial \beta^4}+2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial \alpha^2\,\partial \beta^2}. \qquad (5.45)$$

In the case of the combined method, where the unknowns are the stress function $\varphi=\varphi(\alpha,\ \beta)$ and displacement function $w=w(\alpha,\ \beta)$, the equations of solution are given as follows:

$$\left.\begin{aligned}L_1(D_{jk}-D_{jk}^0)w-L_3(d_{jk})\varphi+\nabla_r\varphi=Z;\\ L_2(C_{jk})\varphi+L_3(d_{jk})w-\nabla_r w=0,\end{aligned}\right\} \qquad (5.46)$$

where

$$L_2(C_{jk})=\frac{C_{11}}{\Omega}\frac{1}{A^4}\frac{\partial^4}{\partial \alpha^4}+\frac{C_{22}}{\Omega}\frac{1}{B^4}\frac{\partial^4}{\partial \beta^4}+\left(\frac{1}{C_{66}}-2\frac{C_{12}}{\Omega}\right)\frac{1}{A^2B^2}\frac{\partial^4}{\partial \alpha^2\,\partial \beta^2}; \qquad (5.47)$$

$$L_3(d_{jk})=\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega}\frac{1}{A^4}\frac{\partial^4}{\partial \alpha^4}+\left(\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}-\right.$$

$$\left.-2\frac{K_{66}}{C_{66}}+\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega}\right)\frac{1}{A^2B^2}\frac{\partial^4}{\partial \alpha^2\,\partial \beta^2}+\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega}\frac{1}{B^4}\frac{\partial^4}{\partial \beta^4}, \qquad (5.48)$$

while for $L_1(D_{jk}-D_{jk}^0)$ we have (5.45) or (5.5.49) ; it will be recalled that D_{jk}^0 are determined from formulas (5.5.44) -(5.5.47) .

The design formulas of the combined method have the following form:

internal forces and moments

$$T_1=\frac{1}{B^2}\frac{\partial^2\varphi}{\partial \beta^2};\quad T_2=\frac{1}{A^2}\frac{\partial^2\varphi}{\partial \alpha^2};\quad S=-\frac{1}{AB}\frac{\partial^2\varphi}{\partial \alpha\,\partial \beta}; \qquad (5.49)$$

$$M_1 = -\frac{D_{11}-D_{11}^0}{A^2}\frac{\partial^2 w}{\partial\alpha^2} - \frac{D_{12}-D_{12}^0}{B^2}\frac{\partial^2 w}{\partial\beta^2} +$$

$$+\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega}\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2}; \qquad (5.50)$$

$$M_2 = -\frac{D_{22}-D_{22}^0}{B^2}\frac{\partial^2 w}{\partial\beta^2} - \frac{D_{12}-D_{12}^0}{A^2}\frac{\partial^2 w}{\partial\alpha^2} +$$

$$+\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega}\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} + \frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega}\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2}; \qquad (5.51)$$

$$H = -2\left(D_{66}-D_{66}^0\right)\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{K_{66}}{C_{66}}\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}; \qquad (5.52)$$

$$N_1 = \frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega}\frac{1}{A^3}\frac{\partial^3\varphi}{\partial\alpha^3} + \left(\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega} -\right.$$

$$\left. -\frac{K_{66}}{C_{66}}\right)\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2} - E_1\left(D_{jk}-D_{jk}^0\right)w; \qquad (5.53)$$

$$N_2 = \frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega}\frac{1}{B^3}\frac{\partial^3\varphi}{\partial\beta^3} + \left(\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega} -\right.$$

$$\left. -\frac{K_{66}}{C_{66}}\right)\frac{1}{BA^2}\frac{\partial^3\varphi}{\partial\beta\,\partial\alpha^2} - E_2\left(D_{jk}-D_{jk}^0\right)w; \qquad (5.54)$$

stresses in the layers

$$\sigma_\alpha^i = \left(B_{11}^i\frac{C_{22}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - \left(B_{11}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{11}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} +$$

$$+\left(B_{11}^i\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega} + B_{12}^i\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega} - \gamma B_{12}^i\right)\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} +$$

$$+\left(B_{11}^i\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega} + B_{12}^i\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega} - \gamma B_{11}^i\right)\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}; \qquad (5.55)$$

$$\sigma_\beta^i = \left(B_{22}^i\frac{C_{11}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \left(B_{22}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{22}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} +$$

$$+\left(B_{22}^i\frac{K_{12}C_{11}-K_{11}C_{12}}{\Omega} + B_{12}^i\frac{K_{11}C_{22}-K_{12}C_{12}}{\Omega} - \gamma B_{12}^i\right)\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} +$$

$$+\left(B_{22}^i\frac{K_{22}C_{11}-K_{12}C_{12}}{\Omega} + B_{12}^i\frac{K_{12}C_{22}-K_{22}C_{12}}{\Omega} - \gamma B_{22}^i\right)\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}; \qquad (5.56)$$

$$\tau_{\alpha\beta}^i = -\frac{B_{66}^i}{C_{66}}\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} + 2B_{66}^i\left(\frac{K_{66}}{C_{66}} - \gamma\right)\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}. \qquad (5.57)$$

As usual, the system of differential equations of the combined method (5.46) may be reduced to a single equivalent eighth-order differential equation of solution. Assuming

$$w = L_2(C_{jk})\Psi; \qquad (5.58)$$

$$\varphi = \nabla_r\Psi - L_3(d_{jk})\Psi, \qquad (5.59)$$

where $\Psi = \Psi(\alpha, \beta)$ is the sought potential function, we identically satisfy the second equation of the system in (5.46) and from the first equation we obtain the sought equation of solution

$$P_1 \frac{1}{A^8} \frac{\partial^8 \Psi}{\partial \alpha^8} + P_3 \frac{1}{A^6 B^2} \frac{\partial^8 \Psi}{\partial \alpha^6 \partial \beta^2} + P_5 \frac{1}{A^4 B^4} \frac{\partial^8 \Psi}{\partial \alpha^4 \partial \beta^4} +$$

$$+ P_4 \frac{1}{A^2 B^6} \frac{\partial^8 \Psi}{\partial \alpha^2 \partial \beta^6} + P_2 \frac{1}{B^8} \frac{\partial^8 \Psi}{\partial \beta^8} + Q_1 \frac{1}{A^6} \frac{\partial^6 \Psi}{\partial \alpha^6} +$$

$$+ Q_3 \frac{1}{A^4 B^2} \frac{\partial^6 \Psi}{\partial \alpha^4 \partial \beta^2} + Q_4 \frac{1}{A^2 B^4} \frac{\partial^6 \Psi}{\partial \alpha^2 \partial \beta^4} + Q_2 \frac{1}{B^6} \frac{\partial^6 \Psi}{\partial \beta^6} +$$

$$+ k_2^2 \frac{1}{A^4} \frac{\partial^4 \Psi}{\partial \alpha^4} + 2 k_1 k_2 \frac{1}{A^2 B^2} \frac{\partial^4 \Psi}{\partial \alpha^2 \partial \beta^2} + k_1^2 \frac{1}{B^4} \frac{\partial^4 \Psi}{\partial \beta^4} = Z, \qquad (5.60)$$

where for coefficients P_j and Q_j we have

$$P_1 = (D_{11} - D^0_{11}) \frac{C_{11}}{\Omega} + \left(\frac{K_{12} C_{11} - K_{11} C_{12}}{\Omega} \right)^2;$$

$$P_2 = (D_{22} - D^0_{22}) \frac{C_{22}}{\Omega} + \left(\frac{K_{12} C_{22} - K_{22} C_{12}}{\Omega} \right)^2;$$

$$P_3 = 2 \left[(D_{12} - D^0_{12}) + 2 (D_{66} - D^0_{66}) \right] \frac{C_{11}}{\Omega} + (D_{11} - D^0_{11}) \left(\frac{1}{C_{66}} - 2 \frac{C_{12}}{\Omega} \right) +$$

$$+ 2 \frac{K_{12} C_{11} - K_{11} C_{12}}{\Omega} \left(\frac{K_{22} C_{11} - K_{12} C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11} C_{22} - K_{12} C_{12}}{\Omega} \right);$$

$$P_4 = 2 \left[(D_{12} - D^0_{12}) + 2 (D_{66} - D^0_{66}) \right] \frac{C_{22}}{\Omega} + (D_{22} - D^0_{22}) \left(\frac{1}{C_{66}} - 2 \frac{C_{12}}{\Omega} \right) +$$

$$+ 2 \frac{K_{12} C_{22} - K_{22} C_{12}}{\Omega} \left(\frac{K_{22} C_{11} - K_{12} C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11} C_{22} - K_{12} C_{12}}{\Omega} \right);$$

$$P_5 = (D_{11} - D^0_{11}) \frac{C_{22}}{\Omega} + 2 \left[(D_{12} - D^0_{12}) + \right.$$

$$\left. + 2 (D_{66} - D^0_{66}) \right] \left(\frac{1}{C_{66}} - 2 \frac{C_{12}}{\Omega} \right) + D_{22} - D^0_{22}) \frac{C_{11}}{\Omega} +$$

$$+ \left(\frac{K_{22} C_{11} - K_{12} C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11} C_{22} - K_{12} C_{12}}{\Omega} \right)^2 +$$

$$+ 2 \frac{(K_{12} C_{11} - K_{11} C_{12})(K_{12} C_{22} - K_{22} C_{12})}{\Omega^2};$$

$$Q_1 = \frac{2}{R_2} \frac{K_{12} C_{11} - K_{11} C_{12}}{\Omega}; \quad Q_2 = \frac{2}{R_1} \frac{K_{12} C_{22} - K_{22} C_{12}}{\Omega};$$

$$Q_3 = \frac{2}{R_1} \frac{K_{12} C_{11} - K_{11} C_{12}}{\Omega} + \frac{2}{R_2} \left(\frac{K_{22} C_{11} - K_{12} C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11} C_{22} - K_{12} C_{12}}{\Omega} \right);$$

$$Q_4 = \frac{2}{R_2} \frac{K_{12} C_{22} - K_{22} C_{12}}{\Omega} + \frac{2}{R_1} \left(\frac{K_{22} C_{11} - K_{12} C_{12}}{\Omega} - 2 \frac{K_{66}}{C_{66}} + \frac{K_{11} C_{22} - K_{12} C_{12}}{\Omega} \right).$$

Inserting into formulas (5.49) -(5.57) for ω and φ their values from (5.58) and (5.59) , we obtain expressions for the design quantities in terms of the sought function $\Psi(\alpha, \beta)$. These expressions will not differ from the corresponding expressions (5.5.67) -(5.5.75) derived for the case of a cylindrical shell.

SECTION 6. EQUATIONS OF SOLUTION AND DESIGN FORMULAS IN THE THEORY OF EXTREMELY SHALLOW SHELLS CONSISTING OF AN ODD NUMBER OF HOMOGENEOUS ANISOTROPIC LAYERS SYMMETRICALLY ARRANGED RELATIVE TO THE MEDIAN SURFACE*

We shall discuss a shell consisting of an odd number (2m + 1) of layers (see Figure 15) . By virtue of the symmetrical structure stiffnesses C_{jk} and D_{jk} are determined from formulas (3.1) and (3.2) . Stiffnesses K_j are all equal to zero (3.3) . consequently, for all coefficients d_{jk} and D^0_{jk} we also have zero values (3.4) .

On this basis we obtain from (5.1) -(5.6) the internal forces and moments

$$T_1 = \left(C_{11}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{16}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{12}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + + \left(\frac{1}{R_1}C_{11} + \frac{1}{R_2}C_{12}\right)w; \qquad (6.1)$$

$$T_2 = \left(C_{12}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(C_{22}\frac{1}{B}\frac{\partial}{\partial\beta} + C_{26}\frac{1}{A}\frac{\partial}{\partial\alpha}\right)v + + \left(\frac{1}{R_2}C_{22} + \frac{1}{R}C_{12}\right)w; \qquad (6.2)$$

$$S = \left(C_{16}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{66}\frac{1}{B}\frac{\partial}{\partial\beta}\right)u + \left(C_{66}\frac{1}{A}\frac{\partial}{\partial\alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial\beta}\right)v + + \left(\frac{1}{R_1}C_{16} + \frac{1}{R_2}C_{26}\right)w; \qquad (6.3)$$

$$N_1 = -E_1(D_{jk})w \quad N_2 = -E_2(D_{jk} \qquad (6.4)$$

$$M_1 = -\left(D_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + D_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + 2D_{16}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right); \qquad (6.5)$$

$$M_2 = -\left(D_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + D_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + 2D_{26}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right); \qquad (6.6)$$

$$H = -\left(2D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + D_{16}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + D_{26}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}\right). \qquad (6.7)$$

Stresses in the layers may be determined by means of formulas:

$$\sigma_\alpha^i = B_{11}^i\frac{1}{A}\frac{\partial u}{\partial\alpha} + B_{12}^i\frac{1}{B}\frac{\partial v}{\partial\beta} + B_{16}^i\left(\frac{1}{B}\frac{\partial u}{\partial\beta} + \frac{1}{A}\frac{\partial v}{\partial\alpha}\right) + + (B_{11}^i k_1 + B_{12}^i k_2)w - \gamma\left(B_{11}^i\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + B_{12}^i\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2} + 2B_{16}^i\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right); \qquad (6.8)$$

---

*S.A. Ambartsumyan, see footnote * on p. 283.

$$\sigma^l_\beta = B^l_{22}\frac{1}{B}\frac{\partial v}{\partial \beta} + B^l_{12}\frac{1}{A}\frac{\partial u}{\partial \alpha} + B^l_{26}\left(\frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}\right) +$$

$$+\left(B^l_{22}k_2 + B^l_{12}k_1\right) w - \gamma\left(B^l_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial} + B_{12}\,{}_{A^2}\frac{\partial^2 w}{\partial \alpha^2} + 2B^l_{26}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\, \partial \beta}\right); \qquad (6.9)$$

$$\tau^l_{\alpha\beta} = B^l_{16}\frac{1}{A}\frac{\partial u}{\partial \alpha} + B^l_{26}\frac{1}{B}\frac{\partial v}{\partial \beta} + B^l_{66}\left(\frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha}\right) +$$

$$+\left(B^l_{16}k_1 + B^l_{26}k_2\right) w - \gamma\left(B^l_{16}\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} + B^l_{26}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + 2B^l_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha\, \partial \beta}\right). \qquad (6.10)$$

Let the sought quantities be displacements $u(\alpha, \beta)$, $v(\alpha, \beta)$, $w(\alpha, \beta)$, in terms of which all the design quantities are determined. Then the system of differential equations of solution is written as follows:

$$\left.\begin{aligned} &L_{11}(C_{jk})\,u + L_{12}(C_{jk})\,v + L_{13}(C_{jk})\,w = -X;\\ &L_{12}(C_{jk})\,u + L_{22}(C_{jk})\,v + L_{23}(C_{jk})\,w = -Y;\\ &L_{13}(C_{jk})\,u + L_{23}(C_{jk})\,v + L_{33}(C_{jk}D_{jk})\,w = Z, \end{aligned}\right\} \qquad (6.11)$$

where for operators $L_{11}(C_{jk})$, $L_{22}(C_{jk})$ and $L_{12}(C_{jk})$ we have (5.8), (5.9), and (5.10), respectively, while the remaining operators are written as:

$$L_{13}(C_{jk}) = (k_1C_{11} + k_2C_{12})\frac{1}{A}\frac{\partial}{\partial \alpha} + (k_1C_{16} + k_2C_{26})\frac{1}{B}\frac{\partial}{\partial \beta}; \qquad (6.12)$$

$$L_{23}(C_{jk}) = (k_2C_{22} + k_1C_{12})\frac{1}{D}\frac{\partial}{\partial \beta} + (k_2C_{26} + k_1C_{16})\frac{1}{A}\frac{\partial}{\partial \alpha}; \qquad (6.13)$$

$$L_{33}(C_{jk}D_{jk}) = \left(k_1^2C_{11} + 2k_1k_2C_{12} + k_2^2C_{22}\right) +$$

$$+ D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial \alpha^4} + 4D_{16}\frac{1}{A^3B}\frac{\partial^4}{\partial \alpha^3\, \partial \beta} +$$

$$+ 2(D_{12} + 2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial \alpha^2\, \partial \beta^2} + 4D_{26}\frac{1}{AB^3}\frac{\partial^4}{\partial \alpha\, \partial \beta^3} + D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial \beta^4}. \qquad (6.14)$$

Solving system (6.11) with given boundary conditions, let us determine functions u, v and w, then by means of formulas (6.1) -(6.10) let us find all the design quantities.

In the special case where X = Y = 0 and Z ≠ 0 the equations of the combined method take the form

$$\left.\begin{aligned} &L_1(D_{jk})\,w + \nabla_r\varphi = Z;\\ &L_2(A_{jk})\,\varphi - \nabla_r w = 0, \end{aligned}\right\} \qquad (6.15)$$

where operator $L_1(D_{jk})$ has the value

$$L_1(D_{jk}) = \frac{D_{11}}{A^4}\frac{\partial^4}{\partial \alpha^4} + 4\frac{D_{16}}{A^3B}\frac{\partial^4}{\partial \alpha^3\, \partial \beta} +$$

$$+ 2(D_{12} + 2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial \alpha^2\, \partial \beta^2} + 4\frac{D_{26}}{AB^3}\frac{\partial^4}{\partial \alpha\, \partial \beta^3} + \frac{D_{22}}{B^4}\frac{\partial^4}{\partial \beta^4}, \qquad (6.16)$$

while for $L_2(A_{jk})$ and ∇_r we have (5.17) and (5.19), respectively.

In this case the design quantities of the problem will be determined through the sought functions $w(\alpha, \beta)$ and $\varphi(\alpha, \beta)$ by means of the following formulas.

Internal forces: T_1, T_2 and S as in (5.20) ; N_1 and N_2 as in (6.4) [where, as usual, E_j is determined from (5.4.25) and (5.4.26)]; the bending and torsional moments M_1, M_2 and H as in (6.5) , (6.6) and (6.7) ; and, finally, the stresses in the layers:

$$\begin{aligned}\sigma_\alpha^i = {} & \left(B_{11}^i A_{12} + B_{12}^i A_{22} + B_{16}^i A_{26}\right) \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} + \\ & + \left(B_{11}^i A_{11} + B_{12}^i A_{12} + B_{16}^i A_{16}\right) \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} - \\ & - \left(B_{11}^i A_{16} + B_{12}^i A_{26} + B_{16}^i A_{66}\right) \frac{1}{AB} \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta} - \\ & - \gamma \left(B_{11}^i \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + B_{12}^i \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} + 2B_{16} \frac{1}{AB} \frac{\partial^2 w}{\partial \alpha\, \partial \beta}\right);\end{aligned} \qquad (\ 6.17)$$

$$\begin{aligned}\sigma_\beta^i = {} & \left(B_{22}^i A_{12} + B_{12}^i A_{11} + B_{26}^i A_{16}\right) \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} + \\ & + \left(B_{22}^i A_{22} + B_{12}^i A_{12} + B_{26}^i A_{26}\right) \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} - \\ & - \left(B_{22}^i A_{26} + B_{12}^i A_{16} + B_{26}^i A_{66}\right) \frac{1}{AB} \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta} - \\ & - \gamma \left(B_{22}^i \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} + B_{12}^i \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + 2B_{26}^i \frac{1}{AB} \frac{\partial^2 w}{\partial \alpha\, \partial \beta}\right);\end{aligned} \qquad (\ 6.18)$$

$$\begin{aligned}\tau_{\alpha\beta}^i = {} & \left(B_{16}^i A_{11} + B_{26}^i A_{12} + B_{66}^i A_{16}\right) \frac{1}{B^2} \frac{\partial^2 \varphi}{\partial \beta^2} + \\ & + \left(B_{16}^i A_{12} + B_{26}^i A_{22} + B_{66}^i A_{26}\right) \frac{1}{A^2} \frac{\partial^2 \varphi}{\partial \alpha^2} - \\ & - \left(B_{16}^i A_{16} + B_{26}^i A_{26} + B_{66}^i A_{66}\right) \frac{1}{AB} \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta} - \\ & - \gamma \left(B_{16}^i \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + B_{26}^i \frac{1}{B^2} \frac{\partial^2 w}{\partial \beta^2} + 2B_{66}^i \frac{1}{AB} \frac{\partial^2 w}{\partial \alpha\, \partial \beta}\right).\end{aligned} \qquad (\ 6.19)$$

The system of equations of solution in the combined method (6.15) may be reduced to a single equivalent eighth-order differential equation of solution. Assuming

$$\left.\begin{aligned} w &= L_2(A_{jk})\Psi, \\ \varphi &= \nabla_r \Psi, \end{aligned}\right\} \qquad (\ 6.20)$$

where, as usual, $\Psi = \Psi(\alpha, \beta)$ is the sought potential function; identically satisfying the second equation of system (6.15) , we obtain from the first equation the sought eighth-order equation of solution, which has the following form:

$$L_0(P_{jk})\Psi + \nabla_r^2 \Psi = Z, \qquad (\ 6.21)$$

where $L_0(P_{jk}) = L_1(D_{jk})\,L_2(A_{jk})$; in expanded form it has the form of (5.8.17), while for ∇_r^2 we have

$$\nabla_r^2 = \nabla_r \nabla_r = k_2^2 \frac{1}{A^4} \frac{\partial^4}{\partial \alpha^4} + k_1^2 \frac{1}{B^4} \frac{\partial^4}{\partial \beta^4} + 2k_1 k_2 \frac{1}{A^2 B^2} \frac{\partial^4}{\partial \alpha^2\, \partial \beta^2}. \tag{6.22}$$

In this formulation the design quantities of the problem are represented in terms of the sought function $\Psi(\alpha, \beta)$ as follows:

internal forces

$$\left.\begin{aligned} T_1 &= \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} \nabla_r \Psi, \\ T_2 &= \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} \nabla_r \Psi, \\ S &= -\frac{1}{AB} \frac{\partial^2}{\partial \alpha\, \partial \beta} \nabla_r \Psi; \end{aligned}\right\} \tag{6.23}$$

$$\left.\begin{aligned} N_1 &= -E_1(D_{jk}) \nabla_r \Psi, \\ N_2 &= -E_2(D_{jk}) \nabla_r \Psi; \end{aligned}\right\} \tag{6.24}$$

bending and torsional moments

$$M_1 = -\left(D_{11} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + D_{12} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + 2D_{16} \frac{1}{AB} \frac{\partial^2}{\partial \alpha\, \partial \beta}\right) L_2(A_{jk}) \Psi; \tag{6.25}$$

$$M_2 = -\left(D_{22} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + D_{12} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + 2D_{26} \frac{1}{AB} \frac{\partial^2}{\partial \alpha\, \partial \beta}\right) L_2(A_{jk}) \Psi; \tag{6.26}$$

$$H = -\left(2D_{66} \frac{1}{AB} \frac{\partial^2}{\partial \alpha\, \partial \beta} + D_{16} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + D_{26} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2}\right) L_2(A_{jk}) \Psi; \tag{6.27}$$

in view of their unwieldiness the stress formulas are not presented here; however, they may easily be derived from (6.17) -(6.19) assuming for w and φ (6.20) ;

tangential displacements

$$\begin{aligned} u = -\Big\{&(k_1 A_{22} - k_2 A_{12}) \frac{1}{A^3} \frac{\partial^3}{\partial \alpha^3} - (2k_1 A_{26} - \\ &- k_2 A_{16}) \frac{1}{A^2 B} \frac{\partial^3}{\partial \alpha^2\, \partial \beta} + [(A_{66} + A_{12}) k_1 - \\ &- k_2 A_{11}] \frac{1}{AB^2} \frac{\partial^3}{\partial \alpha\, \partial \beta^2} - k_1 A_{16} \frac{1}{B^3} \frac{\partial^3}{\partial \beta^3}\Big\} \Psi; \end{aligned} \tag{6.28}$$

$$\begin{aligned} v = -\Big\{&(k_2 A_{11} - k_1 A_{12}) \frac{1}{B^3} \frac{\partial^3}{\partial \beta^3} - (2k_2 A_{16} - \\ &- k_1 A_{26}) \frac{1}{B^2 A} \frac{\partial^3}{\partial \beta^2\, \partial \alpha} + [(A_{66} + A_{12}) k_2 - \\ &- k_1 A_{22}] \frac{1}{BA^2} \frac{\partial^3}{\partial \beta\, \partial \alpha^2} - k_2 A_{26} \frac{1}{A^3} \frac{\partial^3}{\partial \alpha^3}\Big\} \Psi. \end{aligned} \tag{6.29}$$

Thus, with the given boundaries conditions, having the solution of equation (6.21) and using formulas (6.23) -(6.29) as well as (6.17) -(6.20) we may find the values of all the design quantities of the problem.

Orthotropic Shell. Let the shell consist of homogeneous orthotropic layers such that the principle directions of elasticity at each point of each layer coincide with the directions of the corresponding coordinate lines.

Let the sought quantities be the displacements $u(\alpha, \beta)$, $v(\alpha, \beta)$ and $w(\alpha, \beta)$. Then the system of equations of solution will have the form

$$C_{11}\frac{1}{A^2}\frac{\partial^2 u}{\partial\alpha^2}+C_{66}\frac{1}{B^2}\frac{\partial^2 u}{\partial\beta^2}+(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2 v}{\partial\alpha\,\partial\beta}+$$

$$+(k_1C_{11}+k_2C_{12})\frac{1}{A}\frac{\partial w}{\partial\alpha}=-X, \qquad (6.30)$$

$$C_{22}\frac{1}{B^2}\frac{\partial^2 v}{\partial\beta^2}+C_{66}\frac{1}{A^2}\frac{\partial^2 v}{\partial\alpha^2}+(C_{12}+C_{66})\frac{1}{AB}\frac{\partial^2 u}{\partial\alpha\,\partial\beta}+$$

$$+(k_2C_{22}+k_1C_{12})\frac{1}{B}\frac{\partial w}{\partial\beta}=-Y; \qquad (6.31)$$

$$(k_1C_{11}+k_2C_{12})\frac{1}{A}\frac{\partial u}{\partial\alpha}+(k_2C_{22}+k_1C_{12})\frac{1}{B}\frac{\partial v}{\partial\beta}+$$

$$+(k_1^2C_{11}+2k_1k_2C_{12}+k_2^2C_{22})w+L_1(D_{jk})w=Z, \qquad (6.32)$$

where for $L_1(D_{jk})$ we have

$$L_1(D_{jk})=D_{11}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}+D_{22}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4}. \qquad (6.33)$$

For the internal forces and stresses in the shell layers we have

$$T_1=C_{11}\frac{1}{A}\frac{\partial u}{\partial\alpha}+C_{12}\frac{1}{B}\frac{\partial v}{\partial\beta}+(k_1C_{11}+k_2C_{12})w; \qquad (6.34)$$

$$T_2=C_{22}\frac{1}{B}\frac{\partial v}{\partial\beta}+C_{12}\frac{1}{A}\frac{\partial u}{\partial\alpha}+(k_2C_{22}+k_1C_{12})w; \qquad (6.35)$$

$$S=C_{66}\left(\frac{1}{A}\frac{\partial v}{\partial\alpha}+\frac{1}{B}\frac{\partial u}{\partial\beta}\right); \qquad (6.36)$$

$$M_1=-\left(D_{11}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}+D_{12}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}\right); \qquad (6.37)$$

$$M_2=-\left(D_{22}\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}+D_{12}\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}\right); \qquad (6.38)$$

$$H=-2D_{66}\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}; \qquad (6.39)$$

$$N_1=-E_1(D_{jk})w;\quad N_2=-E_2(D_{jk})w, \qquad (6.40)$$

where, as usual, for E_j we have

$$E_1(D_{jk})=D_{11}\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3}+(D_{12}+2D_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}; \qquad (6.41)$$

$$E_2(D_{jk})=D_{22}\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3}+(D_{12}+2D_{66})\frac{1}{BA^2}\frac{\partial^3}{\partial\beta\,\partial\alpha^2}. \qquad (6.42)$$

while for the stresses we have (5.37) -(5.39) .

In the case of the combined method where the sought functions are the stress function $\varphi(\alpha, \beta)$ and the displacement function $w(\alpha, \beta)$, the system of equations of solution has the form:

$$\left.\begin{aligned} L_1(D_{jk})\,w + \nabla_r\varphi = Z, \\ L_2(C_{jk})\,\varphi - \nabla_r w = 0, \end{aligned}\right\} \tag{6.43}$$

where

$$L_2(C_{jk}) = \frac{C_{11}}{\Omega}\frac{1}{A^4}\frac{\partial^4}{\partial\alpha^4} + \frac{C_{22}}{\Omega}\frac{1}{B^4}\frac{\partial^4}{\partial\beta^4} + \\ + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\frac{1}{A^2B^2}\frac{\partial^4}{\partial\alpha^2\partial\beta^2}, \tag{6.44}$$

while for $L_1(D_{jk})$ and ∇_r we have, respectively, (6.33) and (5.19) .

In this case for the internal forces and moments we have (5.49) , (6.37) -(6.40) , while for the stresses in the layers

$$\sigma_\alpha^i = \left(B_{11}^i\frac{C_{22}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - \left(B_{11}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{11}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \\ - \gamma\left(\frac{B_{11}^i}{A^2}\frac{\partial^2 w}{\partial\alpha^2} + \frac{B_{12}^i}{D^2}\frac{\partial^2 w}{\partial\beta^2}\right); \tag{6.45}$$

$$\sigma_\beta^i = \left(B_{22}^i\frac{C_{11}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} - \\ - \left(B_{22}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{22}}{\Omega}\right)\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} - \gamma\left(\frac{B_{22}^i}{B^2}\frac{\partial^2 w}{\partial\beta^2} + \frac{B_{12}^i}{A^2}\frac{\partial^2 w}{\partial\alpha^2}\right); \tag{6.46}$$

$$\tau_{\alpha\beta}^i = -\frac{B_{66}^i}{C_{66}}\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - 2\gamma B_{66}^i\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}. \tag{6.47}$$

In the case where the sought function is the potential function $\Psi = \Psi(\alpha, \beta)$, the differential equation of solution is an eighth-order equation and is represented as follows:

$$P_1\frac{1}{A^8}\frac{\partial^8\Psi}{\partial\alpha^8} + P_3\frac{1}{A^6B^2}\frac{\partial^8\Psi}{\partial\alpha^6\partial\beta^2} + P_5\frac{1}{A^4B^4}\frac{\partial^8\Psi}{\partial\alpha^4\partial\beta^4} + \\ + P_4\frac{1}{A^2B^6}\frac{\partial^8\Psi}{\partial\alpha^2\partial\beta^6} + P_2\frac{1}{B^8}\frac{\partial^8\Psi}{\partial\beta^8} + k_2^2\frac{1}{A^4}\frac{\partial^4\Psi}{\partial\alpha^4} + \\ + 2k_1k_2\frac{1}{A^2B^2}\frac{\partial^4\Psi}{\partial\alpha^2\partial\beta^2} + k_1^2\frac{1}{B^4}\frac{\partial^4\Psi}{\partial\beta^4} = Z. \tag{6.48}$$

where

$$P_1 = \frac{D_{11}C_{11}}{\Omega}, \quad P_3 = 2\frac{C_{11}}{\Omega}(D_{12} + 2D_{66}) + D_{11}\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right);$$

$$P_2 = \frac{D_{22}C_{22}}{\Omega}, \quad P_4 = 2\frac{C_{22}}{\Omega}(D_{12} + 2D_{66}) + D_{22}\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right);$$

$$P_5 = \frac{D_{11}C_{22}}{\Omega} + 2(D_{12} + 2D_{66})\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) + \frac{D_{22}C_{11}}{\Omega}.$$

In terms of the sought function the design quantities of the problem are represented as follows:

$$T_1 = \frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\nabla_r\Psi, \quad T_2 = \frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\nabla_r\Psi; \tag{6.49}$$

$$S = -\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\nabla_r\Psi; \tag{6.50}$$

$$M_1 = -\left(D_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + D_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\right)L_2(C_{jk})\Psi; \tag{6.51}$$

$$M_2 = -\left(D_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + D_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\right)L_2(C_{jk})\Psi; \tag{6.52}$$

$$H = -2D_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(C_{jk})\Psi; \tag{6.53}$$

$$N_1 = -E_1(D_{jk})L_2(C_{jk})\Psi; \quad N_2 = -E_2(D_{jk})L_2(C_{jk})\Psi; \tag{6.54}$$

$$\begin{aligned}\sigma^i_\alpha = \Big[\Big(B_{11}\frac{C_{22}}{\Omega} - B^i_{12}\frac{C_{12}}{\Omega}\Big)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} &- \\ - \Big(B^i_{11}\frac{C_{12}}{\Omega} - B^i_{12}\frac{C_{11}}{\Omega}\Big)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\Big]\nabla_r\Psi &- \\ - \gamma\Big(B^i_{11}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + B^i_{12}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\Big)L_2(C_{jk})\Psi&;\end{aligned} \tag{6.55}$$

$$\begin{aligned}\sigma^i_\beta = \Big[\Big(B^i_{22}\frac{C_{11}}{\Omega} - B^i_{12}\frac{C_{12}}{\Omega}\Big)\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} &- \\ - \Big(B^i_{22}\frac{C_{12}}{\Omega} - B^i_{12}\frac{C_{22}}{\Omega}\Big)\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\Big]\nabla_r\Psi &- \\ - \gamma\Big(B^i_{22}\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2} + B^i_{12}\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}\Big)L_2(C_{jk})\Psi&;\end{aligned} \tag{6.56}$$

$$\tau^i_{\alpha\beta} = -\frac{B^i_{66}}{C_{66}}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}\nabla_r\Psi - 2\gamma B^i_{66}\frac{1}{AB}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(C_{jk})\Psi; \tag{6.57}$$

$$\begin{aligned}u = -\Big\{\Big(k_1\frac{C_{11}}{\Omega} + k_2\frac{C_{12}}{\Omega}\Big)\frac{1}{A^3}\frac{\partial^3}{\partial\alpha^3} &+ \\ + \Big[\Big(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\Big)k_1 - k_2\frac{C_{22}}{\Omega}\Big]\frac{1}{AB^2}\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\Big\}\Psi&;\end{aligned} \tag{6.58}$$

$$\begin{aligned}v = \Big\{\Big(k_2\frac{C_{22}}{\Omega} + k_1\frac{C_{12}}{\Omega}\Big)\frac{1}{B^3}\frac{\partial^3}{\partial\beta^3} &+ \\ + \Big[\Big(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\Big)k_2 - k_1\frac{C_{11}}{\Omega}\Big]\frac{1}{BA^2}\frac{\partial^3}{\partial\beta\,\partial\alpha^2}\Big\}\Psi&;\end{aligned} \tag{6.59}$$

$$w = L_2(C_{jk})\Psi. \tag{6.60}$$

SECTION 7. INTEGRATION OF THE EQUATIONS OF SOLUTION IN THE THEORY OF EXTREMELY SHALLOW ORTHOTROPIC SHELLS

The methods of integration of equations of solution in the theory of extremely shallow shells do not differ from the corresponding methods given for integration of equations of anisotrpoic cylindrical shells. These methods were discussed in some detail in Sections 9 and 10 of Chapter V. Here we present only the results of integration of the equation of solution (6.48) by the method of double trigonometric series and we investigate only one combined example, which is of interest from the standpoint of laminar shells in general.

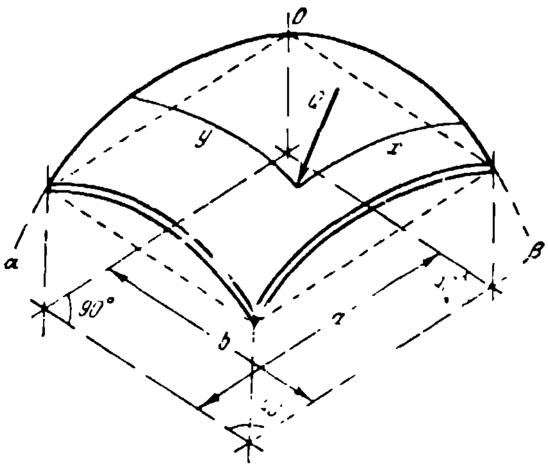

Figure 68

Let an orthotropic symmetrically assembled laminar shell cover a rectangular plane with sides a and b (Figure 68) and sustain a force Q concentrated at point $\alpha = x$, $\beta = y$. Assuming that α and β are absolute coordinates, for the coefficients of the first quadratic form we will have

$$A = 1, \quad B = 1. \qquad (7.1)$$

We assume that the shell is hinge-supported over its entire contour; the boundary conditions will then have the following form:

$$\left.\begin{aligned} &\text{when} \quad \alpha = 0, \ \alpha = a \quad v = w = T_1 = M_1 = 0; \\ &\text{when} \quad \beta = 0, \ \beta = b \quad u = w = T_2 = M_2 = 0. \end{aligned}\right\} \qquad (7.2)$$

We satisfy these conditions by representing the sought function $\Psi(\alpha, \beta)$ in the form

$$\Psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}, \qquad (7.3)$$

where A_{mn} are the sought coefficients.

Concentrated force Q is also represented in the form of an analogous series

$$Z = -Q = -\sum_{m} \sum_{n} a_{mn} \sin \frac{m\pi\alpha}{a} \sin \frac{n\pi\beta}{b}, \qquad (7.4)$$

where

$$a_{mn} = \frac{4Q}{ab} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \qquad (7.5)$$

Inserting the values of Ψ and Z from (7.3) and (7.4), respectively, into (6.48), after certain transformations we obtain for the sought function

$$\Psi = -\frac{4Q}{ab}\frac{a^8}{\pi^8}\sum_m\sum_n\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{\Delta_{mn}}, \tag{7.6}$$

where

$$\Delta_{mn} = P_1 m^8 + P_3 m^6 n^2\lambda^2 + P_5 m^4 n^4\lambda^4 + P_4 m^2 n^6\lambda^6 + \\ + P_2 n^8\lambda^8 + \frac{a^4}{\pi^4}\left(k_2^2 m^4 + 2k_1 k_2 m^2 n^2\lambda^2 + k_1^2 n^4\lambda^4\right). \tag{7.7}$$

The value of P_j is derived in Section 6, while $\lambda = a/b$.

From (6.60), by virtue of (6.44) and (7.6), we obtain for the normal displacement

$$w = -\frac{4Q}{ab}\frac{a^4}{\pi^4}\sum_m\sum_n\frac{\Delta'_{mn}}{\Delta_{mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}, \tag{7.8}$$

where

$$\Delta'_{mn} = \frac{C_{11}}{\Omega}m^4 + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)m^2n^2\lambda^2 + \frac{C_{22}}{\Omega}n^4\lambda^4. \tag{7.9}$$

From (6.55) -(6.57) we obtain for the stresses in the shell layers

$$\sigma_\alpha^i = \frac{4Q\lambda}{\pi^2}\sum_m\sum_n\left\{\frac{a^2}{\pi^2}(k_2 m^2 + k_1 n^2\lambda^2)\left[\left(B_{11}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{11}}{\Omega}\right)m^2 - \right.\right. \\ \left. - \left(B_{11}^i\frac{C_{22}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)n^2\lambda^2\right] - \\ \left. - \gamma\left(B_{11}^i m^2 + B_{12}^i n^2\lambda^2\right)\Delta'_{mn}\right\}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{\Delta_{mn}}; \tag{7.10}$$

$$\sigma_\beta^i = \frac{4Q\lambda}{\pi^2}\sum_m\sum_n\left\{\frac{a^2}{\pi^2}(k_2 m^2 + k_1 n^2\lambda^2)\left[\left(B_{22}^i\frac{C_{12}}{\Omega} - B_{12}^i\frac{C_{22}}{\Omega}\right)n^2\lambda^2 - \right.\right. \\ \left. - \left(B_{22}^i\frac{C_{11}}{\Omega} - B_{12}^i\frac{C_{12}}{\Omega}\right)m^2\right] - \\ \left. - \gamma\left(B_{22}^i n^2\lambda^2 + B_{12}^i m^2\right)\Delta'_{mn}\right\}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{\Delta_{mn}}; \tag{7.11}$$

$$\tau_{\alpha\beta}^i = -\frac{4Q\lambda^2}{\pi^2}B_{66}^i\sum_m\sum_n\left[\frac{a^2}{\pi^2}\frac{1}{C_{66}}(k_2 m^2 + k_1 n^2\lambda^2) - \right. \\ \left. - 2\gamma\Delta'_{mn}\right]\frac{mn}{\Delta_{mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{m\pi\alpha}{a}\cos\frac{n\pi\beta}{b}. \tag{7.12}$$

Regarding the formulas derived here as interference functions, we may transform and obtain the design formulas for other cases of loading of a shell by a normally applied load. For example, in the case of a uniformly distributed load with intensity q we have

$$w = -\frac{16qa^4}{\pi^6}\sum_m\sum_n\frac{\Delta'_{mn}}{mn\Delta_{mn}}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}; \tag{7.13}$$

$$\sigma_\alpha^l = \frac{16qa^2}{\pi^4}\sum_m\sum_n\Big\{\frac{a^2}{\pi^2}(k_2m^2+k_1n^2\lambda^2)\Big[\Big(B_{11}^l\frac{C_{12}}{\Omega}-B_{12}^l\frac{C_{11}}{\Omega}\Big)m^2-$$

$$-\Big(B_{11}^l\frac{C_{22}}{\Omega}-B_{12}^l\frac{C_{12}}{\Omega}\Big)n^2\lambda^2\Big]-$$

$$-\gamma\left(B_{11}^lm^2+B_{12}^ln^2\lambda^2\right)\Delta'_{mn}\Big\}\frac{\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{mn\,\Delta_{mn}}; \qquad (7.14)$$

$$\sigma_\beta^l = \frac{16qa^2}{\pi^4}\sum_m\sum_n\Big\{\frac{a^2}{\pi^2}(k_2m^2+k_1n^2\lambda^2)\Big[\Big(B_{22}^l\frac{C_{12}}{\Omega}-B_{12}^l\frac{C_{22}}{\Omega}\Big)n^2\lambda^2-$$

$$-\Big(B_{22}^l\frac{C_{11}}{\Omega}-B_{12}^l\frac{C_{12}}{\Omega}\Big)m^2\Big]-$$

$$-\gamma\left(B_{22}^ln^2\lambda^2+B_{12}^lm^2\right)\Delta'_{mn}\Big\}\frac{\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{mn\,\Delta_{mn}}; \qquad (7.15)$$

$$\tau_{\alpha\beta}^l = -\frac{16qa^2}{\pi^4}\lambda B_{66}^l\sum_m\sum_n\Big[\frac{a^2}{\pi^2}\frac{1}{C_{66}}(k_2m^2+k_1n^2\lambda^2)-2\gamma\Delta'_{mn}\Big]\times$$

$$\times\frac{\cos\frac{m\pi\alpha}{a}\cos\frac{n\pi\beta}{b}}{\Delta_{mn}} \qquad (7.16)$$

$$(m=1,\ 3,\ 5,\ \ldots;\quad n=1,\ 3,\ 5,\ \ldots).$$

Let us solve an example. We shall investigate the dependence of flexures of three-layer shells on the elastic moduli of the layer materials. Let us investigate a laminar shell with double curvature, freely supported over its entire contour and covering a rectangluar plane (a × b). Let the shell consist of isotropic layers with Poisson ratios equal to zero.

By virtue of (2.12.9) and from (2.13.2) we obtain for the stiffnesses of an isotropic shell with symmetrically arranged layers

$$C=C_{11}=C_{22}=C_{12}+2C_{66}=$$

$$=2\left[\frac{E^{m+1}h_{m+1}}{1-(\nu^{m+1})^2}+\sum_{s=1}^{m}\frac{E^s}{1-(\nu^s)^2}(h_s-h_{s+1})\right]; \qquad (7.17)$$

$$C_{66}=2\left[\frac{E^{m+1}h_{m+1}}{2(1+\nu^{m+1})}+\sum_{s=1}^{m}\frac{E^s}{2(1+\nu^{m+1})}(h_s-h_{s+1})\right]; \qquad (7.18)$$

$$C_{12}=2\left[\frac{E^{m+1}h_{m+1}\nu^{m+1}}{1-(\nu^{m+1})^2}+\sum_{s=1}^{m}\frac{E^s\nu^s}{1-(\nu^s)^2}(h_s-h_{s+1})\right]; \qquad (7.19)$$

$$D=D_{11}=D_{22}=D_{12}+2D_{66}=$$

$$=\frac{2}{3}\left[\frac{E^{m+1}h_{m+1}^3}{1-(\nu^{m+1})^2}+\sum_{s=1}^{m}\frac{E^s}{1-(\nu^s)^2}(h_s^3-h_{s+1}^3)\right]; \qquad (7.20)$$

$$D_{66}=\frac{2}{3}\left[\frac{E^{m+1}h_{m+1}^3}{2(1+\nu^{m+1})}+\sum_{s=1}^{m}\frac{E^s}{2(1+\nu^s)}(h_s^3-h_{s+1}^3)\right]; \qquad (7.21)$$

$$D_{12}=\frac{2}{3}\left[\frac{E^{m+1}h_{m+1}\nu^{m+1}}{1-(\nu^{m+1})^2}+\sum_{s=1}^{m}\frac{E^s\nu^s}{1-(\nu^s)^2}(h_s^3-h_{s+1}^3)\right]; \qquad (7.22)$$

then for the normal displacement of the shell in the case where it is loaded by a concentrated force Q we obtain from (7.8)*

$$w_Q = \frac{4Q}{Dab}\frac{a^4}{\pi^4}\sum_m\sum_n \frac{(m^2+\lambda^2n^2)^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{(m^2+\lambda^2n^2)^4+\frac{C}{D}\frac{a^4}{\pi^4}(k_2m^2+k_1n^2\lambda^2)^2}; \qquad (7.23)$$

in the case of a uniformly distributed load of intensity q we will have from (7.13)

$$w_q = \frac{16qa^4}{D\pi^6}\sum_m\sum_n \frac{(m^2+\lambda^2n^2)^2\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{\left[(m^2+\lambda^2n^2)^4+\frac{C}{D}\frac{a^4}{\pi^4}(k_2m^2+k_1n^2\lambda^2)^2\right]mn}. \qquad (7.24)$$

Furthermore, using the identity

$$\frac{(m^2+\lambda^2n^2)^2}{(m^2+\lambda^2n^2)^4+\frac{C}{D}\frac{a^4}{\pi^4}(k_2m^2+k_1n^2\lambda^2)^2} = \frac{1}{(m^2+n^2\lambda^2)^2} - \frac{\frac{C}{D}\frac{a^4}{\pi^4}(k_2m^2+k_1n^2\lambda^2)^2}{(m^2+\lambda^2n^2)^2\left[(m^2+\lambda^2n^2)^4+\frac{C}{D}\frac{a^4}{\pi^4}(k_2m^2+k_1n^2\lambda^2)^2\right]}, \qquad (7.25)$$

from (7.23) and (7.24) we obtain for the normal displacements of shells in the discussed cases of loading

$$w_Q = \frac{Qa^2}{D}\lambda\left(w_Q^0 - w_Q^k\right); \qquad (7.26)$$

$$w_q = \frac{qa^4}{D}\left(w_q^0 - w_q^k\right), \qquad (7.27)$$

and then

$$w_Q^0 = \frac{4}{\pi^4}\sum_m\sum_n \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{(m^2+\lambda^2n^2)^2}; \qquad (7.28)$$

$$w_q^0 = \frac{16}{\pi^6}\sum_m\sum_n \frac{\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{mn(m^2+\lambda^2n^2)^2}; \qquad (7.29)$$

$$w_Q^k = \frac{4a^4C}{\pi^8D}\sum_m\sum_n \frac{(k_2m^2+k_1n^2\lambda^2)^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{\Delta''_{mn}}; \qquad (7.30)$$

$$w_q^k = \frac{16a^4C}{\pi^{10}D}\sum_m\sum_n \frac{(k_2m^2+k_1n^2\lambda^2)^2\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}}{mn\Delta''_{mn}}, \qquad (7.31)$$

where

$$\Delta''_{mn} = (m^2+\lambda^2n^2)^2\left[(m^2+\lambda^2n^2)^4+\frac{C}{D}\frac{a^4}{\pi^4}(k_2m^2+k_1n^2\lambda^2)^2\right]. \qquad (7.32)$$

The first terms of formulas (7.26) and (7.27) are the flexures of the corresponding isotrophic plane plates. The values of these flexures may be

---

[*] Here and in subsequent discussion we omit the minus sign before w since for the example under discussion the sign of w is of no importance.

taken from tables. The second terms of these formulas, representing the correction for the plate and resulting from the curvature of the shell are calculated by means of rapidly converging double series.

For the sake of concreteness we shall assume

$$k_1 = \frac{3}{4R}, \quad k_2 = \frac{5}{4R}, \quad \lambda = \frac{a}{b} = 1{,}0. \tag{7.33}$$

In (7.30) and (7.31), limiting discussion to the first term of expansion (m = 1, n = 1) and inserting the tabular values of w_Q^0 and w_q^0 from (7.26) and (7.27), we obtain for the flexures of the shell center

$$w_Q = t_Q Q a^2, \quad w_q = t_q q a^4, \tag{7.34}$$

where

$$t_Q = \frac{1}{D}\left(0{,}0116 - \frac{4}{\pi^4}\frac{K}{16+4K}\right)^{*)}; \tag{7.35}$$

$$t_q = \frac{1}{D}\left(0{,}00406 - \frac{16}{\pi^6}\frac{K}{16+4K}\right); \tag{7.36}$$

$K = \frac{C}{D}\frac{a^4}{\pi^4 R^2}$ is a parameter characterizing the shell.

With regard to shell lamination let us examine three different variants.

Variant I. A shell of thickness h = 6δ consists of three layers of identical thickness $\delta_0 = \frac{h}{3}$. The elastic modulus of the middle layer is ten times greater than the elastic moduli of the outer layers (Figure 69).

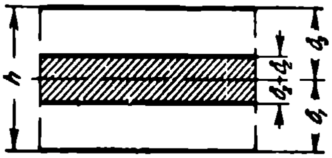

Figure 69

For such a shell we have

$$h_1 = h_3 = 3\delta, \quad \delta_2 = \delta, \quad E_1 = E_3 = E,$$
$$E_2 = 10E, \quad G_{12}^{(1)} = G_{12}^{(3)} = 0{,}5E, \quad G_{12}^{(2)} = 5{,}0E.$$

From (7.17) and (7.20) we find for the stiffnesses

$$C = 24{,}0E\delta, \quad D = 24{,}0E\delta^3;$$

---

*The force Q is applied to the center of the shell.

then

$$K=\frac{1}{\pi^4}\frac{a^4}{R^2\delta^2}.$$

In this case for the coefficients t_j of normal displacements (7.34) of the shell center with different values of $\frac{a^4}{\pi^4 R^2\delta^2}$ we obtain

$\frac{a^4}{\pi^4 R^2\delta^2}$	0,0	5,0	10,0	15,0	20,0	25,0
$t_Q\cdot 10^4$	4,63	2,46	1,79	1,46	1,29	1,17
$t_q\cdot 10^4$	1,69	0,73	0,45	0,33	0,25	0,20

Variant II. A shell of thickness h = 6δ consists of three layers of identical thickness $\delta_0=\frac{h}{3}$. The elastic modulus of the middle layer is ten times greater than the elastic moduli of the outer layers (Figure 70) .

For such a shell we have

$$h_1=h_3=3\delta,\quad \delta_2=\delta,\quad E_1=E_3=10E,\quad E_2=E,$$
$$G_{12}^{(1)}=G_{12}^{(3)}=5{,}0E,\quad G_{12}^{(2)}=0{,}5E.$$

From (7.17) and (7.20) we find for the stiffnesses

$$C=42{,}0E\delta,\quad D=174E\delta^3;$$

then

$$K=0{,}2414\frac{a^4}{\pi^4 R^2\delta^2}.$$

In this case for the coefficients t_j with different values of $\frac{a^4}{\pi^4 R^2\delta^2}$ we obtain

$\frac{a^4}{\pi^4 R^2\delta^2}$	0,0	5,0	10,0	15,0	20,0	25,0
$t_Q\cdot 10^4$	0,67	0,53	0,45	0,39	0,34	0,31
$t_q\cdot 10^4$	0,23	0,18	0,14	0,12	0,10	0,09

Variant III. A shell of thickness h = 6δ consists of three layers, the outer layers having identical thicknesses. The inner (middle) layer is four times thicker than either outer layer. The elastic modulus of the middle layer

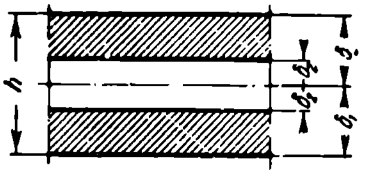

Figure 70

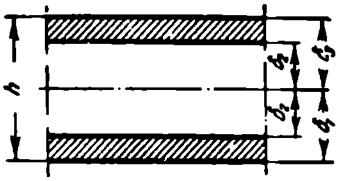

Figure 71

is one tenth of the elastic moduli of the outer layers (Figure 71).

For such a shell we have

$$h_1 = h_3 = 3\delta, \quad \delta_2 = 2\delta, \quad E_1 = E_3 = 10E, \quad E_2 = E,$$
$$G_{12}^{(1)} = G_{12}^{(3)} = 5{,}0E, \quad G_{12}^{(2)} = 0{,}5E.$$

From (7.17) and (7.20) we find for the stiffnesses

$$C = 24{,}0E\delta, \quad D = 132{,}0E\delta^3;$$

then

$$K = 0{,}1818\,\frac{a^4}{\pi^4 R^2 \delta^2}.$$

In this case for the coefficients t_j with different values of $\frac{a^4}{\pi^4 R^2 \delta^2}$ we obtain

$\frac{a^4}{\pi^4 R^2 \delta^2}$	0,0	5,0	10,0	15,0	20,0	25,0
$t_Q \cdot 10^4$	0,88	0,73	0,64	0,56	0,51	0,46
$t_q \cdot 10^4$	0,31	0,25	0,21	0,18	0,16	0,14

Figures 72 and 73 show the plots of these calculations.

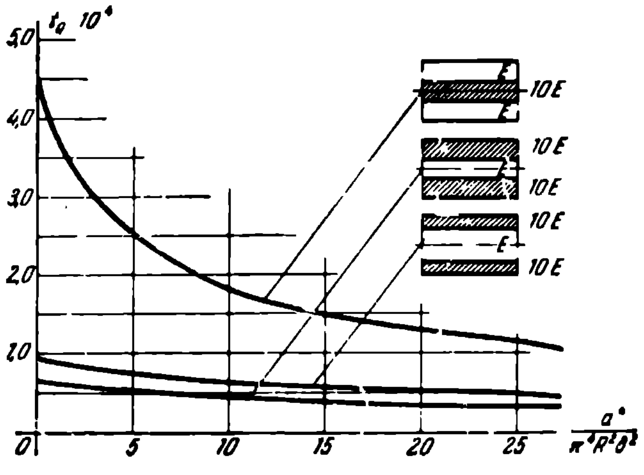

Figure 72

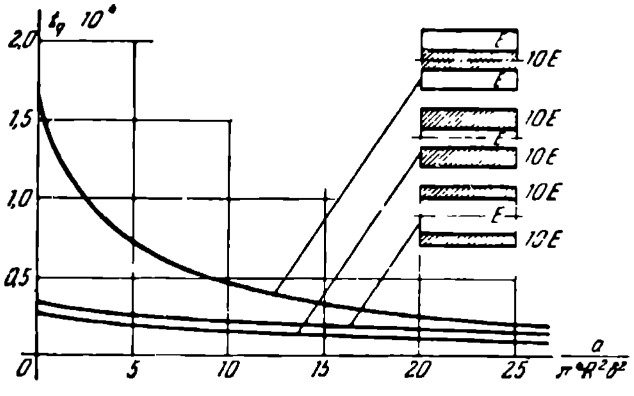

Figure 73

These plots (and tables) show that an increase in the elastic moduli of the outer layers substantially increases the over-all stiffness of the shell. However, considerable effect in the use of large-modulus outer layers is noted in flat and slightly convex plates (with small values of $\frac{a^4}{\pi^4 R^2 \delta^2}$), where the bending parameters are predominant. With a decrease in the bending moments, which occurs upon increasing the rise of the shell or upon transition from concentrated forces to smoothly varying loads, the effectiveness of using large-modulus outer layers decreases.

Comparing the plots for the first and third variants, we note that in using an identical amount of material, in the third variant, in view of the presence of large-modulus outer layers (in comparison with the inner layers), we obtain an extremely effective (in comparison with the first case) structural solution for a three-layer shell. The fact is that the bending stiffness of the shell in the third variant is 5.5 times greater than the bending stiffness of the shell in the first variant even when the shells of both variants have identical tension stiffness.

72808

CHAPTER VII

NEW THEORIES OF ANISOTROPIC SHELLS AND PLATES

SECTION 1. BASIC PREMISES AND HYPOTHESES

Let us examine a thin anisotropic shell of constant thickness h. Let the material of the shell obey the generalized Hooke's law (1.3.1) and at each point let there exist only one plane of axial symmetry parallel to the median surface of the shell. As before (Chapter II), the coordinate surface is taken to be the median surface of the shell referred to the curvilinear orthogonal coordinates α and β coinciding with the lines of principle curvature of the coordinate surface. Let γ represent the distance along the normal from point (α, β) of the median surface to point (α, β, γ) of the shell.

In this chapter we shall not use the hypothesis of the nondeformable normals and the proposed theory will be based on the following assumptions:

a) the distances along the normals (γ) between two points of the shell remain unchanged after deformation;

b) the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ or the corresponding deformations $e_{\alpha\gamma}$ and $e_{\beta\gamma}$ over the thickness of the shell vary according to a given law.*, **, ***

---

*S. A. Ambartsumyan, Toward a theory of bending of anisotropic plates, Izvestiya OTN AN SSSR, No. 5, 1958.

**S. A. Ambartsumyan, Toward a general theory of anistropic shells, PMM. Vol. XXII, No. 2, 1958.

***S. A. Ambartsumyan, D. V. Peshtmaldzhyan, Toward a theory of orthotropic shells and plates, Izvestiya AN ArmSSR (seriya fiz. mat. nauk), Vol XII, No. 1, 1959.

It is to be noted that the theory of isotrophic shells without the hypothesis of nondeformable normals has been dealt with by many authors, for example, N.A. Kil'chevsky,* V.V. Novozhilov and P.M. Finkel'shteyn,** A. Green and W. Zerna,*** E. Reissner,**** I.N. Vekua,***** T. Naghdi******,******* and others; however, these works will not be discussed here.

For greater clarity in the first sections of this chapter the proposed theory will be applied to the case of an orthotropic plate.

SECTION 2. THEORY OF ORTHOTROPIC PLATES

Let an orthotropic plate be referred to a triorthogonal system of rectilinear coordinates (α, β, γ) so that the coordinate plane $\alpha\beta$ coincides with the median plane of the plate and the principle directions of elasticity of the material coincide with the coordinate lines.

The basic assumptions are (a) and (b) as given in the first section and are expressed as follows:

a) we consider approximately

$$e_{\gamma}=0; \tag{2.1}$$

b) the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ are represented in the following form:

$$\tau_{\alpha\gamma}=f_1(\gamma)\varphi(\alpha,\beta)+\frac{\gamma}{h}(X^{+}+X^{-})+\frac{X^{+}-X^{-}}{2}, \tag{2.2}$$

$$\tau_{\beta\gamma}=f_2(\gamma)\psi(\alpha,\beta)+\frac{\gamma}{h}(Y^{+}+Y^{-})+\frac{Y^{+}-Y^{-}}{2}, \tag{2.3}$$

---

*N.A. Kil'chevskiy, Generalization of the modern theory of shells, PMM, Vol. II, No. 4, 1939.

**V.V. Novozhilov, R.M. Finkel'shteyn, Concerning error of the Kirkhoff hypothesis in the theory of shells, PMM, Vol. VII, No. 5, 1943.

***A. Green, W. Zerna, The Equilibrium of Thin Elastic Shells. Quart. J. Mech. Appl. Math., v. III, 1950.

****E. Reissner, Stress Strain Relations in the Theory of Thin Elastic Shells. J. Math. Phys., v. XXXI, 1952, pp. 109-119.

*****I.N. Vekua, On a method of calculation of prismatic shells, Trudy Tbilisskogo matematicheskogo instituta im. A.M. Rzamadze, Vol. XXI, 1955.

******P.M. Naghdi, On the Theory of Thin Elastic Shells. Quart. Appl. Math., v. XIV, No. 4, 1957.

*******P.M. Naghdi, The Effect of Transverse Shear Deformation on the Bending of Elastic Shells of Revolution. Quart. Appl. Math., v. XV, No. 1, 1957.

where $X^+(\alpha,\beta)$, $Y^+(\alpha,\beta)$ and $X^-(\alpha,\beta)$, $Y^-(\alpha,\beta)$ are the components in the axes of a displaceable trihedron (along the directions of the positive tangents to the lines β = const and α = const) of the vectors of intensity of surface loads (Figure 10) applied to the external planes of the plate ($\gamma = 1/2\ h$, $\gamma = -1/2\ h$) (2.5.3) and (2.5.4); $\varphi(\alpha,\beta)$ and $\psi(\alpha,\beta)$ are the arbitrary sought functions of coordinates α and β; $f_i(\gamma)$ are functions characterizing the laws of change in shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ over the thickness of the plate,

wherein $f_i(\pm h/2) = 0$.

Solving the equations for the generalized Hooke's law (1.3.7) we obtain for the stresses σ_α, σ_β, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$, $\tau_{\beta\gamma}$, and

$$\sigma_\alpha = B_{11}e_\alpha + B_{12}e_\beta - A_1\sigma_\gamma; \tag{2.4}$$

$$\sigma_\beta = B_{12}e_\alpha + B_{22}e_\beta - A_2\sigma_\gamma; \tag{2.5}$$

$$\tau_{\alpha\gamma} = B_{55}e_{\alpha\gamma}, \quad \tau_{\beta\gamma} = B_{44}e_{\beta\gamma}; \tag{2.6}$$

$$\tau_{\alpha\beta} = B_{66}e_{\alpha\beta}, \tag{2.7}$$

where, as usual [see (1.3.8), (1.3.9), (1.4.5), (1.4.15)],

$$\begin{gathered}
B_{11} = \frac{a_{22}}{\Omega} = \frac{E_1}{1-\nu_1\nu_2}; \quad B_{22} = \frac{a_{11}}{\Omega} = \frac{E_2}{1-\nu_1\nu_2}; \\
B_{12} = -\frac{a_{12}}{\Omega} = \frac{\nu_2E_1}{1-\nu_1\nu_2} = \frac{\nu_1E_2}{1-\nu_1\nu_2}; \quad B_{66} = \frac{1}{a_{66}} = G_{12}; \\
B_{55} = \frac{1}{a_{55}} = G_{13}; \quad B_{44} = \frac{1}{a_{44}} = G_{23}; \quad \Omega = a_{11}a_{22} - a_{12}^2 = \frac{1-\nu_1\nu_2}{E_1E_2}
\end{gathered} \tag{2.8}$$

and, in addition to these,

$$\begin{aligned}
A_1 &= a_{13}B_{11} + a_{23}B_{12} = -\frac{E_1}{E_3}\,\frac{\nu_{13}+\nu_2\nu_{23}}{1-\nu_1\nu_2}; \\
A_2 &= a_{13}B_{12} + a_{23}B_{22} = -\frac{E_2}{E_3}\,\frac{\nu_{23}+\nu_1\nu_{13}}{1-\nu_1\nu_2}.
\end{aligned} \tag{2.9}$$

Assuming that for a plate in the chosen coordinate system

$$k_1 = \frac{1}{R_1} = 0, \quad k_2 = \frac{1}{R_2} = 0, \quad H_1 = 1, \quad H_2 = 1, \tag{2.10}$$

from (2.1.4)-(2.1.9) we obtain for the deformation components

$$e_\alpha = \frac{\partial u_\alpha}{\partial\alpha}, \quad e_\beta = \frac{\partial u_\beta}{\partial\beta}, \quad e_{\alpha\beta} = \frac{\partial u_\alpha}{\partial\beta} + \frac{\partial u_\beta}{\partial\alpha}, \tag{2.11}$$

$$e_\gamma = \frac{\partial u_\gamma}{\partial\gamma}, \quad e_{\alpha\gamma} = \frac{\partial u_\alpha}{\partial\gamma} + \frac{\partial u_\gamma}{\partial\alpha}, \quad e_{\beta\gamma} = \frac{\partial u_\beta}{\partial\gamma} + \frac{\partial u_\gamma}{\partial\beta}. \tag{2.12}$$

By virtue of (2.1), from the first relationship (2.12) we obtain

$$e_\gamma = \frac{\partial u_\gamma}{\partial\gamma} = 0, \quad u_\gamma = u_\gamma(\alpha, \beta) = w(\alpha, \beta), \tag{2.13}$$

that is, as in the classical theory of plates, displacement u_γ of any point of any plate along direction γ does not depend on γ and for all points of a given normal element is equal to the normal displacement ω of the corresponding point of the median plane.

From (2.6) by virtue of (2.2), (2.3) and (2.8), for the sheer deformations $e_{\alpha,\gamma}$ and $e_{\beta,\gamma}$ we obtain

$$e_{\alpha\gamma} = f_1(\gamma)\Phi_1(\alpha,\beta) + X_1 + \frac{\gamma}{h}X_2; \tag{2.14}$$

$$e_{\beta\gamma} = f_2(\gamma)\Phi_2(\alpha,\beta) + Y_1 + \frac{\gamma}{h}Y_2, \tag{2.15}$$

where we have the following notation:

$$X_1 = a_{55}\frac{X^+ - X^-}{2}, \quad X_2 = a_{55}(X^+ + X^-);$$
$$Y_1 = a_{44}\frac{Y^+ - Y^-}{2}, \quad Y_2 = a_{44}(Y^+ + Y^-); \tag{2.16}$$

$$\Phi_1 = a_{55}\varphi, \quad \Phi_2 = a_{44}\psi. \tag{2.17}$$

Inserting the values of $e_{\alpha,\gamma}$ and $e_{\beta,\gamma}$ from (2.14) and (2.15) into the second and third relationships in (2.12), and also considering (2.13), we obtain

$$\frac{\partial u_\alpha}{\partial\gamma} = -\frac{\partial w}{\partial\alpha} + f_1(\gamma)\Phi_1 + X_1 + \frac{\gamma}{h}X_2,$$
$$\frac{\partial u_\beta}{\partial\gamma} = -\frac{\partial w}{\partial\beta} + f_2(\gamma)\Phi_2 + Y_1 + \frac{\gamma}{h}Y_2. \tag{2.18}$$

Integrating expressions (2.18) for γ over the limits from zero to γ and considering that with $\gamma = 0$ $u_\alpha = u(\alpha,\beta)$ and $u_\beta = v(\alpha,\beta)$, we obtain the following values for the tangential displacements of any point of the plate:

$$u_\alpha = u - \gamma\frac{\partial w}{\partial\alpha} + \gamma X_1 + \frac{\gamma^2}{2h}X_2 + I_{01}(\gamma)\Phi_1, \tag{2.19}$$

$$u_\beta = v - \gamma\frac{\partial w}{\partial\beta} + \gamma Y_1 + \frac{\gamma^2}{2h}Y_2 + I_{02}(\gamma)\Phi_2, \tag{2.20}$$

where

$$I_{01} = \int_0^\gamma f_1(\gamma)\,d\gamma, \quad I_{02} = \int_0^\gamma f_2(\gamma)\,d\gamma, \tag{2.21}$$

$u(\alpha,\beta)$ and $v(\alpha,\beta)$ are the tangential displacements of the corresponding point of the median plane.

Formulas (2.19) and (2.20) show that, in distinction from the classical theory of plates, here the tangential displacements u_α and u_β of any point of the plate at a normal distance of γ from the median are generally nonlinearly dependent on γ. We will also note that, in distinction from the classical theory of thin plates, here the fixing of all points of the median plane (for the purpose of eliminating any displacements and rotations in space) does not reduce the plate to the state of a nondeformable body.

Inserting the values of u $_{\alpha}$ and u$_{\beta}$ from (2.19) and (2.20) into relationships (2.11), we obtain for the corresponding deformation components

$$e_{\alpha}=\frac{\partial u}{\partial\alpha}-\gamma\frac{\partial^2 w}{\partial\alpha^2}+\gamma\frac{\partial X_1}{\partial\alpha}+\frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}+I_{01}\frac{\partial\Phi_1}{\partial\alpha}; \tag{2.22}$$

$$e_{\beta}=\frac{\partial v}{\partial\beta}-\gamma\frac{\partial^2 w}{\partial\beta^2}+\gamma\frac{\partial Y_1}{\partial\beta}+\frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}+I_{02}\frac{\partial\Phi_2}{\partial\beta}; \qquad (2.23$$

$$\begin{aligned}e_{\alpha\beta}=\frac{\partial u}{\partial\beta}+\frac{\partial v}{\partial\alpha}-2\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta}+\gamma\left(\frac{\partial X_1}{\partial\beta}+\frac{\partial Y_1}{\partial\alpha}\right)+&\\ +\frac{\gamma^2}{2h}\left(\frac{\partial X_2}{\partial\beta}+\frac{\partial Y_2}{\partial\alpha}\right)+I_{01}\frac{\partial\Phi_1}{\partial\beta}+I_{02}\frac{\partial\Phi_2}{\partial\alpha}.&\end{aligned} \tag{2.24}$$

From (2.1.10) - (2.1.12), by virtue of (2.10) we have the following equilibrium equations:

$$\frac{\partial\sigma_{\alpha}}{\partial\alpha}+\frac{\partial\tau_{\alpha\beta}}{\partial\beta}+\frac{\partial\tau_{\alpha\gamma}}{\partial\gamma}=0; \tag{2.25}$$

$$\frac{\partial\sigma_{\beta}}{\partial\beta}+\frac{\partial\tau_{\alpha\beta}}{\partial\sigma}+\frac{\partial\tau_{\beta\gamma}}{\partial\gamma}=0; \tag{2.26}$$

$$\frac{\partial\sigma_{\gamma}}{\partial\gamma}+\frac{\partial\tau_{\alpha\gamma}}{\partial\alpha}+\frac{\partial\tau_{\beta\gamma}}{\partial\beta}=0. \tag{2.27}$$

Here the volumetric forces are not considered.

With consideration of (2.8), (2.16) and (2.17), inserting the values of $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ from (2.2) and (2.3) into (2.27) and integrating for γ, we find the

normal stress σ_{γ}

$$\begin{aligned}\sigma_{\gamma}=\chi(\alpha,\beta)-B_{55}\left[I_{01}(\gamma)\frac{\partial\Phi_1}{\partial\alpha}+\gamma\frac{\partial X_1}{\partial\alpha}+\frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}\right]-&\\ -B_{44}\left[I_{02}(\gamma)\frac{\partial\Phi_2}{\partial\beta}+\gamma\frac{\partial Y_1}{\partial\beta}+\frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}\right].&\end{aligned} \tag{2.28}$$

where $\chi(\alpha,\beta)$ is an integration function and may be determined from the conditions at the surfaces of the plate (2.5.3) and (2.5.4), which have the following form:

$$\text{where } \gamma=\frac{h}{2}\quad \sigma_{\gamma}=Z^{+};\ \text{where } \gamma=-\frac{h}{2}\quad \sigma_{\gamma}=-Z^{-}$$

In satisfying these conditions we obtain the following value for $\chi(\alpha,\beta)$:

$$\begin{aligned}\chi(\alpha,\beta)=\frac{B_{55}}{2}\left[I_{01}\left(\frac{h}{2}\right)+I_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_1}{\partial\alpha}+\frac{B_{44}}{2}\left[I_{02}\left(\frac{h}{2}\right)+\right.&\\ \left.+I_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_2}{\partial\beta}+\frac{h}{8}\left(B_{55}\frac{\partial X_2}{\partial\alpha}+B_{44}\frac{\partial Y_2}{\partial\beta}\right)+Z_1,&\end{aligned} \tag{2.29}$$

as well as the following equation:

$$\begin{aligned}B_{55}\left[I_{01}\left(\frac{h}{2}\right)-I_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_1}{\partial\alpha}+B_{44}\left[I_{02}\left(\frac{h}{2}\right)-I_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_2}{\partial\beta}=&\\ =-h\left(B_{55}\frac{\partial X_1}{\partial\alpha}+B_{44}\frac{\partial Y_1}{\partial\beta}\right)-Z_2.&\end{aligned} \tag{2.30}$$

where we have the following notation:

$$Z_1=\frac{Z^+ - Z^-}{2}, \quad Z_2=Z^+ + Z^-. \tag{2.31}$$

Equation (2.30) is the third equilibrium equation (Chapter II, Section 11); this is again confirmed below [see (2.51)].

In (2.4), (2.5) and (2.7), inserting the values of e_α, e_β and $e_{\alpha,\beta}$, respectively, from (2.22), (2.23), and (2.24) as well as the value σ_γ from (2.28) with consideration of (2.29), we obtain for the principal stresses σ_α, σ_β and $\tau_{\alpha\beta}$.

$$\begin{aligned}\sigma_\alpha = {} & B_{11}\left[\frac{\partial u}{\partial\alpha} - \gamma\frac{\partial^2 w}{\partial\alpha^2} + l_{01}(\gamma)\frac{\partial\Phi_1}{\partial\alpha} + \gamma\frac{\partial X_1}{\partial\alpha} + \frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}\right] + \\ & + B_{12}\left[\frac{\partial v}{\partial\beta} - \gamma\frac{\partial^2 w}{\partial\beta^2} + l_{02}(\gamma)\frac{\partial\Phi_2}{\partial\beta} + \gamma\frac{\partial Y_1}{\partial\beta} + \frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}\right] - \\ & - A_1\left\{\frac{B_{55}}{2}\left[l_{01}\left(\frac{h}{2}\right) + l_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_1}{\partial\alpha} + \frac{B_{44}}{2}\left[l_{02}\left(\frac{h}{2}\right) + l_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_2}{\partial\beta} - \right. \\ & - B_{55}\left[l_{01}(\gamma)\frac{\partial\Phi_1}{\partial\alpha} + \gamma\frac{\partial X_1}{\partial\alpha} + \frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}\right] - B_{44}\left[l_{02}(\gamma)\frac{\partial\Phi_2}{\partial\beta} + \right. \\ & \left.\left. + \gamma\frac{\partial Y_1}{\partial\beta} + \frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}\right] + \frac{h}{8}\left(B_{55}\frac{\partial X_2}{\partial\alpha} + B_{44}\frac{\partial Y_2}{\partial\beta}\right) + Z_1\right\};\end{aligned} \tag{2.32}$$

$$\begin{aligned}\sigma_\beta = {} & B_{22}\left[\frac{\partial v}{\partial\beta} - \gamma\frac{\partial^2 w}{\partial\beta^2} + l_{02}(\gamma)\frac{\partial\Phi_2}{\partial\beta} + \gamma\frac{\partial Y_1}{\partial\beta} + \frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}\right] + \\ & + B_{12}\left[\frac{\partial u}{\partial\alpha} - \gamma\frac{\partial^2 w}{\partial\alpha^2} + l_{01}(\gamma)\frac{\partial\Phi_1}{\partial\alpha} + \gamma\frac{\partial X_1}{\partial\alpha} + \frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}\right] - \\ & - A_2\left\{\frac{B_{55}}{2}\left[l_{01}\left(\frac{h}{2}\right) + l_{01}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_1}{\partial\alpha} + \frac{B_{44}}{2}\left[l_{02}\left(\frac{h}{2}\right) + l_{02}\left(-\frac{h}{2}\right)\right]\frac{\partial\Phi_2}{\partial\beta} - \right. \\ & - B_{55}\left[l_{01}(\gamma)\frac{\partial\Phi_1}{\partial\alpha} + \gamma\frac{\partial X_1}{\partial\alpha} + \frac{\gamma^2}{2h}\frac{\partial X_2}{\partial\alpha}\right] - B_{44}\left[l_{02}(\gamma)\frac{\partial\Phi_2}{\partial\beta} + \right. \\ & \left.\left. + \gamma\frac{\partial Y_1}{\partial\beta} + \frac{\gamma^2}{2h}\frac{\partial Y_2}{\partial\beta}\right] + \frac{h}{8}\left(B_{55}\frac{\partial X_2}{\partial\alpha} + B_{44}\frac{\partial Y_2}{\partial\beta}\right) + Z_1\right\};\end{aligned} \tag{2.33}$$

$$\begin{aligned}\tau_{\alpha\beta} = {} & B_{66}\left[\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} - 2\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + l_{01}(\gamma)\frac{\partial\Phi_1}{\partial\beta} + l_{02}(\gamma)\frac{\partial\Phi_2}{\partial\alpha} + \right. \\ & \left. + \gamma\left(\frac{\partial X_1}{\partial\beta} + \frac{\partial Y_1}{\partial\alpha}\right) + \frac{\gamma^2}{2h}\left(\frac{\partial X_2}{\partial\beta} + \frac{\partial Y_2}{\partial\alpha}\right)\right].\end{aligned} \tag{2.34}$$

Inserting the values of stresses σ_α, σ_β, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}\tau_{\beta\gamma}$, respectively, from (2.32), (2.33), (2.34), (2.2) and (2.3) into formulas (2.6.1) - (2.6.5) and considering (2.10), we find for the internal forces and moments

$$\begin{aligned}T_1 = \int\limits_{-h/2}^{h/2}\sigma_\alpha\,d\gamma = {} & B_{11}\left(h\frac{\partial u}{\partial\alpha} + \frac{h^2}{24}\frac{\partial X_2}{\partial\alpha} + l_1\frac{\partial\Phi_1}{\partial\alpha}\right) + \\ & + B_{12}\left(h\frac{\partial v}{\partial\beta} + \frac{h^2}{24}\frac{\partial Y_2}{\partial\beta} + l_2\frac{\partial\Phi_2}{\partial\beta}\right) + A_1\left[B_{55}\left(l_1\frac{\partial\Phi_1}{\partial\alpha} + \frac{h^2}{24}\frac{\partial X_2}{\partial\alpha}\right) + \right. \\ & \left. + B_{44}\left(l_2\frac{\partial\Phi_2}{\partial\beta} + \frac{h^2}{24}\frac{\partial Y_2}{\partial\beta}\right) - h\chi(\alpha, \beta)\right];\end{aligned} \tag{2.35}$$

$$T_2 = \int_{-h/2}^{h/2} \sigma_\beta \, d\gamma = B_{22}\left(h \frac{\partial v}{\partial \beta} + \frac{h^3}{24}\frac{\partial Y_2}{\partial \beta} + I_2 \frac{\partial \Phi_2}{\partial \beta}\right) +$$
$$+ B_{12}\left(h \frac{\partial u}{\partial \alpha} + \frac{h^3}{24}\frac{\partial X_2}{\partial \alpha} + I_1 \frac{\partial \Phi_1}{\partial \alpha}\right) + A_2\left[B_{44}\left(I_2 \frac{\partial \Phi_2}{\partial \beta} + \frac{h^3}{24}\frac{\partial Y_2}{\partial \beta}\right) +\right.$$
$$\left. + B_{55}\left(I_1 \frac{\partial \Phi_1}{\partial \alpha} + \frac{h^3}{24}\frac{\partial X_2}{\partial \alpha}\right) - h\chi(\alpha, \beta)\right]; \qquad (2.36)$$

$$S_{12} = S_{21} = S = \int_{-h/2}^{h/2} \tau_{\alpha\beta} \, d\gamma = B_{66}\left[h\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) +\right.$$
$$\left. + \frac{h^2}{24}\left(\frac{\partial X_2}{\partial \beta} + \frac{\partial Y_2}{\partial \alpha}\right) + I_1 \frac{\partial \Phi_1}{\partial \beta} + I_2 \frac{\partial \Phi_2}{\partial \alpha}\right]; \qquad (2.37)$$

$$M_1 = \int_{-h/2}^{h/2} \sigma_\alpha \gamma \, d\gamma = B_{11}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha} + I_3 \frac{\partial \Phi_1}{\partial \alpha}\right) +$$
$$+ B_{12}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta} + I_4 \frac{\partial \Phi_2}{\partial \beta}\right) + A_1\left[B_{55}\left(I_3 \frac{\partial \Phi_1}{\partial \alpha} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha}\right) +\right.$$
$$\left. + B_{44}\left(I_4 \frac{\partial \Phi_2}{\partial \beta} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta}\right)\right]; \qquad (2.38)$$

$$M_2 = \int_{-h/2}^{h/2} \sigma_\beta \gamma \, d\gamma = B_{22}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta} + I_4 \frac{\partial \Phi_2}{\partial \beta}\right) +$$
$$+ B_{12}\left(-\frac{h^3}{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha} + I_3 \frac{\partial \Phi_1}{\partial \alpha}\right) + A_2\left[B_{44}\left(I_4 \frac{\partial \Phi_2}{\partial \beta} + \frac{h^3}{12}\frac{\partial Y_1}{\partial \beta}\right) +\right.$$
$$\left. + B_{55}\left(I_3 \frac{\partial \Phi_1}{\partial \alpha} + \frac{h^3}{12}\frac{\partial X_1}{\partial \alpha}\right)\right]; \qquad (2.39)$$

$$H_{12} = H_{21} = H = \int_{-h/2}^{h/2} \tau_{\alpha\beta} \gamma \, d\gamma = B_{66}\left[-2\frac{h^3}{12}\frac{\partial^2 w}{\partial \alpha \, \partial \beta} +\right.$$
$$\left. + \frac{h^3}{12}\left(\frac{\partial X_1}{\partial \beta} + \frac{\partial Y_1}{\partial \alpha}\right) + I_3 \frac{\partial \Phi_1}{\partial \beta} + I_4 \frac{\partial \Phi_2}{\partial \alpha}\right]; \qquad (2.40)$$

$$N_1 = \int_{-h/2}^{h/2} \tau_{\alpha\gamma} \, d\gamma = B_{55}(I_5 \Phi_1 + hX_1); \qquad (2.41)$$

$$N_2 = \int_{-h/2}^{h/2} \tau_{\beta\gamma} \, d\gamma = B_{44}(I_6 \Phi_2 + hY_1). \qquad (2.42)$$

where we have the following notation:

$$I_1 = \int_{-h/2}^{h/2} f_{01}(\gamma) \, d\gamma; \qquad I_2 = \int_{-h/2}^{h/2} f_{02}(\gamma) \, d\gamma;$$
$$I_3 = \int_{-h/2}^{h/2} \gamma f_{01}(\gamma) \, d\gamma; \qquad I_4 = \int_{-h/2}^{h/2} \gamma f_{02}(\gamma) \, d\gamma; \qquad (2.43)$$
$$I_5 = \int_{-h/2}^{h/2} f_1(\gamma) \, d\gamma; \qquad I_6 = \int_{-h/2}^{h/2} f_2(\gamma) \, d\gamma.$$

From (2.7.4) and (2.1) we have the following equilibrium equations:

$$\frac{\partial T_1}{\partial \alpha}+\frac{\partial S}{\partial \beta}=-X; \tag{2.44}$$

$$\frac{\partial T_2}{\partial \beta}+\frac{\partial S}{\partial \alpha}=-Y; \tag{2.45}$$

$$\frac{\partial N_1}{\partial \alpha}+\frac{\partial N_2}{\partial \beta}=-Z_2; \tag{2.46}$$

$$\frac{\partial H}{\partial \beta}+\frac{\partial M_1}{\partial \alpha}=N_1; \tag{2.47}$$

$$\frac{\partial H}{\partial \alpha}+\frac{\partial M_2}{\partial \beta}=N_2. \tag{2.48}$$

Inserting the values of the internal forces and moments from (2.35) - (2.42) into equilibrium equations (2.44) - (2.48) and considering (2.16) and (2.17), we obtain

$$\begin{aligned}
&L_{11}(C_{ik})u+L_{12}(C_{ik})v+a_{55}I_1L_{11}(B_{ik})\varphi+a_{44}I_2L_{12}(B_{ik})\psi-\\
&\quad-A_1\Bigl[\Bigl\{\frac{h}{2}\Bigl[I_{01}\Bigl(\frac{h}{2}\Bigr)+I_{01}\Bigl(-\frac{h}{2}\Bigr)\Bigr]-I_1\Bigr\}\frac{\partial^2\varphi}{\partial\alpha^2}+\\
&\quad+\Bigl\{\frac{h}{2}\Bigl[I_{02}\Bigl(\frac{h}{2}\Bigr)+I_{02}\Bigl(-\frac{h}{2}\Bigr)\Bigr]-I_2\Bigr\}\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\Bigr]=-\frac{h^2}{24}[L_{11}(B_{ik})X_2+\\
&\quad+L_{12}(B_{ik})Y_2]-B_{55}X_2+A_1\Bigl[\frac{h^2}{12}\Bigl(B_{55}\frac{\partial^2X_2}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_2}{\partial\alpha\,\partial\beta}\Bigr)+h\frac{\partial Z_1}{\partial\alpha}\Bigr];
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
&L_{22}(C_{ik})v+L_{12}(C_{ik})u+a_{44}I_2L_{22}(B_{ik})\psi+a_{55}I_1L_{12}(B_{ik})\varphi-\\
&\quad-A_2\Bigl[\Bigl\{\frac{h}{2}\Bigl[I_{02}\Bigl(\frac{h}{2}\Bigr)+I_{02}\Bigl(-\frac{h}{2}\Bigr)\Bigr]-I_2\Bigr\}\frac{\partial^2\psi}{\partial\beta^2}+\\
&\quad+\Bigl\{\frac{h}{2}\Bigl[I_{01}\Bigl(\frac{h}{2}\Bigr)+I_{01}\Bigl(-\frac{h}{2}\Bigr)\Bigr]-I_1\Bigr\}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}\Bigr]=-\frac{h^2}{24}[L_{22}(B_{ik})Y_2+\\
&\quad+L_{12}(B_{ik})X_2]-B_{44}Y_2+A_2\Bigl[\frac{h^2}{12}\Bigl(B_{44}\frac{\partial^2Y_2}{\partial\beta^2}+B_{55}\frac{\partial^2X_2}{\partial\alpha\,\partial\beta}\Bigr)+h\frac{\partial Z_1}{\partial\beta}\Bigr];
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
&\Bigl[I_{01}\Bigl(\frac{h}{2}\Bigr)-I_{01}\Bigl(-\frac{h}{2}\Bigr)\Bigr]\frac{\partial\varphi}{\partial\alpha}+\Bigl[I_{02}\Bigl(\frac{h}{2}\Bigr)-I_{02}\Bigl(-\frac{h}{2}\Bigr)\Bigr]\frac{\partial\psi}{\partial\beta}=\\
&\qquad=-Z_2-h\Bigl(B_{55}\frac{\partial X_1}{\partial\beta}+B_{44}\frac{\partial Y_1}{\partial\beta}\Bigr);
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
&L_{13}(D_{ik})w-a_{55}I_3L_{11}(B_{ik})\varphi-a_{44}I_4L_{12}(B_{ik})\psi+\\
&\quad+\Bigl[I_{01}\Bigl(\frac{h}{2}\Bigr)-I_{01}\Bigl(-\frac{h}{2}\Bigr)\Bigr]\varphi-A_1\Bigl(I_3\frac{\partial^2\varphi}{\partial\alpha^2}+I_4\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\Bigr)=\\
&\quad=\frac{h^3}{12}[L_{11}(B_{ik})X_1+L_{12}(B_{ik})Y_1]-hB_{55}X_1+\\
&\qquad+A_1\frac{h^3}{12}\Bigl(B_{55}\frac{\partial^2X_1}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_1}{\partial\alpha\,\partial\beta}\Bigr);
\end{aligned} \tag{2.52}$$

$$\begin{aligned}
&L_{23}(D_{ik})w-a_{44}I_4L_{22}(B_{ik})\psi-a_{55}I_3L_{12}(B_{ik})\varphi+\\
&\quad+\Bigl[I_{02}\Bigl(\frac{h}{2}\Bigr)-I_{02}\Bigl(-\frac{h}{2}\Bigr)\Bigr]\psi-A_2\Bigl(I_4\frac{\partial^2\psi}{\partial\beta^2}+I_3\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}\Bigr)=\\
&\quad=\frac{h^3}{12}[L_{22}(B_{ik})Y_1+L_{12}(B_{ik})X_1]-hB_{44}Y_1+\\
&\qquad+A_2\frac{h^3}{12}\Bigl(B_{44}\frac{\partial^2Y_1}{\partial\beta^2}+B_{55}\frac{\partial^2X_1}{\partial\alpha\,\partial\beta}\Bigr).
\end{aligned} \tag{2.53}$$

where, as usual,

$$L_{11}(B_{ik}) = B_{11}\frac{\partial^2}{\partial\alpha^2} + B_{66}\frac{\partial^2}{\partial\beta^2};$$

$$L_{22}(B_{ik}) = B_{22}\frac{\partial^2}{\partial\beta^2} + B_{66}\frac{\partial^2}{\partial\alpha^2};$$

$$L_{12}(B_{ik}) = (B_{12} + B_{66})\frac{\partial^2}{\partial\alpha\,\partial\beta};$$

$$L_{13}(B_{ik}) = B_{11}\frac{\partial^3}{\partial\alpha^3} + (B_{12} + 2B_{66})\frac{\partial^3}{\partial\alpha\,\partial\beta^2};$$

$$L_{23}(B_{ik}) = B_{22}\frac{\partial^3}{\partial\beta^3} + (B_{12} + 2B_{66})\frac{\partial^3}{\partial\beta\,\partial\alpha^2};$$

$$C_{ik} = hB_{ik}, \quad D_{ik} = \frac{h^3}{12}B_{ik}.$$

Equations (2.49) - (2.53) constitute a complete system of five differential equations in terms of the five sought functions u, v, w, φ, and ψ. Having the values of the sought functions, by means of formulas (2.32) - (2.42) it is not difficult to find the values of the principle stresses and internal forces.

As usual, in solving specific edge problems the applicable boundary conditions are imposed on the differential equations of solutions (2.49) - (2.53) of the plate. The boundary conditions are determined by the usual methods.*,**, ***,****,*****,******,*******

In the general case we assume that the edge of the plate is curvilinear with an external normal n (Figure 74). Then for the internal forces and moments referred to the curvilinear edge (Figure 75) we find

$$T_n = T_1\cos^2\vartheta + T_2\sin^2\vartheta + 2S\sin\vartheta\cos\vartheta;$$
$$S_{nt} = S(\cos^2\vartheta - \sin^2\vartheta) + (T_2 - T_1)\sin\vartheta\cos\vartheta;$$
$$N_n = N_1\cos\vartheta + N_2\sin\vartheta;$$
$$M_n = M_1\cos^2\vartheta + M_2\sin^2\vartheta + 2H\sin\vartheta\cos\vartheta;$$
$$H_{nt} = H(\cos^2\vartheta - \sin^2\vartheta) + (M_2 - M_1)\sin\vartheta\cos\vartheta.$$

---

*S.P. Timoshenko, Plastinki i obolochki, Gostekhizdat, 1948, pp. 98-102.

**E. Reissner, On the Theory of Bending of Elastic Plates, J. Math. Phys., v. XXIII, 1, 1944.

***E. Reissner, On Bending of Elastic Plates. Quart. Appl. Math., v. V, No. 1, 1947.

****L. Bolle, Contribution au problème lineaire de flexion d'une plaque élastique. Parts 1,2. Bulletin technique de la Suisse Romande, pp. 281-285, 293-298, 1947.

*****B.F. Vlasov, On equations in the theory of bending of plates, Izvestiya OTN AN SSSR, No. 12, 1957.

******A.L. Gol'denveyzer, Toward the Reissner theory of bending of plates, Izvestiya OTN AN SSSR, No. 4, 1958.

*******P.M. Ogibalov, Izgib, ustoychivost' i kolebaniya plastinok [Bending, stability and vibrations of plates], Izd-vo MGU, 1958, pp. 50-51.

In addition to these, for the normal and tangential components of displacement of any point of the edge in the plane of the plate we have:

$$u_n = u_\alpha \cos\vartheta + u_\beta \sin\vartheta; \quad u_{nt} = u_\beta \cos\vartheta - u_\alpha \sin\vartheta.$$

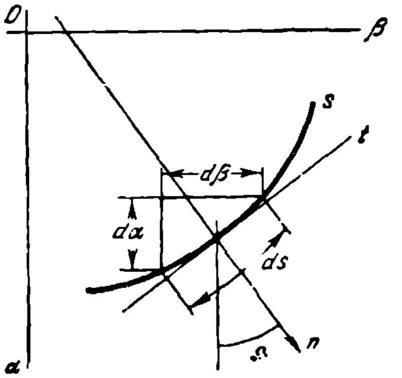

Figure 74

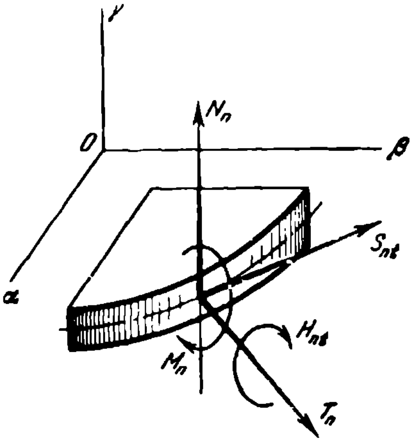

Figure 75

Let us present the uniform boundary conditions for certain fundamental cases:

a) free edge:

$$T_n = 0, \quad S_{nt} = 0, \quad N_n = 0, \quad M_n = 0, \quad H_{nt} = 0;$$

b) freely supported edge

$$w = 0, \quad T_n = 0, \quad S_{nt} = 0, \quad M_n = 0, \quad H_{nt} = 0;$$

c) hinged edge free in the tangential direction

$$w = 0, \quad (u_n)_{\gamma=0} = 0, \quad S_{nt} = 0, \quad M_n = 0, \quad \left(\frac{\partial u_{nt}}{\partial \gamma}\right)_{\gamma=0} = 0;$$

d) hinged edge free in the normal (n) direction

$$w = 0, \quad (u_{nt})_{\gamma=0} = 0, \quad T_n = 0, \quad M_n = 0, \quad \left(\frac{\partial u_{nt}}{\partial \gamma}\right)_{\gamma=0} = 0;$$

e) hinge-fixed edge

$$w = 0, \quad (u_n)_{\gamma=0} = 0, \quad (u_{nt})_{\gamma=0} = 0, \quad M_n = 0, \quad \left(\frac{\partial u_{nt}}{\partial \gamma}\right)_{\gamma=0} = 0;$$

f) fixed edge (fixed element of median surface)

$$w = 0, \quad (u_n)_{\gamma=0} = 0, \quad (u_{nt})_{\gamma=0} = 0, \quad \frac{\partial w}{\partial n} = 0, \quad \left(\frac{\partial u_{nt}}{\partial \gamma}\right)_{\gamma=0} = 0;$$

g) fixed edge (fixed vertical element of edge)

$$w=0,\quad (u_n)_{\gamma=0}=0,\quad (u_{nt})_{\gamma=0}=0,\quad \left(\frac{\partial u_{nt}}{\partial\gamma}\right)_{\gamma=0}=0,\quad \left(\frac{\partial u_n}{\partial\gamma}\right)_{\gamma=0}=0.$$

The corresponding inhomogeneous boundary conditions may be constructed in a manner similar to that in Section 1ʋ, Chapter II. It is easily seen that there are other possible boundary conditions which do not contradict one another.* However, we shall not present them here since it is our opinion that on the basis of those which have already been presented the reader may easily construct boundary conditions of interest to him. The problem of boundary conditions in the theory of anisotropic shells and plates as proposed herein requires additional investigation.

In a work by the author** it has been shown that certain inaccuracies which may be assumed in selection of functions $f_i(\gamma)$ have little effect on the final results for the principal design quantities. Within the proposed theory chance cannot introduce inadmissible errors after a logical choice of functions $f_i(\gamma)$ has been made. (See the appropriate example in Section 6 of this chapter.)

In this connection we assume that**

$$f_1(\gamma)=f_2(\gamma)=f(\gamma), \tag{2.54}$$

where $f(\gamma)$ is a certain logically chosen even function.

By virtue of (2.54), (2.21) and (2.43) we have for the required integrals

$$\left.\begin{aligned} &I_{01}(\gamma)=I_{02}(\gamma)=I_0(\gamma),\quad I_{01}\left(\frac{h}{2}\right)=-I_{01}\left(-\frac{h}{2}\right)=\frac{1}{2}\chi_1,\\ &I_1=I_2=0,\quad I_3=I_4=\chi_2,\quad I_5=I_6=\chi_1.\end{aligned}\right\} \tag{2.55}$$

Then from (2.49) - (2.53) for the discussed special case we obtain the following system of differential equations of solution:

$$\begin{aligned} L_{11}(C_{ik})u+L_{12}(C_{ik})v={}&-\frac{h^3}{24}\left[L_{11}(B_{ik})X_2+L_{12}(B_{ik})Y_2\right]-B_{55}X_2+\\ &+A_1\left[\frac{h^3}{12}\left(B_{55}\frac{\partial^2 X_2}{\partial\alpha^2}+B_{44}\frac{\partial^2 Y_2}{\partial\alpha\,\partial\beta}\right)+h\frac{\partial Z_1}{\partial\alpha}\right];\end{aligned} \tag{2.56}$$

$$\begin{aligned} L_{22}(C_{ik})v+L_{12}(C_{ik})u={}&-\frac{h^3}{24}\left[L_{22}(B_{ik})Y_2+L_{12}(B_{ik})X_2\right]-B_{44}Y_2+\\ &+A_2\left[\frac{h^3}{12}\left(B_{44}\frac{\partial^2 Y_2}{\partial\beta^2}+B_{55}\frac{\partial^2 X_2}{\partial\alpha\,\partial\beta}\right)+h\frac{\partial Z_1}{\partial\beta}\right];\end{aligned} \tag{2.57}$$

---

*S.A. Ambartsumyan, see footnote * and ** on p. 315.

**S.A. Ambartsumyan, see footnote * on p. 315.

$$\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}=-\frac{Z_2}{\chi_1}-\frac{h}{\chi_1}\left(B_{55}\frac{\partial X_1}{\partial\alpha}+B_{44}\frac{\partial Y_1}{\partial\beta}\right); \tag{2.58}$$

$$\begin{gathered}L_{13}(D_{ik})\,w-\chi_2\left[a_{55}L_{11}(B_{ik})\,\varphi+a_{44}L_{12}(B_{ik})\,\psi\right]+\chi_1\varphi=\\ \frac{h^3}{12}\left[L_{11}(B_{ik})\,X_1+L_{12}(B_{ik})\,Y_1\right]-hB_{55}X_1-\\ -A_1\left[\left(h\frac{\chi_2}{\chi_1}-\frac{h^3}{12}\right)\left(B_{55}\frac{\partial^2X_1}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_1}{\partial\alpha\,\partial\beta}\right)+\frac{\chi_2}{\chi_1}\frac{\partial Z_2}{\partial\alpha}\right];\end{gathered} \tag{2.59}$$

$$\begin{gathered}L_{23}(D_{ik})\,w-\chi_2\left[a_{44}L_{22}(B_{ik})\,\psi+a_{55}L_{12}(B_{ik})\,\varphi\right]+\chi_1\psi=\\ =\frac{h^3}{12}\left[L_{22}(B_{ik})\,Y_1+L_{12}(B_{ik})\,X_1\right]-hB_{44}Y_1-\\ -A_2\left[\left(h\frac{\chi_2}{\chi_1}-\frac{h^3}{12}\right)\left(B_{44}\frac{\partial^2Y_1}{\partial\beta^2}+B_{55}\frac{\partial^2X_1}{\partial\alpha\,\partial\beta}\right)+\frac{\chi_2}{\chi_1}\frac{\partial Z_2}{\partial\beta}\right].\end{gathered} \tag{2.60}$$

In deriving equations (2.56) and (2.60) equation (2.58) was used.

In considering that certain inaccuracies in selection of functions $f_i(\gamma)$ cannot introduce inadmissible errors in the proposed theory as well as the well-known fact*,**,***,**** that in the presence of bending the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ over the thickness of the plate vary almost as the equation for a quadric parabola, we assume

$$f_i(\gamma)=\frac{1}{2}\left(\frac{h^2}{4}-\gamma^2\right). \tag{2.61}$$

Then from (2.21) and (2.43) we obtain for the required integrals

$$f_0(\gamma)=\frac{\gamma}{2}\left(\frac{h^2}{4}-\frac{\gamma^2}{3}\right),\quad \chi_1=\frac{h^3}{12},\quad \chi_2=\frac{h^5}{120}. \tag{2.62}$$

By virtue of (2.62) we obtain from (2.56) - (2.60) the following system of differential equations of solution:

$$\begin{gathered}L_{11}(C_{ik})\,u+L_{12}(C_{ik})\,v=-\frac{h^2}{24}\left[L_{11}(B_{ik})\,X_2+L_{12}(B_{ik})\,Y_2\right]-B_{55}X_2+\\ +A_1\left[\frac{h^2}{12}\left(B_{55}\frac{\partial^2X_2}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_2}{\partial\alpha\,\partial\beta}\right)+h\frac{\partial Z_1}{\partial\alpha}\right];\end{gathered} \tag{2.63}$$

---

*B. G. Galerkin, Napryazhennoye sostoyaniye pri izgibe pryamougol'noy plity po teorii tolstykh plit i teorii plit tonkikh [Bending stress of a rectangular plate in the theory of thick plates and theory of thin plates], Soch. [Collection], Vol. I, Izd. AN SSSR, 1952, pp. 347-362.

**A. I. Lur'ye, Prostranstvennyye zadachi teorii uprugosti, Gostekhizdat, 1955, pp. 200-218.

***B. F. Vlasov, On one case of bending of a thick rectangular plate, Vestnik MGU, No. 2, 1957.

****A. P. Melkonyan, On the bending of a thick two-layer plate, Izvestiya AN ArmSSR (seriya fiz.-mat. nauk), Vol. XII, No. 2, 1959.

$$L_{22}(C_{ik})v+L_{12}(C_{ik})u=-\frac{h^2}{24}\left[L_{22}(B_{ik})Y_2+L_{12}(B_{ik})X_2\right]-B_{44}Y_2+$$
$$+A_2\left[\frac{h^2}{12}\left(B_{44}\frac{\partial^2Y_2}{\partial\beta^2}+B_{55}\frac{\partial^2X_2}{\partial\alpha\,\partial\beta}\right)+h\frac{\partial Z_1}{\partial\beta}\right]; \tag{2.64}$$
$$\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}=-\frac{12}{h^3}Z_2-\frac{12}{h^2}\left(B_{55}\frac{\partial X_1}{\partial\alpha}+B_{44}\frac{\partial Y_1}{\partial\beta}\right); \tag{2.65}$$
$$L_{13}(D_{ik})w-\frac{h^2}{10}\left[a_{55}L_{11}(D_{ik})\varphi+a_{44}L_{12}(D_{ik})\psi\right]+\frac{h^3}{12}\varphi=$$
$$=\frac{h^3}{12}\left[L_{11}(B_{ik})X_1+L_{12}(B_{ik})Y_1\right]-hB_{55}X_1-$$
$$-A_1\left[\frac{h^3}{60}\left(B_{55}\frac{\partial^2X_1}{\partial\alpha^2}+B_{44}\frac{\partial^2Y_1}{\partial\beta\,\partial\alpha}\right)+\frac{h^2}{10}\frac{\partial Z_2}{\partial\alpha}\right]; \tag{2.66}$$
$$L_{23}(D_{ik})w-\frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\psi+a_{55}L_{12}(D_{ik})\varphi\right]+\frac{h^3}{12}\psi=$$
$$=\frac{h^3}{12}\left[L_{22}(B_{ik})Y_1+L_{12}(B_{ik})X_1\right]-hB_{44}Y_1-$$
$$-A_2\left[\frac{h^3}{60}\left(B_{44}\frac{\partial^2Y_1}{\partial\beta^2}+B_{55}\frac{\partial^2X_1}{\partial\alpha\,\partial\beta}\right)+\frac{h^2}{10}\frac{\partial Z_2}{\partial\beta}\right]. \tag{2.67}$$

In the discussed special cases of (2.54) and (2.61) the formulas for internal forces (2.35) - (2.42) and stresses (2.32) - (2.34) are somewhat simplified. To avoid repetition we shall not present these formulas here. However, let us state that in performing the corresponding calculations we must use formulas (2.35) - (2.42) and (2.32) - (2.34), but in addition in each case (2.34) and (2.61) it is necessary to take into account the corresponding values of the required integrals (2.55) and (2.62).

Of great practical interest is the case wherein $X^{\pm}=0$, $Y^{\pm}=0$, that is, when the plate is loaded only by a normally applied load $Z^{\mp}$. In this case the equations of solution (2.49) - (2.53), (2.56) - (2.60), (2.63) - (2.67) and their corresponding design formulas are substantially simplified. For example, in the case of (2.61) the system of equations in (2.63) - (2.67) takes the following form:

$$L_{11}(C_{ik})u+L_{12}(C_{ik})v=hA_1\frac{\partial Z_1}{\partial\alpha}; \tag{2.68}$$
$$L_{22}(C_{ik})v+L_{12}(C_{ik})u=hA_2\frac{\partial Z_1}{\partial\beta}; \tag{2.69}$$
$$\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}=-\frac{12}{h^3}Z_2; \tag{2.70}$$
$$L_{13}(D_{ik})w-\frac{h^2}{10}\left[a_{55}L_{11}(D_{ik})\varphi+a_{44}L_{12}(D_{ik})\psi\right]+$$
$$+\frac{h^3}{12}\varphi=-A_1\frac{h^2}{10}\frac{\partial Z_2}{\partial\alpha}; \tag{2.71}$$
$$L_{23}(D_{ik})w-\frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\psi+a_{55}L_{12}(D_{ik})\varphi\right]+$$
$$+\frac{h^3}{12}\psi=-A_2\frac{h^2}{10}\frac{\partial Z_2}{\partial\beta}. \tag{2.72}$$

while from (2.35) - (2.42) and (2.32) - (2.34) we obtain for the internal forces and stresses corresponding to the system in (2.68) - (2.72)

$$T_1 = C_{11}\frac{\partial u}{\partial \alpha} + C_{12}\frac{\partial v}{\partial \beta} - hA_1Z_1; \tag{2.73}$$

$$T_2 = C_{22}\frac{\partial v}{\partial \beta} + C_{12}\frac{\partial u}{\partial \alpha} - hA_2Z_1; \tag{2.74}$$

$$S = C_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right); \tag{2.75}$$

$$\begin{aligned} M_1 = {} & -D_{11}\frac{\partial^2 w}{\partial \alpha^2} - D_{12}\frac{\partial^2 w}{\partial \beta^2} + \\ & + \frac{h^3}{10}\left(D_{11}a_{55}\frac{\partial \varphi}{\partial \alpha} + D_{12}a_{44}\frac{\partial \psi}{\partial \beta}\right) - \frac{h^3}{10}A_1Z_2; \end{aligned} \tag{2.76}$$

$$\begin{aligned} M_2 = {} & -D_{22}\frac{\partial^2 w}{\partial \beta^2} - D_{12}\frac{\partial^2 w}{\partial \alpha^2} + \\ & + \frac{h^3}{10}\left(D_{22}a_{44}\frac{\partial \psi}{\partial \beta} + D_{12}a_{55}\frac{\partial \varphi}{\partial \alpha}\right) - \frac{h^3}{10}A_2Z_2; \end{aligned} \tag{2.77}$$

$$H = D_{66}\left[-2\frac{\partial^2 w}{\partial \alpha\,\partial \beta} + \frac{h^3}{10}\left(a_{55}\frac{\partial \varphi}{\partial \beta} + a_{44}\frac{\partial \psi}{\partial \alpha}\right)\right]; \tag{2.78}$$

$$N_1 = \frac{h^3}{12}\varphi, \quad N_2 = \frac{h^3}{12}\psi; \tag{2.79}$$

$$\begin{aligned} \sigma_\alpha = {} & B_{11}\left(\frac{\partial u}{\partial \alpha} - \gamma\frac{\partial^2 w}{\partial \alpha^2}\right) + B_{12}\left(\frac{\partial v}{\partial \beta} - \gamma\frac{\partial^2 w}{\partial \beta^2}\right) + \\ & + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left(B_{11}a_{55}\frac{\partial \varphi}{\partial \alpha} + B_{12}a_{44}\frac{\partial \psi}{\partial \beta}\right) - \\ & - A_1\left[Z_1 + 6\left(\frac{1}{4}\frac{\gamma}{h} - \frac{1}{3}\frac{\gamma^3}{h^3}\right)Z_2\right]; \end{aligned} \tag{2.80}$$

$$\begin{aligned} \sigma_\beta = {} & B_{22}\left(\frac{\partial v}{\partial \beta} - \gamma\frac{\partial^2 w}{\partial \beta^2}\right) + B_{12}\left(\frac{\partial u}{\partial \alpha} - \gamma\frac{\partial^2 w}{\partial \alpha^2}\right) + \\ & + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left(B_{22}a_{44}\frac{\partial \psi}{\partial \beta} + B_{12}a_{55}\frac{\partial \varphi}{\partial \alpha}\right) - \\ & - A_2\left[Z_1 + 6\left(\frac{1}{4}\frac{\gamma}{h} - \frac{1}{3}\frac{\gamma^3}{h^3}\right)Z_2\right]; \end{aligned} \tag{2.81}$$

$$\begin{aligned} \tau_{\alpha\beta} = {} & B_{66}\left[\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} - 2\gamma\frac{\partial^2 w}{\partial \alpha\,\partial \beta} + \right. \\ & \left. + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left(a_{55}\frac{\partial \varphi}{\partial \beta} + a_{44}\frac{\partial \psi}{\partial \alpha}\right)\right]. \end{aligned} \tag{2.82}$$

SECTION 3. THEORY OF BENDING OF A PLATE POSSESSING CYLINDRICAL ANISOTROPY

Proceeding in the same manner as in the previous section, we may easily construct a theory for the bending of a plate possessing cylindrical anisotropy.

After S.G. Lekhnitskiy,* we assume that a plate with cylindrical anisotropy is at the same time orthotropic, wherein all radial planes passing through the axis of anisotropy are planes of elastic symmetry. We assume that the pole of anisotropy - the point of intersection of the axis of anisotropy and the median plane (which is taken perpendicular to this axis) - is the origin of the cylindrical system of coordinates r, θ , γ, that the γ axis is directed along the axis of anisotropy and that the β axis (the polar axis) is arbitrarily directed in the median plane. Finally, we assume that the pole of anisotropy may lie within the plate or outside it (Figure 76).

---

*S.G. Lekhnitskiy, Anizotropnyye plastinki, Gostekhizdat, 1957, pp. 276-278.

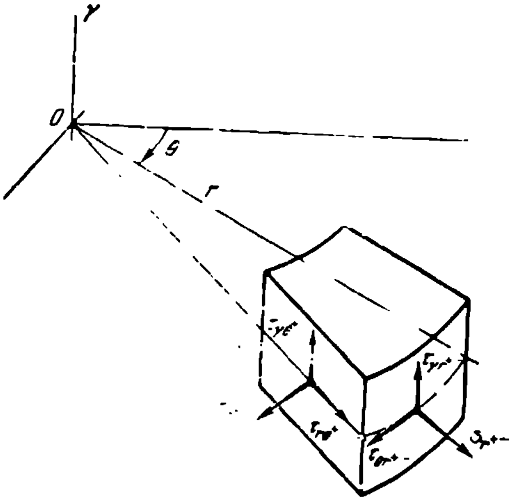

Figure 76

Let us derive the final formulas for the design stresses and internal forces as well as the equations of solution of the problem, wherein we have (2.61) and assume that X± = 0, Y± = 0.

The formulas for the stresses (Figure 76):

$$\sigma_r = B_r\left(\frac{\partial u}{\partial r} - \gamma\frac{\partial^2 w}{\partial r^2}\right) + B_r\nu_\theta\left[\frac{\partial v}{r\,\partial\theta} + \frac{u}{r} - \gamma\left(\frac{\partial^2 w}{r^2\,\partial\theta^2} + \frac{1}{r}\,\frac{\partial w}{\partial r}\right)\right] +$$
$$+ \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left[B_r a_r\left(\frac{\partial\varphi}{\partial r} + \nu_\theta\frac{\varphi}{r}\right) +\right.$$
$$\left.+ 3_r\nu_\theta a_\theta\frac{\partial\psi}{r\,\partial\theta}\right] - A_1\left[Z_1 + 6\left(\frac{1}{4}\,\frac{\gamma}{h} - \frac{1}{3}\,\frac{\gamma^3}{h^3}\right)Z_2\right]; \qquad (3.1)$$

$$\sigma_\theta = B_\theta\left[\frac{\partial v}{r\,\partial\theta} + \frac{u}{r} - \gamma\left(\frac{\partial^2 w}{r^2\,\partial\theta^2} + \frac{1}{r}\,\frac{\partial w}{\partial r}\right)\right] + B_\theta\nu_r\left(\frac{\partial u}{\partial r} - \gamma\frac{\partial^2 w}{\partial r^2}\right) +$$
$$+ \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left[B_\theta a_\theta\frac{\partial\psi}{r\,\partial\theta} + B_\theta a_r\left(\frac{\varphi}{r} + \nu_r\frac{\partial\varphi}{\partial r}\right)\right] -$$
$$- A_2\left[Z_1 + 6\left(\frac{1}{4}\,\frac{\gamma}{h} - \frac{1}{3}\,\frac{\gamma^3}{h^3}\right)Z_2\right]; \qquad (3.2)$$

$$\tau_r = B_k\left\{\frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} - 2\gamma\frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial\theta}\right) + \right.$$
$$\left. + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{1}{3}\right)\left[a_r\frac{\partial\varphi}{r\,\partial\theta} + a_\theta\left(\frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right)\right]\right\}. \quad (3.3)$$

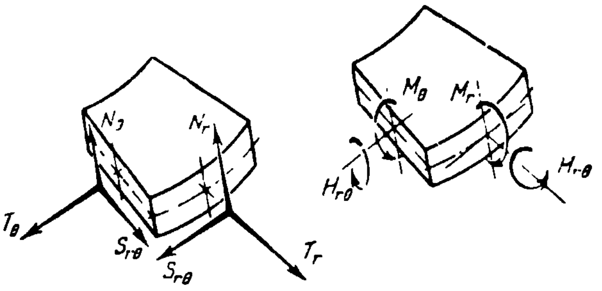

Figure 77

The internal forces and moments (Figure 77):

$$T_r = C_r\frac{\partial u}{\partial r} + C_r\nu_\theta\frac{1}{r}\left(\frac{\partial v}{\partial\theta} + u\right) - hA_1Z_1; \quad (3.4)$$

$$T_\theta = C_\theta\frac{1}{r}\left(\frac{\partial v}{\partial\theta} + u\right) + C_\theta\nu_r\frac{\partial u}{\partial r} - hA_2Z_1; \quad (3.5)$$

$$S_{\theta r} = C_k\left[\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right]; \quad (3.6)$$

$$M_r = -D_r\frac{\partial^2 w}{\partial r^2} - D_r\nu_\theta\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial w}{\partial r}\right) +$$
$$+ \frac{h^2}{10}\left[D_r a_r\left(\frac{\partial\varphi}{\partial r} + \nu_\theta\frac{\varphi}{r}\right) + D_r\nu_\theta a_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right] - \frac{h^2}{10}A_1Z_2; \quad (3.7)$$

$$M_\theta = -D_\theta\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial w}{\partial r}\right) - D_\theta\nu_r\frac{\partial^2 w}{\partial r^2} +$$
$$+ \frac{h^2}{10}\left[D_\theta a_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + D_\theta a_r\left(\frac{\varphi}{r} + \nu_r\frac{\partial\varphi}{\partial r}\right)\right] - \frac{h^2}{10}A_2Z_2; \quad (3.8)$$

$$H = D_k\left\{-2\frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial\theta}\right) + \frac{h^2}{10}\left[a_r\frac{\partial\varphi}{r\,\partial\theta} + a_\theta\left(\frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right)\right]\right\}; \quad (3.9)$$

$$N_r = \frac{h^3}{12}\varphi, \quad N_\theta = \frac{h^3}{12}\psi. \quad (3.10)$$

The system of equations of solution:

$$C_r\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) + C_k\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2} - C_\theta\frac{u}{r} + (C_\theta\nu_r + C_k)\frac{\partial^2 v}{\partial r\,\partial\theta} -$$
$$-(C_\theta + C_k)\frac{1}{r}\frac{\partial v}{\partial\theta} = hA_1\frac{\partial}{\partial r}(rZ_1) - hA_2Z_1; \quad (3.11)$$

$$C_\theta\frac{1}{r}\frac{\partial^2 v}{\partial\theta^2} + C_k\left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) + (C_r\nu_\theta + C_k)\frac{\partial^2 u}{\partial r\,\partial\theta} +$$
$$+(C_\theta + C_k)\frac{1}{r}\frac{\partial u}{\partial\theta} = hA_2\frac{\partial Z_1}{\partial\theta}; \quad (3.12)$$

$$\frac{1}{r}\frac{\partial (r\varphi)}{\partial r}+\frac{1}{r}\frac{\partial \psi}{\partial \theta}=-\frac{12}{h^3}Z_\gamma; \tag{3.13}$$

$$\begin{aligned}&D_r\frac{\partial}{\partial r}\left(r\frac{\partial^2 w}{\partial r^2}\right)+D_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right)-D_\theta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)-\\ &-\frac{h^2}{10}\left\{a_r\left[D_r\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)+D_k\frac{1}{r}\frac{\partial^2 \varphi}{\partial \theta^2}-D_\theta\frac{\varphi}{r}\right]+\right.\\ &\left.+a_\theta\left[(D_\theta\nu_r+D_k)\frac{\partial^2 \psi}{\partial r\,\partial \theta}-(D_\theta+D_k)\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right]\right\}+\frac{h^3}{12}r\varphi=\\ &=A_2\frac{h^2}{10}Z_2-A_1\frac{h^2}{10}\frac{\partial}{\partial r}(rZ_2);\end{aligned} \tag{3.14}$$

$$\begin{aligned}&D_\theta\left(\frac{1}{r^2}\frac{\partial^3 w}{\partial \theta^3}+\frac{1}{r}\frac{\partial^3 w}{\partial r\,\partial \theta}\right)+D_{r\theta}\frac{\partial^3 w}{\partial r^2\,\partial \theta}-\frac{h^2}{10}\left\{a_\theta\left[D_\theta\frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2}+\right.\right.\\ &\left.+D_k\left(r\frac{\partial^2 \psi}{\partial r^2}+\frac{\partial \psi}{\partial r}-\frac{\psi}{r}\right)\right]+a_r\left[(D_r\nu_\theta+D_k)\frac{\partial^2 \varphi}{\partial r\,\partial \theta}+\right.\\ &\left.\left.+(D_\theta+D_k)\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right]\right\}+\frac{h^3}{12}r\psi=-A_2\frac{h^2}{10}\frac{\partial Z_2}{\partial \theta}.\end{aligned} \tag{3.15}$$

In the given formulas and equations: D_r, D_θ, and C_r, C_θ are, respectively, the bending stiffness and the elongation stiffness for directions r and θ; D_k is the shear stiffness:

$$\left.\begin{aligned}&D_r=\frac{h^3}{12}B_r, \quad D_\theta=\frac{h^3}{12}B_\theta, \quad D_k=\frac{h^3}{12}B_k,\\ &C_r=hB_r, \qquad C_\theta=hB_\theta, \qquad C_k=hB_k,\\ &\qquad D_{r\theta}=D_r\nu_\theta+2D_k,\end{aligned}\right\} \tag{3.16}$$

where

$$B_r=\frac{E_r}{1-\nu_r\nu_\theta}, \quad B_\theta=\frac{E_\theta}{1-\nu_r\nu_\theta}, \quad B_k=G_{r\theta}; \tag{3.17}$$

$$a_r=\frac{1}{B_{r\gamma}}=\frac{1}{G_{r\gamma}}, \qquad a_\theta=\frac{1}{B_{\theta\gamma}}=\frac{1}{G_{\theta\gamma}}; \tag{3.18}$$

$$A_1=-\frac{E_r}{E_\gamma}\frac{\nu_{r\gamma}+\nu_\theta\nu_{\theta\gamma}}{1-\nu_r\nu_\theta}, \quad A_2=-\frac{E_\theta}{E_\gamma}\frac{\nu_{\theta\gamma}+\nu_r\nu_{r\gamma}}{1-\nu_r\nu_\theta}. \tag{3.19}$$

In these formulas: E_r, E_θ, E_γ are the elastic moduli in the radial (r), tangential (θ) and normal (γ) directions, respectively; $\nu_r=\nu_{\theta r}$, $\nu_\theta=\nu_{r\theta}$, $\nu_{r\gamma}$, $\nu_{\theta\gamma}$ are Poisson's ratios ($E_r\nu_\theta=E_\theta\nu_r$ etc.); $G_{r\theta}$, $G_{r\gamma}$, $G_{\theta\gamma}$ are the shear moduli for directions r and θ, r and γ, θ and γ, respectively. In addition, $\varphi=\varphi(r, \theta)$, $\psi=\psi(r, \theta)$, $w=w(r, \theta)$.

SECTION 4. APPROXIMATE THEORY OF AN ANISOTROPIC PLATE CONSIDERING TRANSVERSE SHEAR*

In distinction from the general theory of orthotropic plates presented in Section 2 of this chapter, we here present a more approximate theory of plates which, in addition to basic premises (a) and (b), contains another assumption, namely: the normal stresses σ_γ at areas parallel to the median plane may be disregarded in comparison with other stresses.

Without disturbing the general validity of our remarks, let us discuss the case in which an orthotropic plate is loaded only by a normally applied load $Z^\pm \neq 0$ ($X^\pm = 0$, $Y^\pm = 0$). It is assumed that $f_1(\gamma) = f_2(\gamma) = f(\gamma)$, where $f(\gamma)$ is an even function, as well as that with $\gamma = 0$ $u_\alpha = u = 0$, $u_\beta = v = 0$.

Then for the stresses we have

$$\tau_{\alpha\gamma} = f(\gamma)\varphi(\alpha, \beta), \quad \tau_{\beta\gamma} = f(\gamma)\psi(\alpha, \beta); \tag{4.1}$$

$$\sigma_\alpha = B_{11}e_\alpha + B_{12}e_\beta; \tag{4.2}$$

$$\sigma_\beta = B_{22}e_\beta + B_{12}e_\alpha; \tag{4.3}$$

$$\tau_{\alpha\beta} = B_{66}e_{\alpha\beta}. \tag{4.4}$$

Comparing (4.2) and (4.3) with (2.4) and (2.5), it is easily seen that the additional hypothesis concerning the negligibility of stresses σ_γ in comparison with other stresses formally reduces to the assumption $A_1 = 0$, $A_2 = 0$.

For greater strictness let it be pointed out that in disregarding the normal stresses σ_γ, we are not proceeding in a formally consistent manner. The fact is that in the hypothesis of nondeformable normals the correction due to consideration of stresses σ_γ, is of the same order (h^2/a^2) as the correction due to consideration of the influence of transverse shear, which in the theory proposed in this section we do not disregard. However, this inconsistency, as will be seen from the discussed examples (Section 6), in the majority of cases of practical importance cannot introduce unacceptable error.

---

*S.A. Ambartsumyan, see footnote * on p. 315.

We will point out that in many cases the phenomena associated with stress σ_{γ} must be taken into consideration. Then, in the design formulas and equations it is necessary to leave all terms with factors A_1 and A_2 (see Sections 2, 3, 5, 6, 7).

From (2.19) and (2.20), by virtue of (2.17) and (2.55) we obtain for the tangential displacements

$$\left.\begin{aligned} u_{\alpha} &= -\gamma\frac{\partial w}{\partial\alpha} + I_0(\gamma)\,a_{55}\varphi; \\ u_{\beta} &= -\gamma\frac{\partial w}{\partial\beta} + I_0(\gamma)\,a_{44}\psi. \end{aligned}\right\} \tag{4.5}$$

For the normal displacements w, as before, we have (2.13).

From (2.32) - (2.34), we obtain for the principal design stresses

$$\sigma_{\alpha} = -\gamma\left(B_{11}\frac{\partial^2 w}{\partial\alpha^2} + B_{12}\frac{\partial^2 w}{\partial\beta^2}\right) + I_0(\gamma)\left(B_{11}a_{55}\frac{\partial\varphi}{\partial\alpha} + B_{12}a_{44}\frac{\partial\psi}{\partial\beta}\right); \tag{4.6}$$

$$\sigma_{\beta} = -\gamma\left(B_{22}\frac{\partial^2 w}{\partial\beta^2} + B_{12}\frac{\partial^2 w}{\partial\alpha^2}\right) + I_0(\gamma)\left(B_{22}a_{44}\frac{\partial\psi}{\partial\beta} + B_{12}a_{55}\frac{\partial\varphi}{\partial\alpha}\right); \tag{4.7}$$

$$\tau_{\alpha\beta} = -2\gamma B_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + I_0(\gamma)\,B_{66}\left(a_{55}\frac{\partial\varphi}{\partial\beta} + a_{44}\frac{\partial\psi}{\partial\beta}\right). \tag{4.8}$$

From (2.35) - (2.37) and (2.41), (2.42) we have for the internal forces

$$T_1 = T_2 = S = 0, \quad N_1 = \chi_1\varphi, \quad N_2 = \chi_1\psi. \tag{4.9}$$

From (2.38) - (2.40) we obtain for the bending and torsional moments

$$M_1 = -\left(D_{11}\frac{\partial^2 w}{\partial\alpha^2} + D_{12}\frac{\partial^2 w}{\partial\beta^2}\right) + \chi_2\left(B_{11}a_{55}\frac{\partial\varphi}{\partial\alpha} + B_{12}a_{44}\frac{\partial\psi}{\partial\beta}\right); \tag{4.10}$$

$$M_2 = -\left(D_{22}\frac{\partial^2 w}{\partial\beta^2} + D_{12}\frac{\partial^2 w}{\partial\alpha^2}\right) + \chi_2\left(B_{22}a_{44}\frac{\partial\psi}{\partial\beta} + B_{12}a_{55}\frac{\partial\varphi}{\partial\alpha}\right); \tag{4.11}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + \chi_2 B_{66}\left(a_{55}\frac{\partial\varphi}{\partial\beta} + a_{44}\frac{\partial\psi}{\partial\alpha}\right). \tag{4.12}$$

Finally, from (2.58) - (2.60), expanding the linear operators, we obtain the complete system of three differential equations in terms of the three sought functions w, φ, ψ:

$$\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\psi}{\partial\beta} = -\frac{Z_2}{\chi_1}; \tag{4.13}$$

$$\begin{aligned} D_{11}\frac{\partial^3 w}{\partial\alpha^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial\alpha\,\partial\beta^2} - \chi_2\Big[B_{11}a_{55}\frac{\partial^2\varphi}{\partial\alpha^2} + B_{66}a_{55}\frac{\partial^2\varphi}{\partial\beta^2} + {} \\ + (B_{12} + B_{66})\,a_{44}\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\Big] + \chi_1\varphi = 0; \end{aligned} \tag{4.14}$$

$$D_{22} \frac{\partial^3 w}{\partial \beta^3} + (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial \beta\, \partial \alpha^2} - \chi_2 \Big[B_{22} a_{44} \frac{\partial^2 \psi}{\partial \beta^2} + B_{66} a_{44} \frac{\partial^2 \psi}{\partial \alpha^2} + \\ + (B_{12} + B_{66}) a_{55} \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta} \Big] + \chi_1 \psi = 0. \tag{4.15}$$

Assuming

$$w = \Big\{ \Big(\frac{12}{h^3} \chi_2 \Big)^2 a_{44} a_{55} \Big[D_{11} D_{66} \frac{\partial^4}{\partial \alpha^4} + (D_{11} D_{22} - 2D_{12} D_{66} - D_{12}^2) \frac{\partial^4}{\partial \alpha^2\, \partial \beta^2} + \\ + D_{22} D_{66} \frac{\partial^4}{\partial \beta^4} \Big] - \frac{12}{h^3} \chi_1 \chi_2 \Big[(a_{55} D_{11} + a_{44} D_{66}) \frac{\partial^2}{\partial \alpha^2} + \\ + (a_{44} D_{22} + a_{55} D_{66}) \frac{\partial^2}{\partial \beta^2} \Big] + \chi_1^2 \Big\} \Phi; \tag{4.16}$$

$$\varphi = \Big\{ \frac{12}{h^3} \chi_2 a_{44} \Big[D_{11} D_{66} \frac{\partial^5}{\partial \alpha^5} + (D_{11} D_{22} - 2D_{12} D_{66} - D_{12}^2) \frac{\partial^5}{\partial \alpha^3\, \partial \beta^2} + \\ + D_{22} D_{66} \frac{\partial^5}{\partial \alpha\, \partial \beta^4} \Big] - \chi_1 \Big[D_{11} \frac{\partial^3}{\partial \alpha^3} + (D_{12} + 2D_{66}) \frac{\partial^3}{\partial \alpha\, \partial \beta^2} \Big] \Big\} \Phi; \tag{4.17}$$

$$\psi = \Big\{ \frac{12}{h^3} \chi_2 a_{55} \Big[D_{22} D_{66} \frac{\partial^5}{\partial \beta^5} + (D_{11} D_{22} - 2D_{12} D_{66} - D_{12}^2) \frac{\partial^5}{\partial \alpha^2\, \partial \beta^3} + \\ + D_{11} D_{66} \frac{\partial^5}{\partial \beta\, \partial \alpha^4} \Big] - \chi_1 \Big[D_{22} \frac{\partial^3}{\partial \beta^3} + (D_{12} + 2D_{66}) \frac{\partial^3}{\partial \beta\, \partial \alpha^2} \Big] \Big\} \Phi, \tag{4.18}$$

we identically satisfy equations (4.14) and (4.15) and from (4.13) we obtain the differential equation of solution of the problem in terms of function $\Phi = \Phi(\alpha, \beta)$:

$$\Big[D_{11} \frac{\partial^4}{\partial \alpha^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial \alpha^2\, \partial \beta^2} + D_{22} \frac{\partial^4}{\partial \beta^4} \Big] \Phi - \frac{12}{h^3} \frac{\chi_2}{\chi_1} \Big\{ a_{44} D_{11} D_{66} \frac{\partial^6}{\partial \alpha^6} + \\ + \big[a_{44} (D_{11} D_{22} - 2D_{12} D_{66} - D_{12}^2) + a_{55} D_{11} D_{66} \big] \frac{\partial^6}{\partial \alpha^4\, \partial \beta^2} + \\ + \big[a_{55} (D_{11} D_{22} - 2D_{12} D_{66} - D_{12}^2) + a_{44} D_{22} D_{66} \big] \frac{\partial^6}{\partial \beta^4\, \partial \alpha^2} + \\ + a_{55} D_{22} D_{66} \frac{\partial^6}{\partial \beta^6} \Big\} \Phi = \frac{Z_2}{\chi_1^2}. \tag{4.19}$$

Thus, the problem of transverse shear of an orthotropic plate has reduced to determining a certain potential function $\Phi(\alpha, \beta)$ which satisfies the sixth-order differential equation (4.19) and the boundary conditions presented in Section 2 of this chapter.

Generalizing the results obtained in this section for the more general case of anisotropy (that is, the case in which the material of the plate has at each point only one plane of elastic symmetry parallel to the median plane of the plate), we obtain:

for the tangential displacements

$$u_\alpha = -\gamma \frac{\partial w}{\partial \alpha} + f_0(\gamma)(a_{55} \varphi + a_{45} \psi); \tag{4.20}$$

$$u_\beta = -\gamma \frac{\partial w}{\partial \beta} + f_0(\gamma)(a_{44} \psi + a_{45} \varphi) \tag{4.21}$$

for the normal displacement w (2.13);

for the principal design stresses

$$\begin{aligned}\sigma_\alpha = & -\gamma\left(B_{11}\frac{\partial^2 w}{\partial\alpha^2}+B_{12}\frac{\partial^2 w}{\partial\beta^2}+2B_{16}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right)+\\ & +I_0(\gamma)\left[(B_{11}a_{55}+B_{16}a_{44})\frac{\partial\varphi}{\partial\alpha}+(B_{12}a_{45}+B_{16}a_{55})\frac{\partial\varphi}{\partial\beta}+\right.\\ & \left.+(B_{12}a_{44}+B_{16}a_{45})\frac{\partial\psi}{\partial\beta}+(B_{11}a_{45}+B_{16}a_{44})\frac{\partial\psi}{\partial\alpha}\right];\end{aligned} \tag{4.22}$$

$$\begin{aligned}\sigma_\beta = & -\gamma\left(B_{22}\frac{\partial^2 w}{\partial\beta^2}+B_{12}\frac{\partial^2 w}{\partial\alpha^2}+2B_{26}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right)+\\ & +I_0(\gamma)\left[(B_{22}a_{44}+B_{26}a_{45})\frac{\partial\psi}{\partial\beta}+(B_{12}a_{45}+B_{26}a_{44})\frac{\partial\psi}{\partial\alpha}+\right.\\ & \left.+(B_{12}a_{55}+B_{26}a_{45})\frac{\partial\varphi}{\partial\alpha}+(B_{22}a_{45}+B_{26}a_{55})\frac{\partial\varphi}{\partial\beta}\right];\end{aligned} \tag{4.23}$$

$$\begin{aligned}\tau_{\alpha\beta} = & -\gamma\left(B_{12}\frac{\partial^2 w}{\partial\alpha^2}+B_{26}\frac{\partial^2 w}{\partial\beta^2}+2B_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right)+\\ & +I_0(\gamma)\left[(B_{16}a_{55}+B_{66}a_{45})\frac{\partial\varphi}{\partial\alpha}+(B_{26}a_{45}+B_{66}a_{55})\frac{\partial\varphi}{\partial\beta}+\right.\\ & \left.+(B_{26}a_{44}+B_{66}a_{45})\frac{\partial\psi}{\partial\beta}+(B_{16}a_{45}+B_{66}a_{44})\frac{\partial\psi}{\partial\alpha}\right];\end{aligned} \tag{4.24}$$

for the internal forces and moments

$$N_1=\chi_1\varphi,\quad N_2=\chi_1\psi; \tag{4.25}$$

$$\begin{aligned}M_1 = & -\left(D_{11}\frac{\partial^2 w}{\partial\alpha^2}+D_{12}\frac{\partial^2 w}{\partial\beta^2}+2D_{16}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right)+\\ & +\frac{12}{h^3}\chi_2\left[(D_{11}a_{55}+D_{16}a_{45})\frac{\partial\varphi}{\partial\alpha}+(D_{12}a_{45}+D_{16}a_{55})\frac{\partial\varphi}{\partial\beta}+\right.\\ & \left.+(D_{12}a_{44}+D_{16}a_{45})\frac{\partial\psi}{\partial\beta}+(D_{11}a_{45}+D_{16}a_{44})\frac{\partial\psi}{\partial\alpha}\right];\end{aligned} \tag{4.26}$$

$$\begin{aligned}M_2 = & -\left(D_{22}\frac{\partial^2 w}{\partial\beta^2}+D_{12}\frac{\partial^2 w}{\partial\alpha^2}+2D_{26}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right)+\\ & +\frac{12}{h^3}\chi_2\left[(D_{22}a_{44}+D_{26}a_{45})\frac{\partial\psi}{\partial\beta}+(D_{12}a_{45}+D_{26}a_{44})\frac{\partial\psi}{\partial\alpha}+\right.\\ & \left.+(D_{12}a_{55}+D_{26}a_{45})\frac{\partial\varphi}{\partial\alpha}+(D_{22}a_{45}+D_{26}a_{55})\frac{\partial\varphi}{\partial\beta}\right];\end{aligned} \tag{4.27}$$

$$\begin{aligned}H = & -\left(D_{16}\frac{\partial^2 w}{\partial\alpha^2}+D_{26}\frac{\partial^2 w}{\partial\beta^2}+2D_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right)+\\ & +\frac{12}{h^3}\chi_2\left[(D_{16}a_{55}+D_{66}a_{45})\frac{\partial\varphi}{\partial\alpha}+(D_{26}a_{45}+D_{66}a_{55})\frac{\partial\varphi}{\partial\beta}+\right.\\ & \left.+(D_{26}a_{44}+D_{66}a_{45})\frac{\partial\psi}{\partial\beta}+(D_{16}a_{45}+D_{66}a_{44})\frac{\partial\psi}{\partial\alpha}\right].\end{aligned} \tag{4.28}$$

And, finally, we obtain the complete system of three differential equations in terms of the three unknown functions w, φ, ψ:

$$\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}=-\frac{Z_1}{\chi_1}; \tag{4.29}$$

$$\begin{aligned}
&D_{11}\frac{\partial^3 w}{\partial\alpha^3}+3D_{16}\frac{\partial^3 w}{\partial\alpha^2\,\partial\beta}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial\alpha\,\partial\beta^2}+\\
&+D_{26}\frac{\partial^3 w}{\partial\beta^3}-\frac{12}{h^3}\chi_1\Big\{(D_{11}a_{55}+D_{16}a_{45})\frac{\partial^2\varphi}{\partial\alpha^2}+[(D_{12}+D_{66})\,a_{45}+\\
&+2D_{16}a_{55}]\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}+(D_{26}a_{45}+D_{66}a_{55})\frac{\partial^2\varphi}{\partial\beta^2}+(D_{11}a_{45}+\\
&+D_{16}a_{44})\frac{\partial^2\psi}{\partial\alpha^2}+[(D_{12}+D_{66})\,a_{44}+2D_{16}a_{45}]\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}+\\
&\qquad\qquad+(D_{26}a_{44}+D_{66}a_{45})\frac{\partial^2\psi}{\partial\beta^2}\Big\}+\chi_1\varphi=0; \qquad (4.30)
\end{aligned}$$

$$\begin{aligned}
&D_{22}\frac{\partial^3 w}{\partial\beta^3}+3D_{26}\frac{\partial^3 w}{\partial\beta^2\,\partial\alpha}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial\beta\,\partial\alpha^2}+\\
&+D_{16}\frac{\partial^3 w}{\partial\alpha^3}-\frac{12}{h^3}\chi_2\Big\{(D_{22}a_{44}+D_{26}a_{45})\frac{\partial^2\psi}{\partial\beta^2}+[(D_{12}+D_{66})\,a_{45}+\\
&+2D_{26}a_{44}]\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}+(D_{16}a_{45}+D_{66}a_{44})\frac{\partial^2\psi}{\partial\alpha^2}+\\
&+(D_{22}a_{45}+D_{26}a_{55})\frac{\partial^2\varphi}{\partial\beta^2}+[(D_{12}+D_{66})\,a_{55}+2D_{26}a_{45}]\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}+\\
&\qquad\qquad+(D_{16}a_{55}+D_{66}a_{45})\frac{\partial^2\varphi}{\partial\alpha^2}\Big\}+\chi_1\psi=0. \qquad (4.31)
\end{aligned}$$

Here, as usual,

$$C_{ik}=hB_{ik};\qquad D_{ik}=\frac{h^3}{12}B_{ik};$$

$$B_{11}=\frac{a_{22}a_{66}-a_{26}^2}{\Omega};\qquad B_{22}=\frac{a_{11}a_{66}-a_{16}^2}{\Omega};$$

$$B_{12}=\frac{a_{16}a_{26}-a_{12}a_{66}}{\Omega};\qquad B_{66}=\frac{a_{11}a_{22}-a_{12}^2}{\Omega};$$

$$B_{16}=\frac{a_{12}a_{26}-a_{22}a_{16}}{\Omega};\qquad B_{26}=\frac{a_{12}a_{16}-a_{11}a_{26}}{\Omega};$$

$$\Omega=(a_{11}a_{22}-a_{12}^2)\,a_{66}+2a_{12}a_{16}a_{26}-a_{11}a_{26}^2-a_{22}a_{16}^2;$$

a_{ik} are the elastic constants.

We will note that the boundary conditions presented for the general theory in this case remain unchanged.

Proceeding from the basic premises of the theory described in this section, the reader may easily write out all the design formulas and equations of solution for the previously discussed special cases. For this purpose, in all formulas and equations presented in the previous sections of this chapter it is sufficient to formally assume $A_1 = 0$ and $A_2 = 0$.

SECTION 5. ANOTHER APPROXIMATE THEORY OF ANISOTROPIC PLATES

For the same orthotropic plate which was discussed in Sections 2 and 4 of this chapter let us assume:*.**,***,****

a) the distances along the normal (γ) between two points of a plate before and after deformation remain unchanged;

b) normal stresses σ_γ at areas parallel to the median plane may be disregarded in comparison with other stresses;

c) in determining deformations $e_{\alpha\gamma}$ and $e_{\beta\gamma}$ we assume that the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ do not differ from the corresponding stresses ($\tau^0_{\alpha\gamma}$ and $\tau^0_{\beta\gamma}$) as determined from the hypothesis of nondeformable normals, that is, they do not differ from the corresponding stresses in the classical theory of bending of anisotropic plates.

In virtue of these assumptions we obtain for the stresses*****

$$\tau_{\alpha\gamma} = \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right)\varphi_0; \quad \tau_{\beta\gamma} = \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right)\psi_0; \tag{5.1}$$

$$\tau_{\alpha\beta} = B_{66} e_{\alpha\beta}; \tag{5.2}$$

$$\sigma_\alpha = B_{11} e_\alpha + B_{12} e_\beta, \quad \sigma_\beta = B_{22} e_\beta + B_{12} e_\alpha, \tag{5.3}$$

where

$$\varphi_0 = -\left[B_{11}\frac{\partial^3 w_0}{\partial \alpha^3} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial \alpha\, \partial \beta^2}\right]; \tag{5.4}$$

$$\psi_0 = -\left[B_{22}\frac{\partial^3 w_0}{\partial \beta^3} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial \beta\, \partial \alpha^2}\right]; \tag{5.5}$$

---

*S.A. Ambartsumyan, Toward calculation of two-layer orthotropic shells, Izvestiya OTN AN SSSR, No. 7, 1957.

**S.A. Ambartsumyan, On two methods of calculation of two-layer orthotropic shells, Izvestiya AN ArmSSR (seriya fiz.-mat. nauk), Vol X, No. 2, 1957.

***S.A. Ambartsumyan, Toward the problem of a nonlinear theory of anisotropic plates, DAN ArmSSR, Vol. XXIV, No. 4, 1957.

****S.A. Ambartsumyan, D.V. Peshtmaldzhyan, On a nonlinear theory of shallow orthotropic shells, Izvestiya AN ArmSSR (seriya fiz.-mat. nauk), Vol, XI, No. 1, 1958.

*****S.G. Lekhnitskiy, see p. 248 in the book cited in parentheses on p. 328.

w_0 is the normal displacement of the corresponding orthotropic plate as determined from the hypothesis of nondeformable normals. Here, as in Section 4, it is assumed that $X^{\pm} = 0$ and $Y^{\pm} = 0$.

Proceeding in the same manner as in Sections 2 and 4 of this chapter, we obtain for the displacements, deformations and stresses

$$u_\alpha = -\gamma \frac{\partial w}{\partial \alpha} + I_0 a_{55}\varphi_0, \quad u_\beta = -\gamma \frac{\partial w}{\partial \beta} + I_0 a_{44}\psi_0; \tag{5.6}$$

for the normal displacement we have (2.13)

$$\left.\begin{aligned} e_\alpha &= -\gamma \frac{\partial^2 w}{\partial \alpha^2} + I_0 a_{55}\frac{\partial \varphi_0}{\partial \alpha}, \quad e_\beta = -\gamma \frac{\partial^2 w}{\partial \beta^2} + I_0 a_{44}\frac{\partial \psi_0}{\partial \beta}; \\ e_{\alpha\beta} &= -2\gamma \frac{\partial^2 w}{\partial \alpha\, \partial \beta} + I_0\left(a_{55}\frac{\partial \varphi_0}{\partial \beta} + a_{44}\frac{\partial \psi_0}{\partial \alpha}\right); \end{aligned}\right\} \tag{5.7}$$

$$\sigma_\alpha = -\gamma\left(B_{11}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}\frac{\partial^2 w}{\partial \beta^2}\right) + I_0\left(B_{11}a_{55}\frac{\partial \varphi_0}{\partial \alpha} + B_{12}a_{44}\frac{\partial \psi_0}{\partial \beta}\right); \tag{5.8}$$

$$\sigma_\beta = -\gamma\left(B_{22}\frac{\partial^2 w}{\partial \beta^2} + B_{12}\frac{\partial^2 w}{\partial \alpha^2}\right) + I_0\left(B_{22}a_{44}\frac{\partial \psi_0}{\partial \beta} + B_{12}a_{55}\frac{\partial \varphi_0}{\partial \alpha}\right); \tag{5.9}$$

$$\tau_{\alpha\beta} = -B_{66}\left[2\gamma \frac{\partial^2 w}{\partial \alpha\, \partial \beta} + I_0\left(a_{55}\frac{\partial \varphi_0}{\partial \beta} + a_{44}\frac{\partial \psi_0}{\partial \alpha}\right)\right], \tag{5.10}$$

where, as in (2.62), $I_0 = \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)$.

Moreover, in virtue of (2.38) - (2.42) and (5.1), (5.8) - (5.10) as well as (2.62), we obtain for the internal forces and moments

$$M_1 = -\left(D_{11}\frac{\partial^2 w}{\partial \alpha^2} + D_{12}\frac{\partial^2 w}{\partial \beta^2}\right) + \frac{h^2}{10}\left(D_{11}a_{55}\frac{\partial \varphi_0}{\partial \alpha} + D_{12}a_{44}\frac{\partial \psi_0}{\partial \beta}\right); \tag{5.11}$$

$$M_2 = -\left(D_{22}\frac{\partial^2 w}{\partial \beta^2} + D_{12}\frac{\partial^2 w}{\partial \alpha^2}\right) + \frac{h^2}{10}\left(D_{22}a_{44}\frac{\partial \psi_0}{\partial \beta} + D_{12}a_{55}\frac{\partial \varphi_0}{\partial \alpha}\right); \tag{5.12}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial \alpha\, \partial \beta} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi_0}{\partial \beta} + a_{44}\frac{\partial \psi_0}{\partial \alpha}\right); \tag{5.13}$$

$$N_1 = \frac{h^3}{12}\varphi_0, \quad N_2 = \frac{h^3}{12}\psi_0. \tag{5.14}$$

From equilibrium equations (2.46) - (2.48), eliminating the transverse forces N_1 and N_2, we obtain

$$\frac{\partial^2 M_1}{\partial \alpha^2} + 2\frac{\partial^2 H}{\partial \alpha\, \partial \beta} + \frac{\partial^2 M_2}{\partial \beta^2} = -Z_2. \tag{5.15}$$

Inserting the values of moments from (5.11) - (5.13) into (5.15), we obtain one differential equation of solution in terms of one unknown function w:

$$D_{11}\frac{\partial^4 w}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}=Z_2+$$

$$+\frac{h^2}{10}\left\{a_{55}\left[D_{11}\frac{\partial^3}{\partial\alpha^3}+(D_{12}+2D_{66})\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right]\varphi_0+\right.$$

$$\left.+a_{44}\left[D_{22}\frac{\partial^3}{\partial\beta^3}+(D_{12}+2D_{66})\frac{\partial^3}{\partial\beta\,\partial\alpha^2}\right]\psi_0\right\}. \qquad (5.16)$$

Finally, from (5.4) and (5.5), inserting the values of φ_0 and ψ_0 into (5.16), we write the equation of solution in slightly different form:

$$D_{11}\frac{\partial^4 w}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}=$$

$$=Z_2-\frac{h^5}{120}\left\{a_{55}\frac{\partial^2}{\partial\alpha^2}\left[B_{11}^2\frac{\partial^4 w_0}{\partial\alpha^4}+2B_{11}(B_{12}+2B_{66})\frac{\partial^4 w_0}{\partial\alpha^2\partial\beta^2}+\right.\right.$$

$$\left.+(B_{12}+2B_{66})^2\frac{\partial^4 w_0}{\partial\beta^4}\right]+a_{44}\frac{\partial^2}{\partial\beta^2}\left[B_{22}^2\frac{\partial^4 w_0}{\partial\beta^4}+\right.$$

$$\left.\left.+2B_{22}(B_{12}+2B_{66})\frac{\partial^4 w_0}{\partial\beta^2\,\partial\alpha^2}+(B_{12}+2B_{66})^2\frac{\partial^4 w_0}{\partial\alpha^4}\right]\right\}. \qquad (5.17)$$

Thus, in this variant of the theory the problem of transverse bending of an orthotropic plate has been reduced to the determination of only function $w(\alpha,\beta)$ (function $w_0(\alpha,\beta)$ and functions $\varphi_0(\alpha,\beta)$ and $\psi_0(\alpha,\beta)$ are thereby known from the classical theory of bending of an anisotropic plate). This function satisfies the fourth-order differential equation (5.17) or (5.16), the left numbers of which do not differ from the left member of the well-known equation in the classical theory of bending of an orthotropic plate,* and satisfies the boundary conditions, which also do not differ from the boundary conditions of the classical theory. As an example let us present the homogeneous boundary conditions for four principal cases* (see Chapter II, Section 10):

a) free edge

$$M_n=0,\quad N_n+\frac{\partial H_{nt}}{\partial s}=0;$$

b) hinge-fastened edge

$$w=0,\quad M_n=0;$$

c) fixed edge (fixed element of median surface)

$$w=0,\quad \frac{\partial w}{\partial n}=0;$$

---

*S.G. Lekhnitskiy, see p. 248 of work cited in footnote on p. 328.

d) fixed edge (fixed vertical element of edge)

$$w = 0, \quad \left(\frac{\partial u_{nt}}{\partial \gamma}\right)_{\gamma=0} = 0.$$

In the case where the plate is cylindrically anisotropic (see Section 3) the theory proposed herein gives us the following design formulas and equations:

principal design stresses

$$\sigma_r = -\gamma\left[B_r \frac{\partial^2 w}{\partial r^2} + \nu_\theta B_r\left(\frac{\partial^2 w}{r^2 \partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right] + \\ + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left[B_r a_r\left(\frac{\partial\varphi_0}{\partial r} + \nu_\theta \frac{\varphi_0}{r}\right) + \nu_\theta B_r a_\theta \frac{\partial\psi_0}{r\,\partial\theta}\right]; \qquad (5.18)$$

$$\sigma_\theta = -\gamma\left[B_\theta\left(\frac{\partial^2 w}{r^2 \partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \nu_r B_\theta \frac{\partial^2 w}{\partial r^2}\right] + \\ + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left[B_\theta a_\theta \frac{\partial\psi_0}{r\,\partial\theta} + \nu_r B_\theta a_r\left(\frac{\varphi_0}{r} + \nu_r \frac{\partial\varphi_0}{\partial r}\right)\right]; \qquad (5.19)$$

$$\tau_{\theta r} = -\gamma 2B_k \frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial\theta}\right) + \frac{1}{2}\left(\gamma\frac{h^2}{4} - \frac{\gamma^3}{3}\right)\left[a_r \frac{\partial\varphi_0}{r\,\partial\theta} + a_\theta\left(\frac{\partial\psi_0}{\partial r} - \frac{\psi_0}{r}\right)\right]; \qquad (5.20)$$

internal bending and torsional moments

$$M_r = -D_r \frac{\partial^2 w}{\partial r^2} - \nu_\theta D_r \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial w}{\partial r}\right) + \\ + \frac{h^2}{10}\left[D_r a_r\left(\frac{\partial\varphi_0}{\partial r} + \nu_\theta \frac{\varphi_0}{r}\right) + \nu_\theta D_r a_\theta \frac{1}{r}\frac{\partial\psi_0}{\partial\theta}\right]; \qquad (5.21)$$

$$M_\theta = -D_\theta \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial w}{\partial r}\right) - \nu_r D_\theta \frac{\partial^2 w}{\partial r^2} + \\ + \frac{h^2}{10}\left[D_\theta a_\theta \frac{1}{r}\frac{\partial\psi_0}{\partial\theta} + D_\theta a_r\left(\frac{\varphi_0}{r} + \nu_r \frac{\partial\varphi_0}{\partial r}\right)\right]; \qquad (5.22)$$

$$H = D_k\left\{-2\frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial\theta}\right) + \frac{h^2}{10}\left[a_r \frac{\partial\varphi_0}{r\,\partial\theta} + a_\theta\left(\frac{\partial\psi_0}{\partial r} - \frac{\psi_0}{r}\right)\right]\right\} \qquad (5.23)$$

and, finally, the equation of solution of the problem

$$D_r \frac{\partial^4 w}{\partial r^4} + 2D_{r\theta}\frac{1}{r^2}\frac{\partial^4 w}{\partial r^2 \partial\theta^2} + D_\theta \frac{1}{r^4}\frac{\partial^4 w}{\partial\theta^4} + \\ + 2D_r \frac{1}{r}\frac{\partial^3 w}{\partial r^3} - 2D_{r\theta}\frac{1}{r^3}\frac{\partial^3 w}{\partial r\,\partial\theta^2} - D_\theta \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \\ + 2(D_\theta + D_{r\theta})\frac{1}{r^4}\frac{\partial^2 w}{\partial\theta^2} + D_\theta \frac{1}{r^3}\frac{\partial w}{\partial r} = \\ = Z_2 + \frac{h^2}{10}\left\{a_r\left[D_r\left(\frac{2}{r}\frac{\partial^2\varphi_0}{\partial r^2} + \frac{\partial^3\varphi_0}{\partial r^3}\right) + D_{r\theta}\frac{1}{r^2}\frac{\partial^3\varphi_0}{\partial r\,\partial\theta^2} + \right.\right. \\ + D_\theta\left(\frac{1}{r^3}\frac{\partial^2\varphi_0}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial\varphi_0}{\partial r} + \frac{\varphi_0}{r^3}\right)\Big] + a_\theta\Big[D_{r\theta}\frac{1}{r}\frac{\partial^3\psi_0}{\partial r^2 \partial\theta} + \\ \left.\left. + D_\theta\left(\frac{1}{r^3}\frac{\partial^3\psi_0}{\partial\theta^3} - \frac{1}{r^2}\frac{\partial^2\psi_0}{\partial r\,\partial\theta} + \frac{1}{r^3}\frac{\partial\psi_0}{\partial\theta}\right)\right]\right\}. \qquad (5.24)$$

In all these formulas and equations for functions φ_0 amd ψ_0 we have*

$$\varphi_0 = -\left[B_r\left(\frac{\partial^3 w_0}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w_0}{\partial r^2}\right) + B_{\theta r}\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial w_0}{\partial r} - \frac{w_0}{r}\right) - \right.$$
$$\left. - B_\theta\frac{1}{r^2}\left(\frac{\partial w_0}{\partial r} + \frac{1}{r}\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]; \tag{5.25}$$

$$\psi_0 = -\left[B_{r\theta}\frac{1}{r}\frac{\partial^3 w_0}{\partial r^2\,\partial\theta} + B_\theta\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial w_0}{\partial r} + \frac{1}{r}\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]. \tag{5.26}$$

This theory may be refined on the basis of the results obtained in Section 2 of this chapter. The fact is that without introducing any significant complication we may omit the assumption of negligibility of σ_γ in comparison with other stresses; then from (2.80) - (2.82) and (5.8) - (5.10) we obtain for the design stresses

$$\sigma_\alpha = -\gamma\left(B_{11}\frac{\partial^2 w}{\partial\alpha^2} + B_{12}\frac{\partial^2 w}{\partial\beta^2}\right) + I_0\left(B_{11}a_{55}\frac{\partial\varphi_0}{\partial\alpha} + B_{12}a_{44}\frac{\partial\psi_0}{\partial\beta}\right) -$$
$$- A_1\left[Z_1 + \left(\frac{3}{2}\frac{\gamma}{h} - 2\frac{\gamma^3}{h^3}\right)Z_2\right]; \tag{5.27}$$

$$\sigma_\beta = -\gamma\left(B_{22}\frac{\partial^2 w}{\partial\beta^2} + B_{12}\frac{\partial^2 w}{\partial\alpha^2}\right) + I_0\left(B_{22}a_{44}\frac{\partial\psi_0}{\partial\beta} + B_{12}a_{55}\frac{\partial\varphi_0}{\partial\alpha}\right) -$$
$$- A_2\left[Z_1 + \left(\frac{3}{2}\frac{\gamma}{h} - 2\frac{\gamma^3}{h^3}\right)Z_2\right]; \tag{5.28}$$

$$\tau_{\alpha\beta} = -B_{66}\left[2\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + I_0\left(a_{55}\frac{\partial\varphi_0}{\partial\beta} + a_{44}\frac{\partial\psi_0}{\partial\alpha}\right)\right]. \tag{5.29}$$

From (2.76) - (2.79) and (5.11) - (5.14) we have for the moments and transverse forces

$$M_1 = -\left(D_{11}\frac{\partial^2 w}{\partial\alpha^2} + D_{12}\frac{\partial^2 w}{\partial\beta^2}\right) +$$
$$+ \frac{h^2}{10}\left(D_{11}a_{55}\frac{\partial\varphi_0}{\partial\alpha} + D_{12}a_{44}\frac{\partial\psi_0}{\partial\beta}\right) - \frac{h^2}{10}A_1Z_2; \tag{5.30}$$

$$M_2 = -\left(D_{22}\frac{\partial^2 w}{\partial\beta^2} + D_{12}\frac{\partial^2 w}{\partial\alpha^2}\right) +$$
$$+ \frac{h^2}{10}\left(D_{22}a_{44}\frac{\partial\psi_0}{\partial\beta} + D_{12}a_{55}\frac{\partial\varphi_0}{\partial\alpha}\right) - \frac{h^2}{10}A_2Z_2, \tag{5.31}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial\varphi_0}{\partial\beta} + a_{44}\frac{\partial\psi_0}{\partial\alpha}\right); \tag{5.32}$$

$$N_1 = \frac{h^3}{12}\varphi_0, \quad N_2 = \frac{h^3}{12}\psi_0 \tag{5.33}$$

---

*S.G. Lekhnitskiy, see pp. 277-278 of work cited in footnote on p. 328.

and finally, from (5.15) and by virtue of (5.30) - (5.32) we obtain the differential equation of solution of the problem*

$$
\begin{aligned}
&D_1\frac{\partial^4 w}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}=\\
&=Z_2+\frac{h^2}{10}\left\{a_{55}\left[D_{11}\frac{\partial^3\varphi_0}{\partial\alpha^3}+(D_{12}+2D_{66})\frac{\partial^3\varphi_0}{\partial\alpha\,\partial\beta^2}\right]+a_{44}\left[D_{22}\frac{\partial^3\psi_0}{\partial\beta^3}+\right.\right.\\
&\qquad\left.\left.+(D_{12}+2D_{66})\frac{\partial^3\psi_0}{\partial\beta\,\partial\alpha^2}\right]\right\}-\frac{h^2}{10}\left(A_1\frac{\partial^2 Z_2}{\partial\alpha^2}+A_2\frac{\partial^2 Z_2}{\partial\beta^2}\right).
\end{aligned}
\tag{5.34}
$$

Examining the formulas for design quantities (5.27) - (5.33) and the equation of solution (5.34) of the refined variant of the theory, we note that they differ from the corresponding formulas (5.8) - (5.14) and equation of solution (5.16) of the unrefined theory only in the terms which contain factors A_1 and A_2.

SECTION 6 EXAMPLES OF PLATE CALCULATIONS

Without impairing the general validity of the calculation procedure, we shall dwell chiefly on transversely isotropic plates. In all examples it is assumed that at each point of the plate the plane of isotropy is parallel to the median plane of the plate. In this connection, for the elastic constants of the plate material we have (Chapter II, Section 14)

$$
\left.
\begin{aligned}
&B_{11}=B_{22}=B_{12}+2B_{66}=\frac{E}{1-\nu^2}=\bar{E},\quad B_{12}=\nu\bar{E},\\
&B_{66}=\frac{1-\nu}{2}\bar{E},\quad B_{12}+B_{66}=\frac{1+\nu}{2}\bar{E},\\
&a_{55}=a_{44}=\frac{1}{G'},\quad a_{13}=-\frac{\nu_{13}}{E_3}=-\frac{\nu'}{E'},\\
&A_1=A_2=a_{13}(B_{11}+B_{12})=-\frac{\bar{E}}{E'}\nu'(1+\nu),
\end{aligned}
\right\}
\tag{6.1}
$$

where E is the elastic modulus for directions in the plane of isotropy; E' is the elastic modulus for directions perpendicular to the median plane; ν is the Poisson

---

*In the special case of an isotropic plate, in using equation (2.46), the equation of solution (5.34) takes the form

$$
\frac{Eh^3}{12(1-\nu^2)}\Delta\Delta w=Z-\frac{h^2}{10}\frac{2-\nu(1-\nu^2)}{(1-\nu)}\Delta Z,\quad \Delta=\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2};
$$

this equation almost coincides with the corresponding equation derived by S.A. Alekseyev for bending of a thick plate. (See S.A. Alekseyev, Bending of thick plates, Trudy VVIA im. N.Ye. Zhukovskogo, No. 312, 1949).

ratio in the plane of isotropy; ν' is the Poisson ratio characterizing the contraction in the plane of isotropy in the presence of elongation in direction γ; G' is the shear modulus characterizing distortion of the angles between directions in the plane of isotropy and direction γ.

A. Rectangular Plate. By virtue of (6.1) the system of equations in (2.70) - (2.72) in the case of a transversally isotropic plate is rewritten as:

$$\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}=-\frac{12}{h^3}Z; \tag{6.2}$$

$$\frac{h^3}{12}\bar{E}\left(\frac{\partial^3 w}{\partial\alpha^3}+\frac{\partial^3 w}{\partial\alpha\,\partial\beta^2}\right)-\frac{h^5}{120}\frac{\bar{E}}{G'}\left(\frac{\partial^2\varphi}{\partial\alpha^2}+\frac{1-\nu}{2}\frac{\partial^2\varphi}{\partial\beta^2}+\frac{1+\nu}{2}\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\right)+\frac{h^3}{12}\varphi=$$
$$=\frac{\bar{E}}{E'}\nu'(1+\nu)\frac{h^2}{10}\frac{\partial Z}{\partial\alpha}; \tag{6.3}$$

$$\frac{h^3}{12}\bar{E}\left(\frac{\partial^3 w}{\partial\beta^3}+\frac{\partial^3 w}{\partial\beta\,\partial\alpha^2}\right)-\frac{h^5}{120}\frac{\bar{E}}{G'}\left(\frac{\partial^2\psi}{\partial\beta^2}+\frac{1-\nu}{2}\frac{\partial^2\psi}{\partial\alpha^2}+\frac{1+\nu}{2}\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}\right)+\frac{h^3}{12}\psi=$$
$$=\frac{\bar{E}}{E'}\nu'(1+\nu)\frac{h^2}{10}\frac{\partial Z}{\partial\beta}. \tag{6.4}$$

In these equations it is assumed that $Z^+ = Z$, $Z^- = 0$.

Here we have presented those equations which describe transverse bending of the plate, since it is our purpose to compare only the normal displacements of the plate as determined by means of various theories; the complete system of equations in (2.68) and (2.69) deals with the plane problem of the plate and is of independent interest.

1. Let a rectangular plate ($a \times b$) be hinge-supported over its entire contour and sustain a load with the distribution described by (Figure 78)

$$Z=q\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \tag{6.5}$$

where q is the load intensity at the center ($\alpha = a/2$, $\beta = b/2$) of the plate. Assuming

$$\varphi=B\cos\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b},\quad \psi=C\sin\frac{\pi\alpha}{a}\cos\frac{\pi\beta}{b},$$
$$w=A\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \tag{6.6}$$

we satisfy the conditions for hinge support and from the system of equations (6.2) - (6.4), in virtue of (6.5) and (6.6), for coefficient A we obtain

$$A=\frac{q\left[1+\frac{\pi^2}{10}h^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{\bar{E}}{G'}-\nu'(1+\nu)\frac{\bar{E}}{E'}\right)\right]}{\pi^4\frac{h^3}{12}\bar{E}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}; \tag{6.7}$$

then for flexure of the plate center we obtain

$$w = w_0\left\{1 + \frac{\pi^2}{10}h^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left[\frac{\overline{E}}{G'} - \nu'(1+\nu)\frac{\overline{E}}{E'}\right]\right\}, \tag{6.8}$$

where $w_0 = \frac{12a^4b^4q}{\pi^4h^3\overline{E}(a^2+b^2)^2}$ is the flexure of the plate center as determined by means of the classical theory of plates.*

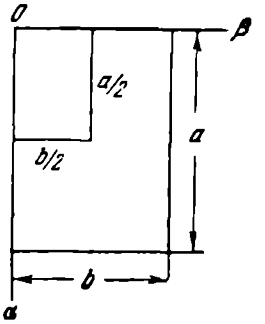

Figure 78

In the special case of a square plate (a = b) we have for the flexure

$$w = w_0\left\{1 + \frac{\pi^2}{5}\frac{h^2}{a^2}\left[\frac{\overline{E}}{G'} - \nu'(1+\nu)\frac{\overline{E}}{E'}\right]\right\}, \tag{6.9}$$

where $w_0 = 3a^4q/\pi^4h^3\overline{E}$.

Let us solve the above example of a rectangular, transversally isotropic plate by means of the theory described in Section 4 of this chapter. For this purpose we should write out the system of equations in (4.13) - (4.15), but we omit this since the resulting system differs from the system in (6.2) - (6.4) only in the right members of equations (6.3) and (6.4), which in this case are equal to zero, and equations (6.3) and (6.4) of system (6.2) - (6.4), disregarding the influence of normal stress σ_γ, become homogeneous. Formally speaking, the terms which contain factor ν' vanish. Then it is evident that for the normal displacement of the center of a square plate we obtain

$$w = w_0\left(1 + \frac{\pi^2}{5}\frac{h^2}{a^2}\frac{\overline{E}}{G'}\right). \tag{6.10}$$

---

*S.P. Timoshenko see p. 116 of work cited in footnote * on p. 323.

The same problem for a rectangular, transversally isotropic plate may also be solved by means of the theories discussed in Section 5 of this chapter. From (5.34), in virtue of (2.46), (5.33) and (6.1), for a transversely isotropic plate we obtain the following equation of solution:

$$\frac{h^3}{12}\bar{E}\Delta\Delta w = Z - \frac{h^2}{10}\left[\frac{\bar{E}}{G'} - \nu'(1+\nu)\frac{\bar{E}}{E'}\right]\Delta Z, \tag{6.11}$$

where

$$\Delta = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}. \tag{6.12}$$

Assuming

$$w = A\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \tag{6.13}$$

we satisfy the conditions for hinge support (for this theory) and from equation (6.11), in virtue of (6.5) and (6.13), we obtain for coefficient A precisely the same expression as before, that is, (6.7). Then for the normal displacement of a square plate as before, we have (6.9). In the case where we disregard the influence of σ_γ, that is, we use equation (5.17), which in the case of a transversely isotropic plate has the following form:

$$\frac{h^3}{12}\bar{E}\Delta\Delta w = Z - \frac{h^2}{10}\frac{\bar{E}}{G'}\Delta Z, \tag{6.14}$$

for the normal displacement we obtain the previous formula (6.10). Hence we conclude that the obviously simple theory discussed in Section 5, taking into account the effect of transverse shear and normal stresses σ_γ, may provide sufficiently accurate results.

Examining design formulas (6.9) and (6.10), we easily see that at certain values of ratios $\bar{E}/G'$, $\bar{E}/E'$ and h/a the results of calculation of normal displacements according to the classical theory of plates may differ somewhat from the corresponding values of normal displacements calculated from the proposed theories.

The results of calculation of the maximum normal displacement with different values of ratios E/G', E/E' and with two variants of ratio h/a (h/a = 1/5, 1/10) are listed in the following tables. In these calculations it was also assumed that $\nu = 0$, $\nu' = 0.25$.

E/E' \ E/G'	0,0	2,0	5,0	10,0
0,0	1,00	1,158	1,395	1,789
1,0	0,980	1,138	1,375	1,770
5,0	0,901	1,059	1,296	1,691
10,0	0,803	0,961	1,197	1,592

$h/a = \frac{1}{5}$

E/E' \ E/G'	0,0	2,0	5,0	10,0
0,0	1,00	1,039	1,099	1,198
1,0	0,995	1,035	1,094	1,193
5,0	0,975	1,015	1,074	1,173
10,0	0,951	0,990	1,049	1,148

$h/a = \frac{1}{10}$

The factor for the data listed in the tables is the corresponding value of w_0, that is, the maximum flexure as determined from the classical theory. In the first line of each table we list the results of calculation from formula (6.10), that is, for the case where the influence of normal stress σ_r is disregarded and in the remaining lines of the tables we list the results of calculation from formula (6.9).

It is easily seen from the tables that even for a somewhat thin plate (h/a = 1/5) the classical theory (E/E' = 0, E/G' = 0) may yield substantial error at large values of E/G'. It is obvious that the error in the classical theory is decreased with a decrease in the derived relative thickness of the plate (h*), which in the case of a transversely isotropic plate has the form

$$h^* = \frac{h^2}{a^2}\left[\frac{\bar{E}}{G'} - \nu'(1+\nu)\frac{\bar{E}}{E'}\right]. \qquad (6.15)$$

We will note that the derived relative thickness of the plate depends to a great extent on the geometric and mechanical (elastic) characteristics of the plate. Finally, comparing formulas (6.9) and (6.10), we easily see that the correction introduced by normal stress σ_r, is small in the case of an isotropic plate. For example, when h/a = 1/5, $\nu = 0.3$ this correction has a value of approximately 3%, whereas the correction with consideration of the influence of transverse shear is considerable and is as high as 20%.

2. In certain cases of boundary conditions and loading the theories proposed herein may also be used for calculation of relatively thick plates. By way of illustration we list the values of maximum flexure of a square (a × a) isotropic plate hinge-supported over its entire contour (h/a = 1/3, $\nu = 0.3$) as calculated from the theories given herein, that is, from formulas (6.9) and (6.10) and from the precise theory.* As before, it is assumed that the plate sustains a load which over the surface of the plate ($\gamma = h/2$) has a distribution described by (6.5). The results of the calculations are listed below:

	Precise theory*	From formula (6.9)	From formula (6.10)	Classical theory of plates***
$w\frac{E}{qh}$	3,492	3,498	3,692	2,270

In examining this table it is easily seen that even in the case of such a thick plate (h/a = 1/3) the theories proposed herein result in insignificant errors. Thus, the theory described in Section 2 (formula 337 (6.9)) yields an error of only 0.18 percent; the theory described in Section 4 (formula (6.10)) yields an error of 5.73 percent. As was to be expected, the classical theory of thin plates here gives an error as high as 35 percent.

3. In discussing the presented problems we have assumed that

$$f^1(\gamma) = f_1(\gamma) = f_2(\gamma) = \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right), \tag{6.16}$$

by virtue of which we have (2.62), that is

$$f_0(\gamma) = \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right), \quad \chi_1 = \frac{h^3}{12}, \quad \chi_2 = \frac{h^5}{120}.$$

However, as was stated above, the behavior of f(γ) as given in (6.16) in the general case only approximately describes the actual conditions. In this connection there naturally arises the question of what results will be obtained from the theories proposed herein (in addition to the theories described in Section 5) if f (γ) takes other values. In order to answer this let us discuss two more variants of f(γ).

---

*B.F. Vlasov, see footnote *** on p. 326.

**S.P. Timoshenko, see p. 116 of work cited in footnote * on p. 325.

The second variant (assuming (6.16) to have been the first variant):

$$f^{\mathrm{II}}(\gamma)=\frac{1}{2}\left(\frac{h^4}{16}-\gamma^4\right), \tag{6.17}$$

by virtue of which, from (2.21), (2.43) and (2.45), we have

$$f_0(\gamma)=\frac{\gamma}{2}\left(\frac{h^4}{16}-\frac{\gamma^4}{5}\right) \quad \chi_1=\frac{h^5}{40}, \quad \chi_2=\frac{h^7}{420}.$$

The third variant is not quite natural, but in the given case it is required as a limiting case:

$$f^{\mathrm{III}}(\gamma)=\frac{1}{2}, \tag{6.18}$$

by virtue of which, as in the second variant,

$$f_0(\gamma)=\frac{\gamma}{2}, \quad \chi_1=\frac{h}{2}, \quad \chi_2=\frac{h^3}{24}.$$

For a comparison of the results obtained in the examination of these three variants of f(γ) — (6.16), (6.17) and (6.18) — let us turn to the previous problem of a square (a × a) transversely isotropic plate freely supported over its entire contour and loaded by forces (6.5). For brevity of notation and without disturbing the general validity, let us examine the theory in which the influence of normal stress σ_γ is disregarded, that is, the theory described in

Section 4 of this chapter.

Proceeding as before, for the normal displacement of the plate in the three discussed variants of f(γ) we obtain (in the case of the first variant the value of w is rewritten from (6.10))

$$\left.\begin{aligned} w^{\mathrm{I}} &= w_0\left(1+\frac{1}{5}h^*\right); \\ w^{\mathrm{II}} &= w_0\left(1+\frac{4}{21}h^*\right); \\ w^{\mathrm{III}} &= w_0\left(1+\frac{1}{6}h^*\right), \end{aligned}\right\} \tag{6.19}$$

where for the derived thickness we have

$$h^*=\frac{\pi^2 E}{G'(1-\nu^2)}\frac{h^2}{a^2}. \tag{6.20}$$

The results of calculation of ratio w^i/w_0 for different values of h* are listed in tabular form:

h^*	0,5	1,0	2,0	5,0	10,0
w^{I}/w_0	1,100	1,200	1,400	2,000	3,000
$w^{II}\ w_0$	1,095	1,191	1,381	1,952	2,905
w^{III}/w_0	1,083	1,167	1,333	1,833	2,667

In examining the table we see that all three variants provide almost identical values for flexure. Even in cases where the values of flexure from the classical theory differ from the values according to the proposed theory by a factor of 2 or 3, the discussed natural variants of f(γ), (6.16) and (6.17), differ from the limiting case (6.18) by only 8.0-10.0 percent. Variants (6.16) and (6.17) differ from another by only 2.5-3.0 percent.

Examination of the other design quantities of a plate* reveals that the same conditions pertain. Thus, we conclude that in the proposed theories a logical choice of function f(γ) will not be subject to fortuitous introduction of inadmissible errors. We are of the opinion that function f(γ) is more naturally chosen in the form (6.16).

B. Circular Plate. We shall assume that the origin of coordinates (α, β, γ) coincides with the center of the circular plate. The plate is loaded by a uniformly distributed load of intensity q
by a uniformly distributed load of intensity q ($Z^+ = q$, $Z^- = 0$) (Figure 79).

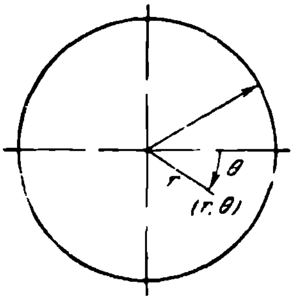

Figure 79

By virtue of (6.1), (3.16) - (3.19) the system of equations in (3.11) - (3.15) in the case of a transversely isotropic plate is rewritten as:

---

*S.A. Ambartsumyan, see footnote * on p. 315.

$$\left.\begin{aligned} & r\frac{d^2u}{dr^2}+\frac{du}{dr}-\frac{u}{r}=0; \quad \frac{d}{dr}(r\varphi)=-\frac{12}{h^3}rq; \\ & D\left[\frac{d}{dr}\left(r\frac{d^2w}{dr^2}\right)-\frac{1}{r}\frac{dw}{dr}\right]+\frac{h^3}{12}r\varphi- \\ & \qquad -\frac{h^2}{10}\frac{D}{G'}\left[r\frac{d^2\varphi}{dr^2}+\frac{d\varphi}{dr}-\frac{\varphi}{r}\right]=0, \end{aligned}\right\} \tag{6.21}$$

where $D=D_r=D_\theta=\frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical stiffness.

From (3.4) θ (3.10) we obtain for the internal forces

$$T_r=\bar{E}h\frac{du}{dr}+\nu\bar{E}h\frac{u}{r}-hA\frac{q}{2}; \tag{6.22}$$

$$T_\theta=\bar{E}h\frac{u}{r}+\nu\bar{E}h\frac{du}{dr}-hA\frac{q}{2}; \tag{6.23}$$

$$\begin{aligned} M_r=&-\frac{\bar{E}h^3}{12}\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)-A\frac{h^2}{10}q+ \\ &+\frac{h^2}{10}\frac{\bar{E}h^3}{12G'}\left(\frac{d\varphi}{dr}+\nu\frac{\varphi}{r}\right); \end{aligned} \tag{6.24}$$

$$\begin{aligned} M_\theta=&-\frac{\bar{E}h^3}{12}\left(\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}\right)-A\frac{h^2}{10}q+ \\ &+\frac{h^2}{10}\frac{\bar{E}h^3}{12G'}\left(\frac{\varphi}{r}+\nu\frac{d\varphi}{dr}\right). \end{aligned} \tag{6.25}$$

From (3.1) and (3.2) we obtain for the stresses

$$\begin{aligned} \sigma_r=&\bar{E}\left[\frac{du}{dr}-\gamma\frac{d^2w}{dr^2}+\frac{1}{2G'}\left(\gamma\frac{h^2}{4}-\frac{\gamma^3}{3}\right)\frac{d\varphi}{dr}\right]+ \\ &+\nu\bar{E}\left[\frac{u}{r}-\frac{\gamma}{r}\frac{dw}{dr}+\frac{1}{2G'}\left(\gamma\frac{h^2}{4}-\frac{\gamma^3}{3}\right)\frac{\varphi}{r}\right]- \\ &-A\left(\frac{1}{2}+\frac{3\gamma}{2h}-2\frac{\gamma^3}{h^3}\right)q; \end{aligned} \tag{6.26}$$

$$\begin{aligned} \sigma_\theta=&\bar{E}\left[\frac{u}{r}-\frac{\gamma}{r}\frac{dw}{dr}+\frac{1}{2G'}\left(\gamma\frac{h^2}{4}-\frac{\gamma^3}{3}\right)\frac{\varphi}{r}\right]+ \\ &+\nu\bar{E}\left[\frac{du}{dr}-\gamma\frac{d^2w}{dr^2}+\frac{1}{2G'}\left(\gamma\frac{h^2}{4}-\frac{\gamma^3}{3}\right)\frac{d\varphi}{dr}\right]- \\ &-A\left(\frac{1}{2}+\frac{3\gamma}{2h}-2\frac{\gamma^3}{h^3}\right)q. \end{aligned} \tag{6.27}$$

In these formulas, as in (6.1), $A=A_1=A_2=-\frac{E}{E'}\nu'(1+\nu)$. .

From the second and third equations in system (6.21) we easily obtain

$$\varphi=-\frac{12}{h^3}\frac{q}{2}r+\frac{C}{r}, \tag{6.28}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=\frac{qr}{2D}. \tag{6.29}$$

C is an integration constant which is equal to zero from the condition of the finiteness of function φ at the center of the plate. Equation (6.29) does not differ from the well-known equation of classical theory for transverse bending of a circular, uniformly loaded plate.* Equation (6.29) was derived from the fact that q = const. Thus, the discussed problem of a circular plate essentially reduces to two independent differential equations:

$$r\frac{d^2u}{dr^2}+\frac{du}{dr}-\frac{u}{r}=0; \tag{6.30}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=\frac{qr}{2D}. \tag{6.31}$$

In order to determine the normal displacement w let us use equation (6.31). By successive integration of equation (6.31) we obtain

$$w=\frac{qr^4}{64D}+C_1\frac{r^2}{4}+C_2\ln r+C_3. \tag{6.32}$$

From the condition that the flexure of the plate center is finite it must be assumed that $C_2 = 0$. The remaining integration constants are determined from the conditions of fixing of the plate edge. Let us discuss three variants of fixing.

1. Freely Supported Edge. In this case the following boundary conditions must be fulfilled:

$$\text{for } r=a \quad w=0, \quad M_r=0. \tag{6.33}$$

On the basis of (6.32), (6.28), (6.24) and satisfying conditions (6.43), we obtain for the normal displacement of the plate

$$w=\frac{qa^4}{64D}\left[\frac{r^4}{a^4}-2\frac{3+\nu}{1+\nu}\frac{r^2}{a^2}+\frac{5+\nu}{1+\nu}\right]+ \\ +\frac{qh^2}{20D}\left(\frac{\bar{E}}{2G'}+A\right)(a^2-r^2). \tag{6.34}$$

Hence for the flexure of the plate center we obtain

$$w=\frac{qa^4}{64D}\,\frac{5+\nu}{1+\nu}\left[1+\frac{8h^2}{5(1-\nu)(5+\nu)a^2}\left(\frac{E}{G'}-2\nu'\frac{E}{E'}\right)\right], \tag{6.35}$$

where the first term $\left(\frac{qa^4}{64D}\,\frac{5+\nu}{1+\nu}\right)$ is the flexure of the plate center as derived from the classical theory.**

---

*S. P. Timoshenko, see p. 64 of work cited in footnote * on p. 323.

**S. P. Timoshenko, see p. 67 of work cited in footnote 1 on p. 323.

2. Fixed Edge With Fastened Element of Median Surface. In this case the following boundary conditions must be satisfied:

where $$r = a \quad w = 0, \quad \frac{\partial w}{\partial r} = 0. \tag{6.36}$$

On the basis of (6.32) and upon satisfying the boundary conditions (6.36), we obtain for the flexure of the plate

$$w = \frac{q}{64D}(a^2 - r^2)^2, \tag{6.37}$$

and for the flexure of the plate center

$$w = \frac{qa^4}{64D}. \tag{6.38}$$

It is interesting to note that the last two formulas completely coincide with the corresponding formulas in the classical theory.*

3. Fixed Edge With Fastened Vertical Element of Edge. In this case the following boundary conditions must be satisfied:

where $$r = a \quad w = 0, \quad \left(\frac{\partial u_r}{\partial \gamma}\right)_{\gamma=0} = 0. \tag{6.39}$$

On the basis of the initial assumptions of the problem as well as from formulas (2.19), (6.28), (6.32) and upon satisfying the boundary conditions in (6.39), we obtain for the flexure of the plate

$$w = \frac{q}{64D}(a^2 - r^2)^2 + \frac{3q}{8hG'}(a^2 - r^2), \tag{6.40}$$

while for the flexure of the plate center

$$w = \frac{qa^4}{64D}\left(1 + 2\frac{\overline{E}}{G'}\frac{h^2}{a^2}\right). \tag{6.41}$$

If the same problems are solved by means of the theory described in Section 4 of this chapter, that is, by means of the theory which disregards the influence of normal stress σ_γ, it is evident that for the flexure of the center of a circular plate which is freely supported over its contour we obtain

$$w = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left(1 + \frac{8h^2E}{5(1-\nu)(5+\nu)a^2G'}\right). \tag{6.42}$$

---

*See pp. 65-66 of work cited in footnote * on p. 323.

For the two variants of fixed plates formulas (6.38) and (6.41) remain unchanged since they are not affected by a change in ratio E/E'.

These same problems may be solved by use of the theory described in the fifth section of this chapter. Without presenting the details (these are elementary and have been repeated more than once), we shall point out that all the design formulas coincide completely with the corresponding formulas derived by means of other theories,* that is, with formulas (6.34), (6.35), (6.37), (6.38), (6.40), (6.41) and (6.42).

Examining design formulas (6.35) and (6.41), as in the case of a rectangular plate, we notice that for certain values of ratios $\overline{E}/G'$, $\overline{E}/E'$ and h/a the normal displacements as determined from the classical theory of plates may differ substantially from the corresponding normal displacements as found from the proposed theories. A marked diversity in the normal displacements is obtained in the case of the second variant of a fixed plate (that is, when we have a fixed edge with fastened vertical element). By way of illustration we list the results of several calculations of normal displacement in accordance with formula (6.41):

h/a E/G'	$^1/_3$	$^1/_5$	$^1/_{10}$
0,0	1,000	1,000	1,000
2,6	1,635	1,229	1,057
5,0	2,221	1,440	1,110
10,0	3,442	1,879	1,220

the factor for the data listed in the table is $qa^4/64D$. The calculations assume that $\nu = 0.3$. The table clearly shows that the classical theory (E/G'; = 0) may yield substantially lower values for the flexures of the plate.

By the examples of circular plates it may be shown that the theories proposed herein may under certain cases of boundary conditions and load be used for calculation of relatively thick plates. By way of illustration let us discuss circular isotropic plates for which there are precise solutions of the problems presented here. We shall compare the values of normal displacement of the centers of the discussed circular plates.

---

*S.A. Ambartsumyan, D.V. Peshtmaldzhyan, see footnote *** on p. 315.

In the case of free support we have:

a) with consideration of the influence of σ_γ (6.35)

$$w = \frac{qa^4}{64D}\,\frac{5+\nu}{1+\nu}\left[1 + \frac{16}{5(1-\nu)(5+\nu)}\,\frac{h^2}{a^2}\right]; \tag{6.43}$$

b) without consideration of σ_γ (6.42)

$$w = \frac{qa^4}{64D}\,\frac{5+\nu}{1+\nu}\left[1 + \frac{16(1+\nu)}{5(1-\nu)(5+\nu)}\,\frac{h^2}{a^2}\right]. \tag{6.44}$$

In the case of fixing:

a) fastened element of median surface

$$w = \frac{qa^4}{64D}; \tag{6.45}$$

b) fastened vertical element of edge

$$w = \frac{qa^4}{64D}\left(1 + \frac{4}{1-\nu}\,\frac{h^2}{a^2}\right). \tag{6.46}$$

For flexure of the center of the median plane we have (from the precise theory of plates)* in the case of free support

$$w = \frac{qa^4}{64D}\,\frac{5+\nu}{1+\nu}\left[1 + \frac{2(8+\nu+\nu^2)}{5(1-\nu)(5+\nu)}\,\frac{h^2}{a^2}\right]. \tag{6.47}$$

In the case of fixing the results obtained from the precise theory coincide with formulas (6.45) and (6.46).

Thus, a numerical comparison is required only in the case of free support. Let us assume that h/a = 1/3 and = 0.3. The results of the calculations are listed in tabular form:

	Precise theory	From formula (6.43)	From formula (6.44)	Classical theory of plates
$\frac{64D(1+\nu)}{qa^4(5+\nu)}w$	1,1005	1,0958	1,1245	1,0000

---

*A. Love, Matematicheskaya teoriya uprugosti, Moscow, 1935, pp. 504-507.

In examining the table we note that in the case where the classical theory of plates yields an error of the order of 9 percent (in the case of an isotropic plate), the theories proposed here in one case [formula (6.44)] yield an error of the order of 2 percent and in the other case [formula (6.43)] less than 1 percent.

For the sake of thoroughness let us also describe the procedure for determining stresses σ_α and σ_β. Considering for the examples of a circular plate discussed in this section that when r = 0 u = 0 and solving differential equation (6.30), we obtain for the tangential displacement

$$u = cr. \tag{6.48}$$

The integration constant c (it is evident that in integration of (6.30) we have two integration constants, one of which, in virtue of the conditions at the center of the plate, vanishes) is easily found from the condition of fastening of the edge of the plate. In the case of free support, when r = a T_r = 0, we obtain from (6.22) and (6.48)

$$c = \frac{Aq}{2\bar{E}(1+\nu)}, \quad u = \frac{Aq}{2\bar{E}(1+\nu)}r. \tag{6.49}$$

If for a freely supported edge we assume that when $r = a$ $\sigma_r = 0$, for the integration constant c (and thereby also for displacement u) we obtain (6.48). In the case of a fixed edge (for both variants of fastening), when r = a u = 0 and we obtain from (6.48)

$$c = 0, \quad u = 0. \tag{6.50}$$

Thus, for each example, having the values of u [(6.49), (6.50)], w [(6.35), (6.37), (6.40)] and φ [(6.28)], the stress formulas are easily obtained.

Let us derive the stress formulas only for the central normal element of a plate, that is, for r = 0.

In the case of free support:

$$\sigma_r = \sigma_\theta = q(1+\nu)\frac{\gamma}{h}\left[\frac{3}{4}\left(\frac{3+\nu}{1+\nu}\frac{a^2}{h^2} - \frac{1}{5}\frac{\bar{E}}{G'} + \frac{2}{5}\nu'\frac{\bar{E}}{E'}\right) + \right.$$
$$\left. + \frac{\gamma^2}{h^2}\left(\frac{\bar{E}}{G'} - 2\nu'\frac{\bar{E}}{E'}\right)\right]. \tag{6.51}$$

In the case of fixing:

a) fastened element of median surface

$$\sigma_r = \sigma_\theta = q(1+\nu)\left[\frac{\nu'}{2}\frac{\bar{E}}{E'} + \frac{3}{4}\frac{\gamma}{h}\left(\frac{a^2}{h^2} - \frac{\bar{E}}{G'} + 2\nu'\frac{\bar{E}}{E'}\right) + \right.$$
$$\left. + \frac{\gamma^3}{h^3}\left(\frac{\bar{E}}{G'} - 2\nu'\frac{\bar{E}}{E'}\right)\right]; \tag{6.52}$$

b) fastened vertical element of edge

$$\sigma_r = \sigma_\theta = q(1+\nu)\left[\frac{\nu'}{2}\frac{\bar{E}}{E'} + \frac{3}{4}\frac{\gamma}{h}\left(\frac{a^2}{h^2} + 2\nu'\frac{\bar{E}}{E'}\right) + \right.$$
$$\left. + \frac{\gamma^3}{h^3}\left(\frac{\bar{E}}{G'} - 2\nu'\frac{\bar{E}}{E'}\right)\right]. \tag{6.53}$$

These stresses are in excellent correspondence with the stresses determined from the theory of thick plates.

Examining the results obtained here (Sections 2-6), we note that the theory of anisotropic plates, based on the hypothesis of nondeformable normals, requires corrections in the case of substantial anisotropy of the material of the plate. The fact is that the classical theory of anisotropic plates (that is, the theory which is based on the hypothesis of nondeformable normals), having an error of the order of h^2/a^2 in comparison with unity, as was to be expected, is completely unaffected by ratios of the type B_{ik}/B_{55}, B_{ik}/B_{44} and B_{ik}/B_{33}, which in the more precise theories (the theories dealing with quantities of the order of h^2/a^2) of anisotropic plates occur with an numerical coefficient and factor h^2/a^2 and may have rather large numerical values. This also explains the fact that the "correction" for the classical theory for substantially anisotropic plates (by which term we here and henceforth mean plates for which ratios of type B_{ik}/B_{55}, etc are of the order of 5-15) may be considerable.

The so-called "correction" to the classical theory of plates, due to consideration of transverse shear and normal stress σ_γ, is of the order of h^2/a^2 and depends on the form and relative thickness of the plate, on the boundary conditions, on the type of load, and, finally, on the numerical values of ratios of type B_{ik}/B_{55}, etc. In view of the presence of such a variety of parameters affecting the magnitude of correction for the classical theory of plates, it is almost impossible here (at least in this work) to indicate the specific limits of applicability of the hypothesis of nondeformable normals. It may be stated only that the hypothesis of nondeformable normals is applicable in those cases when $\lambda \frac{h^\gamma}{a^\gamma}$ (where γ depends on the above-mentioned parameters) is negligibly small in comparison with unity.

In order to determine the specific applicability (with one or another accuracy) of the hypothesis of nondeformable normals it is necessary that each specific class of problems be subjected to test calculation. The theory described in Section 5 of this chapter is extremely useful from this standpoint. Inserting the values of φ_0 and ψ_0 as obtained from the classical theory into the design formulas and equations of solution in Section 5, it is not difficult to ascertain that maximum value of the relative thickness of the plate h/a at which the discussed plate may, with a given accuracy, be calculated by means of the classical theory of plates.

However, as numerous calculations performed at the Institute of Mathematics and Mechanics (Academy of Sciences, Armenian SSR) have shown, the classical theory of plates may be used with sufficiently high accuracy (of the order of 5-10 percent) for calculation of the majority of actual plates if, in the case of isotropic or negligibly anisotropic plates (ratios of type B_{ik}/B_{55}, etc are less than three) h/a, that is, ε (see Chapter II, Section 1) is less than 0.1, while in the case of a substantially anisotropic plate (ratios of type B_{ik}/B_{55}, etc., are of the order of 10-15) h/a is less than 0.05. We will note that here there are no limitations placed on anisotropy in the plane $\gamma = 0$.

The presented numerical limits of applicability of the hypothesis of nondeformable normals are approximate and are pessimum (minimum). We will note that the values of h/a were found (and in subsequent discussion will be found for shells) on the basis of an analysis of the displacements of numerous and various anisotropic plates

In the case where the ratios of type B_{ik}/B_{55}, etc are less than unity, it is possible that the hypothesis of nondeformable normals may be applicable with a high degree of design accuracy for all real problems of anisotropic plates.

SECTION 7. THEORY OF EXTREMELY SHALLOW ORTHOTROPIC SHELLS

As in the preceding chapter (Section 4), we shall discuss an extremely shallow orthotropic shell. It is assumed that the shell under discussion is so shallow and the coordinate system is so chosen that for the first quadratic form of the median surface we may write

$$ds^2 = d\alpha^2 + d\beta^2, \tag{7.1}$$

that is, we assume that the internal geometry of the median surface of an extremely shallow shell of non-zero Gaussian curvature does not differ from the Euclidian geometry of a plane. Hence, with the chosen coordinate system we have for the coefficients of the first quadratic form

$$A = 1, \quad B = 1. \tag{7.2}$$

Furthermore, with the same accuracy it is considered that the principal curvatures of a median surface of an extremely shallow shell behave as constant quantities:

$$k_1 = \text{const}, \quad k_2 = \text{const}. \tag{7.3}$$

In view of the substantial shallowness of the shell, it is evident without any additional remarks that quantities of the order of hk_j will be disregarded in comparison with unity.* In addition it will be kept in mind that the thickness of the shell h is extremely small but finite.

The theory of extremely shallow orthotropic shells, considering the influence of transverse shear and normal stress σ_γ, is constructed on the basis of assumptions (a) and (b) given in the first section of this chapter. These assumptions, as in the case of a plate, are given in the following form:

a) We approximately consider

$$e_\gamma = 0; \tag{7.4}$$

b) The shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ are given as follows:

$$\left.\begin{aligned} \tau_{\alpha\gamma} &= \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right)\varphi, \\ \tau_{\beta\gamma} &= \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right)\psi. \end{aligned}\right\} \tag{7.5}$$

---

*A detailed list of literature on these problems may be found in the footnotes of the preceding (sixth) chapter.

where, as before, $\varphi=\varphi(\alpha, \beta)$ and $\psi=\psi(\alpha, \beta)$ are arbitrary unknown functions of coordinates α, β.

It is here assumed that the shell is loaded only by a normally applied load $Z = Z^+$ and that shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ in the direction of the thickness of the shell vary as for a quadratic parabola* (see the third example of a rectangular plate).

From the equations of the generalized Hooke's Law (1.3.7) we have for the stresses

$$\sigma_\alpha = B_{11}e_\alpha + B_{12}e_\beta - A_1\sigma_\gamma; \tag{7.6}$$

$$\sigma_\beta = B_{12}e_\alpha + B_{22}e_\beta - A_2\sigma_\gamma; \tag{7.7}$$

$$\tau_{\alpha\gamma} = B_{55}e_{\alpha\gamma}, \quad \tau_{\beta\gamma} = B_{44}e_{\beta\gamma}; \tag{7.8}$$

$$\tau_{\alpha\beta} = B_{66}e_{\alpha\beta}, \tag{7.9}$$

where, as usual, for elastic coefficients B_{ik} and A_i we have (2.8) and (2.9).

From (2.1.4)-(2.1.9) we obtain for the deformation components

$$e_\alpha = \frac{\partial u_\alpha}{\partial \alpha} + k_1 u_\gamma; \quad e_\beta = \frac{\partial u_\beta}{\partial \beta} + k_2 u_\gamma; \tag{7.10}$$

$$e_\gamma = \frac{\partial u_\gamma}{\partial \gamma}; \quad e_{\alpha\beta} = \frac{\partial u_\alpha}{\partial \beta} + \frac{\partial u_\beta}{\partial \alpha}; \tag{7.11}$$

$$e_{\alpha\gamma} = \frac{\partial u_\alpha}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \alpha}, \quad e_{\beta\gamma} = \frac{\partial u_\beta}{\partial \gamma} + \frac{\partial u_\gamma}{\partial \beta}. \tag{7.12}$$

By means of (7.4) we obtain from the first relationship in (7.11)

$$e_\gamma = \frac{\partial u_\gamma}{\partial \gamma} = 0, \quad u_\gamma = u_\gamma(\alpha, \beta) = w(\alpha, \beta), \tag{7.13}$$

that is, as in the case of a plate (2.13) or the classical theory of extremely shallow shells, the normal displacement u_γ of any point of the shell does not depend on γ and for all points of a given normal element is equal to the normal displacement w of the corresponding point of the median surface of the shell.

From (7.8) on the basis of (7.5) and (2.8) we obtain for the shear deformations $e_{\alpha\gamma}$ and $e_{\beta\gamma}$

---

*For arbitrary f (γ) the theory of shallow shells has been discussed in the work by S.A. Ambartsumyan and D.V. Peshtmaldghyan (see footnote 3 on p. 315.

$$e_{\alpha\gamma} = \frac{a_{55}}{2}\left(\frac{h^2}{4} - \gamma^2\right)\varphi, \quad e_{\beta\gamma} = \frac{a_{44}}{2}\left(\frac{h^2}{4} - \gamma^2\right)\psi. \tag{7.14}$$

Inserting into relationships (7.12) the values of $e_{\alpha\gamma}$, $e_{\beta\gamma}$ and u_γ from (7.14) and (7.13) and integrating for γ from zero to γ, also considering that when $\gamma = 0$ $u_\alpha = u(\alpha, \beta)$ and $u_\beta = v(\alpha, \beta)$ for the tangential displacements of any point of the shell we obtain

$$u_\alpha = u - \gamma\frac{\partial w}{\partial \alpha} + \gamma\frac{a_{55}}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\varphi; \tag{7.15}$$

$$u_\beta = v - \gamma\frac{\partial w}{\partial \beta} + \gamma\frac{a_{44}}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\psi. \tag{7.16}$$

where $u = u(\alpha, \beta)$ and $v = v(\alpha, \beta)$ are the tangential displacements of the corresponding point of the median surface of the shell.

From (7.15) and (7.16), as from (2.19) and (2.20), it follows that in distinction from the classical theory of extremely shallow shells* the tangential displacements u_α and u_β at any point removed from the median surface of the shell by normal distance γ depend on γ nonlinearly. Inserting the values of u_α, u_β and u_γ from (7.15), (7.16) and (7.13) into relationships (7.10) and (7.11), we have for the deformation components

$$e_\alpha = \frac{\partial u}{\partial \alpha} - \gamma\frac{\partial^2 w}{\partial \alpha^2} + k_1 w + \gamma\frac{a_{55}}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\frac{\partial \varphi}{\partial \alpha}; \tag{7.17}$$

$$e_\beta = \frac{\partial v}{\partial \beta} - \gamma\frac{\partial^2 w}{\partial \beta^2} + k_2 w + \gamma\frac{a_{44}}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\frac{\partial \psi}{\partial \beta}; \tag{7.18}$$

$$e_{\alpha\beta} = \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} - 2\gamma\frac{\partial^2 w}{\partial \alpha\,\partial \beta} + \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(a_{55}\frac{\partial \varphi}{\partial \beta} + a_{44}\frac{\partial \psi}{\partial \alpha}\right). \tag{7.19}$$

Inserting the values of deformations e_α, e_β and $e_{\alpha\beta}$ into (7.6), (7.7) and (7.9), we obtain for the stresses

$$\begin{aligned}\sigma_\alpha = {} & B_{11}\frac{\partial u}{\partial \alpha} + B_{12}\frac{\partial v}{\partial \beta} - \gamma\left(B_{11}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}\frac{\partial^2 w}{\partial \beta^2}\right) + \\ & + (k_1 B_{11} + k_2 B_{12})w + \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{11}a_{55}\frac{\partial \varphi}{\partial \alpha} + B_{12}a_{44}\frac{\partial \psi}{\partial \beta}\right) - A_1\sigma_\gamma;\end{aligned} \tag{7.20}$$

---

*As before, the classical theory is understood to be that theory which is based on the hypothesis of nondeformable normals.

$$\sigma_\beta = B_{22}\frac{\partial v}{\partial\beta} + B_{12}\frac{\partial u}{\partial\alpha} - \gamma\left(B_{22}\frac{\partial^2 w}{\partial\beta^2} + B_{12}\frac{\partial^2 w}{\partial\alpha^2}\right) +$$

$$+ (k_2B_{22} + k_1B_{12})w + \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{22}a_{44}\frac{\partial\psi}{\partial\beta} + B_{12}a_{55}\frac{\partial\varphi}{\partial\alpha}\right) - A_2\sigma_0; \qquad (7.21)$$

$$\tau_{\alpha\beta} = B_{66}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}\right) - 2B_{66}\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta} +$$

$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{66}a_{55}\frac{\partial\varphi}{\partial\beta} + B_{66}a_{44}\frac{\partial\psi}{\partial\alpha}\right). \qquad (7.22)$$

Inserting the values of stresses σ_α, σ_β, $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ into equilibrium equation (1.1.12) and integrating for γ, therein considering that when $\gamma = h/2$ $\sigma_\gamma = Z = Z^+$ and when $\gamma = -h/2$ $\sigma_\gamma = 0$, with the accuracy of the theory of extremely shallow shells we obtain for stresses

$$\sigma_\gamma = \frac{Z}{2} + \gamma\left[(k_1B_{11} + k_2B_{12})\frac{\partial u}{\partial\alpha} + (k_2B_{22} +\right.$$

$$\left. + k_1B_{12})\frac{\partial v}{\partial\beta} + (k_1^2B_{11} + 2k_1k_2B_{12} + k_2^2B_{22})\,w\right] +$$

$$+ \frac{1}{2}\left(\frac{h^2}{4} - \gamma^2\right)\left[(k_1B_{11} + k_2B_{12})\frac{\partial^2 w}{\partial\alpha^2} + (k_2B_{22} + k_1B_{12})\frac{\partial^2 w}{\partial\beta^2}\right] -$$

$$- \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(\frac{\partial\varphi}{\partial\alpha} + \frac{\partial\psi}{\partial\beta}\right). \qquad (7.23)$$

Inserting the values of stresses from (7.5), (7.20)-(7.23) into formulas (2.6.1)-(2.6.5) and keeping in mind that here we have an extremely shallow single-layer shell, we obtain for the internal forces and moments

$$T_1 = C_{11}\left(\frac{\partial u}{\partial\alpha} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial\beta} + k_2 w\right) - A_1T^0; \qquad (7.24)$$

$$T_2 = C_{22}\left(\frac{\partial v}{\partial\beta} + k_2 w\right) + C_{12}\left(\frac{\partial u}{\partial\alpha} + k_1 w\right) - A_2T^0; \qquad (7.25)$$

$$S = C_{66}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha}\right); \qquad (7.26)$$

$$M_1 = -D_{11}\frac{\partial^2 w}{\partial\alpha^2} - D_{12}\frac{\partial^2 w}{\partial\beta^2} +$$

$$+ \frac{h^2}{10}\left(a_{55}D_{11}\frac{\partial\varphi}{\partial\alpha} + a_{44}D_{12}\frac{\partial\psi}{\partial\beta}\right) - A_1M^0; \qquad (7.27)$$

$$M_2 = -D_{22}\frac{\partial^2 w}{\partial\beta^2} - D_{12}\frac{\partial^2 w}{\partial\alpha^2} +$$

$$+ \frac{h^2}{10}\left(a_{44}D_{22}\frac{\partial\psi}{\partial\beta} + a_{55}D_{12}\frac{\partial\varphi}{\partial\alpha}\right) - A_2M^0; \qquad (7.28)$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial\varphi}{\partial\beta} + \alpha_{44}\frac{\partial\psi}{\partial\alpha}\right); \qquad (7.29)$$

$$N_1 = \frac{h^3}{12}\varphi, \quad N_2 = \frac{h^3}{12}\psi, \qquad (7.30)$$

where we have the following notation:

$$T^0 = \frac{h}{2} Z + (k_1 D_{11} + k_2 D_{12}) \frac{\partial^2 w}{\partial \alpha^2} + (k_2 D_{22} + k_1 D_{12}) \frac{\partial^2 w}{\partial \beta^2}; \tag{7.31}$$

$$\begin{aligned} M^0 = {} & (k_1 D_{11} + k_2 D_{12}) \frac{\partial u}{\partial \alpha} + (k_2 D_{22} + k_1 D_{12}) \frac{\partial v}{\partial \beta} + \\ & + (k_1^2 D_{11} + 2 k_1 k_2 D_{12} + k_2^2 D_{22}) w - \frac{h^5}{120} \left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} \right). \end{aligned} \tag{7.32}$$

From (2.74) we have the following equilibrium equations

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = 0, \quad \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = 0; \tag{7.33}$$

$$(k_1 T_1 + k_2 T_2) - \frac{\partial N_1}{\partial \alpha} - \frac{\partial N_2}{\partial \beta} = Z; \tag{7.34}$$

$$\frac{\partial M_1}{\partial \alpha} + \frac{\partial H}{\partial \beta} = N_1, \quad \frac{\partial M_2}{\partial \beta} + \frac{\partial H}{\partial \alpha} = N_2. \tag{7.35}$$

Inserting the values of internal forces and moments from (7.24)-(7.30) into equilibrium equations(7.33)-(7.35), we obtain the following system of differential equations of solution in the theory of extremely shallow shells:

$$\begin{aligned} & L_{11}(C_{ik}) u + L_{12}(C_{ik}) v + (k_1 C_{11} + k_2 C_{12}) \frac{\partial w}{\partial \alpha} - \\ & - A_1 \left[(k_1 D_{11} + k_2 D_{12}) \frac{\partial^3 w}{\partial \alpha^3} + (k_2 D_{22} + k_1 D_{12}) \frac{\partial^3 w}{\partial \alpha \, \partial \beta^2} \right] = A_1 \frac{h}{2} \frac{\partial Z}{\partial \alpha}; \end{aligned} \tag{7.36}$$

$$\begin{aligned} & L_{22}(C_{ik}) v + L_{12}(C_{ik}) u + (k_2 C_{22} + k_1 C_{12}) \frac{\partial w}{\partial \beta} - \\ & - A_2 \left[(k_2 D_{22} + k_1 D_{12}) \frac{\partial^3 w}{\partial \beta^3} + (k_1 D_{11} + k_2 D_{12}) \frac{\partial^3 w}{\partial \beta \, \partial \alpha^2} \right] = A_2 \frac{h}{2} \frac{\partial Z}{\partial \beta}; \end{aligned} \tag{7.37}$$

$$\begin{aligned} & (k_1 C_{11} + k_2 C_{12}) \frac{\partial u}{\partial \alpha} + (k_2 C_{22} + k_1 C_{12}) \frac{\partial v}{\partial \beta} + \\ & \quad + (k_1^2 C_{11} + 2 k_1 k_2 C_{12} + k_2^2 C_{22}) w - \frac{h^3}{12} \left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} \right) - \\ & \quad - (k_1 A_1 + k_2 A_2) \left[(k_1 D_{11} + k_2 D_{12}) \frac{\partial^2 w}{\partial \alpha^2} + \right. \\ & \quad \left. + (k_2 D_{22} + k_1 D_{12}) \frac{\partial^2 w}{\partial \beta^2} \right] = Z \left[1 + \frac{h}{2} (k_1 A_1 + k_2 A_2) \right]; \end{aligned} \tag{7.38}$$

$$\begin{aligned} & L_{13}(D_{ik}) w - \frac{h^2}{10} [a_{55} L_{11}(D_{ik}) \varphi + a_{44} L_{12}(D_{ik}) \psi] + \frac{h^3}{12} \varphi + \\ & \quad + A_1 \left[(k_1 D_{11} + k_2 D_{12}) \frac{\partial^2 u}{\partial \alpha^2} + (k_2 D_{22} + k_1 D_{12}) \frac{\partial^2 v}{\partial \alpha \, \partial \beta} + \right. \\ & \left. + (k_1^2 D_{11} + 2 k_1 k_2 D_{12} + k_2^2 D_{22}) \frac{\partial w}{\partial \alpha} - \frac{h^5}{120} \left(\frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \alpha \, \partial \beta} \right) \right] = 0; \end{aligned} \tag{7.39}$$

$$\begin{aligned} & L_{23}(D_{ik}) w - \frac{h^2}{10} \left[a_{44} L_{22}(D_{ik}) \psi + a_{55} L_{12}(D_{ik}) \varphi \right] + \\ & + \frac{h^3}{12} \psi + A_2 \left[(k_2 D_{22} + k_1 D_{12}) \frac{\partial^2 v}{\partial \beta^2} + (k_1 D_{11} + k_2 D_{12}) \frac{\partial^2 u}{\partial \alpha \, \partial \beta} + \right. \\ & \left. + (k_1^2 D_{11} + 2 k_1 k_2 D_{12} + k_2^2 D_{22}) \frac{\partial w}{\partial \beta} - \frac{h^5}{120} \left(\frac{\partial^2 \psi}{\partial \beta^2} + \frac{\partial^2 \varphi}{\partial \alpha \, \partial \beta} \right) \right] = 0, \end{aligned} \tag{7.40}$$

where for the linear operators and stiffnesses we have the known expressions

$$
\begin{aligned}
L_{11}(C_{jk}) &= C_{11}\frac{\partial^2}{\partial\alpha^2} + C_{66}\frac{\partial^2}{\partial\beta^2};\\
L_{22}(C_{jk}) &= C_{22}\frac{\partial^2}{\partial\beta^2} + C_{66}\frac{\partial^2}{\partial\alpha^2};\\
L_{12}(C_{jk}) &= (C_{12} + C_{66})\frac{\partial^2}{\partial\alpha\,\partial\beta};\\
L_{13}(D_{jk}) &= D_{11}\frac{\partial^3}{\partial\alpha^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial\alpha\,\partial\beta^2};\\
L_{23}(D_{jk}) &= D_{22}\frac{\partial^3}{\partial\beta^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial\beta\,\partial\alpha^2};\\
C_{jk} &= hB_{jk}, \quad D_{jk} = \frac{h^3}{12}B_{jk}.
\end{aligned}
$$

Equations (7.36)-(7.40) constitute a complete system of five differential equations of solution in terms of the five sought functions u, v, w, φ, ψ. In solving specific edge problems the differential equations of solution (7.36)-(7.40), as usual, are supplemented by the boundary conditions given in Section 2 of this chapter.

The principal equations in the theory of extremely shallow shells may be represented in the form of the combined method. The third equation of compatibility of deformation (3.3) in the case of an extremely shallow shell, as is known, has the form

$$k_2\varkappa_1 + k_1\varkappa_2 + \frac{\partial^2\varepsilon_2}{\partial\alpha^2} - \frac{\partial^2\omega}{\partial\alpha\,\partial\beta} + \frac{\partial^2\varepsilon_1}{\partial\beta^2} = 0. \qquad (7.41)$$

This equation, representing the compatibility of the median surface of the shell, also must remain unchanged in the case of the theory proposed herein, which theory considers the influence of transverse shear and normal stress σ_γ.

In examining formulas (7.24)-(7.28) the careful reader will probably have noticed that the coefficient of C_{11} contains the relative deformation of the median surface ε_1 the coefficient of C_{22} contains the relative deformation ε_2 the coefficient of C_{66} contains the shear of the median surface ω, the coefficient of D_{11} contains the parameter characterizing the change in curvature of the median surface $\varkappa_1$ and, finally, the coefficient of D_{22} contains the change in curvature $\varkappa_2$ (see formulas (4.10) in Chapter VI). Considering this, the equation of compatibility (7.41) may be written as follows:

$$\begin{gathered}\frac{C_{22}}{\Omega}\frac{\partial^2 T_1}{\partial\beta^2}-\frac{C_{12}}{\Omega}\frac{\partial^2 T_1}{\partial\alpha^2}+\frac{C_{11}}{\Omega}\frac{\partial^2 T_2}{\partial\alpha^2}-\frac{C_{12}}{\Omega}\frac{\partial^2 T_2}{\partial\beta^2}-\\ -\frac{1}{C_{66}}\frac{\partial^2 S}{\partial\alpha\,\partial\beta}-k_2\frac{\partial^2 w}{\partial\alpha^2}-k_1\frac{\partial^2 w}{\partial\beta^2}+\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)\frac{\partial^2 T^0}{\partial\alpha^2}+\\ +\left(A_1\frac{C_{22}}{\Omega}-A_2\frac{C_{12}}{\Omega}\right)\frac{\partial^2 T^0}{\partial\beta^2}=0.\end{gathered}\tag{7.42}$$

Introducing the stress function F = F (α, β), in terms of which the internal forces T_1, T_2 and S are represented by relationships

$$T_1=\frac{\partial^2 F}{\partial\beta^2},\quad T_2=\frac{\partial^2 F}{\partial\alpha^2},\quad S=-\frac{\partial^2 F}{\partial\alpha\,\partial\beta},\tag{7.43}$$

we identically satisfy the first two equilibrium equations (7.33) and from the remaining three equilibrium equations (7.34), (7.35) and equation of compatibility of deformation (7.42), by virtue of (7.27)-(7.30) and (7.43) we obtain the following system of differential equations in the combined method:

$$\nabla_r F-\frac{h^3}{12}\left(\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}\right)=Z;\tag{7.44}$$

$$\begin{gathered}L_{13}(D_{ik})\,w-\frac{h^2}{10}\left[a_{55}L_{11}(D_{ik})\,\varphi+a_{44}L_{12}(D_{ik})\,\psi\right]+\\ +\frac{h^3}{12}\varphi+A_1\Big[(k_1D_{11}+k_2D_{12})\frac{\partial^2 u}{\partial\alpha^2}+(k_2D_{22}+k_1D_{12})\frac{\partial^2 v}{\partial\alpha\,\partial\beta}+\\ +(k_1^2D_{11}+2k_1k_2D_{12}+k_2^2D_{22})\frac{\partial w}{\partial\alpha}-\frac{h^5}{120}\left(\frac{\partial^2\varphi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\alpha\,\partial\beta}\right)\Big]=0;\end{gathered}\tag{7.45}$$

$$\begin{gathered}L_{23}(D_{ik})\,w-\frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\,\psi+a_{55}L_{12}(D_{ik})\,\varphi\right]+\\ +\frac{h^3}{12}\psi+A_2\Big[(k_2D_{22}+k_1D_{12})\frac{\partial^2 v}{\partial\beta^2}+(k_1D_{11}+k_2D_{12})\frac{\partial^2 u}{\partial\alpha\,\partial\beta}+\\ +(k_1^2D_{11}+2k_1k_2D_{12}+k_2^2D_{22})\frac{\partial w}{\partial\beta}-\frac{h^5}{120}\left(\frac{\partial^2\psi}{\partial\beta^2}+\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta}\right)\Big]=0;\end{gathered}\tag{7.46}$$

$$\begin{gathered}L_2(C_{ik})\,F-\nabla_r w+\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)(k_1D_{11}+k_2D_{12})\frac{\partial^4 w}{\partial\alpha^4}+\\ +\Big[\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)(k_2D_{22}+k_1D_{12})+\\ +\left(A_1\frac{C_{22}}{\Omega}-A_2\frac{C_{12}}{\Omega}\right)(k_1D_{11}+k_2D_{12})\Big]\frac{\partial^4 w}{\partial\alpha^2\,\partial\beta^2}+\\ +\left(A_1\frac{C_{22}}{\Omega}-A_2\frac{C_{12}}{\Omega}\right)(k_2D_{22}+k_1D_{12})\frac{\partial^4 w}{\partial\beta^4}=\\ =-\frac{h}{2}\left[\left(A_2\frac{C_{11}}{\Omega}-A_1\frac{C_{12}}{\Omega}\right)\frac{\partial^2 Z}{\partial\alpha^2}+\left(A_1\frac{C_{22}}{\Omega}-A_2\frac{C_{12}}{\Omega}\right)\frac{\partial^2 Z}{\partial\beta^2}\right],\end{gathered}\tag{7.47}$$

where, along with the linear differential operators L_{11}, L_{22}, L_{12}, L_{13} and L_{23}, we have the operators $L_2(C_{ik})$, as in (6.6.44), and ∇_r as in (6.5.19).

Thus, the problem of an extremely shallow shell has been reduced in the combined method to a system of four differential equations F, ω, φ and ψ.

The design quantities in the case of the combined method may also be determined by means of the formulas derived above: for the design stresses we have (7.20)-(7.22); for the internal forces we have (7.30), (7.43); for the moments we have (7.27)-(7.29). By virtue of (7.24), (7.25), (7.31) and (7.43), the tangential displacements (u, v) which enter into the above formulas for the design quantities may be determined from the following equations:

$$\frac{\partial u}{\partial \alpha} = \frac{C_{22}}{\Omega}\frac{\partial^2 F}{\partial \beta^2} - \frac{C_{12}}{\Omega}\frac{\partial^2 F}{\partial \alpha^2} - k_1 w + \left(A_1 \frac{C_{22}}{\Omega} - A_2 \frac{C_{12}}{\Omega}\right) \times$$
$$\times \left[\frac{h}{2} Z + (k_1 D_{11} + k_2 D_{12}) \frac{\partial^2 w}{\partial \alpha^2} + (k_2 D_{22} + k_1 D_{12}) \frac{\partial^2 w}{\partial \beta^2}\right]; \qquad (7.48)$$

$$\frac{\partial v}{\partial \beta} = \frac{C_{11}}{\Omega}\frac{\partial^2 F}{\partial \alpha^2} - \frac{C_{12}}{\Omega}\frac{\partial^2 F}{\partial \beta^2} - k_2 w + \left(A_2 \frac{C}{\Omega} - A_1 \frac{C_{12}}{\Omega}\right) \times$$
$$\times \left[\frac{h}{2} Z + (k_2 D_{22} + k_1 D_{12}) \frac{\partial^2 w}{\partial \beta^2} + (k_1 D_{11} + k_2 D_{12}) \frac{\partial^2 w}{\partial \alpha^2}\right]. \qquad (7.49)$$

SECTION 8. APPROXIMATE THEORY OF AN EXTREMELY SHALLOW SHELL CONSIDERING TRANSVERSE SHEAR

In a manner similar to that for the case of an anisotropic plate (Section 4), we here give a more approximate theory for an extremely shallow anisotropic shell, which, in addition to basic assumptions (a) and (b), is based on still another assumption: the normal stresses σ_γ at areas parallel to the median surface of the shell may be disregarded in comparison with other stresses.

As is seen from the results in Section 4, the principal formulas and relationships of this approximate theory of extremely shallow shells may be derived from the corresponding formulas and relationships of the preceding section (Section 6) if we formally assume

$$A_1 = 0, \quad A_2 = 0. \qquad (8.1)$$

Considering (8.1), we obtain from (7.36)-(7.40) the following system of differential equations of solution in the proposed theory:

$$L_{11}(C_{ik}) u + L_{12}(C_{ik}) v + (k_1 C_{11} + k_2 C_{12}) \frac{\partial w}{\partial \alpha} = 0; \qquad (8.2)$$

$$L_{22}(C_{ik}) v + L_{12}(C_{ik}) u + (k_2 C_{22} + k_1 C_{12}) \frac{\partial w}{\partial \beta} = 0; \qquad (8.3)$$

$$(k_1 C_{11} + k_2 C_{12}) \frac{\partial u}{\partial \alpha} + (k_2 C_{22} + k_1 C_{12}) \frac{\partial v}{\partial \beta} +$$
$$+ (k_1^2 C_{11} + 2k_1 k_2 C_{12} + k_2^2 C_{22}) w - \frac{h^3}{12}\left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\right) = Z; \qquad (8.4)$$

$$L_{13}(D_{ik}) w - \frac{h^2}{10}[a_{55} L_{11}(D_{ik}) \varphi + a_{44} L_{12}(D_{ik}) \psi] + \frac{h^3}{12} \varphi = 0; \qquad (8.5)$$

$$L_{23}(D_{ik}) w - \frac{h^2}{10}[a_{44} L_{22}(D_{ik}) \psi + a_{55} L_{12}(D_{ik}) \varphi] + \frac{h^3}{12} \psi = 0. \qquad (8.6)$$

For the design quantities we have:

stresses ((7.20)-(7.22))

$$\sigma_\alpha = B_{11}\frac{\partial u}{\partial \alpha} + B_{12}\frac{\partial v}{\partial \beta} - \gamma\left(B_{11}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}\frac{\partial^2 w}{\partial \beta^2}\right) + $$
$$+ (k_1 B_{11} + k_2 B_{12})\, w + \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{11}a_{55}\frac{\partial \varphi}{\partial \alpha} + B_{12}a_{44}\frac{\partial \psi}{\partial \beta}\right); \tag{8.7}$$

$$\sigma_\beta = B_{22}\frac{\partial v}{\partial \beta} + B_{12}\frac{\partial u}{\partial \alpha} - \gamma\left(B_{22}\frac{\partial^2 w}{\partial \beta^2} + B_{12}\frac{\partial^2 w}{\partial \alpha^2}\right) + $$
$$+ (k_2 B_{22} + k_1 B_{12})\, w + \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{22}a_{44}\frac{\partial \psi}{\partial \beta} + B_{12}a_{55}\frac{\partial \varphi}{\partial \alpha}\right); \tag{8.8}$$

$$\tau_{\alpha\beta} = B_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) - 2B_{66}\gamma\frac{\partial^2 w}{\partial \alpha\, \partial \beta} + $$
$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{66}a_{55}\frac{\partial \varphi}{\partial \beta} + B_{66}a_{44}\frac{\partial \psi}{\partial \alpha}\right); \tag{8.9}$$

internal forces ((7.24)-(7.26) and (7.30))

$$T_1 = C_{11}\left(\frac{\partial u}{\partial \alpha} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial \beta} + k_2 w\right), \quad N_1 = \frac{h^3}{12}\varphi; \tag{8.10}$$

$$T_2 = C_{22}\left(\frac{\partial v}{\partial \beta} + k_2 w\right) + C_{12}\left(\frac{\partial u}{\partial \alpha} + k_1 w\right), \quad N_2 = \frac{h^3}{12}\psi; \tag{8.11}$$

$$S = C_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right); \tag{8.12}$$

moments ((7.27)-(7.29))

$$M_1 = -D_{11}\frac{\partial^2 w}{\partial \alpha^2} - D_{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^2}{10}\left(a_{55}D_{11}\frac{\partial \varphi}{\partial \alpha} + a_{44}D_{12}\frac{\partial \psi}{\partial \beta}\right), \tag{8.13}$$

$$M_2 = -D_{22}\frac{\partial^2 w}{\partial \beta^2} - D_{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^2}{10}\left(a_{44}D_{22}\frac{\partial \psi}{\partial \beta} + a_{55}D_{12}\frac{\partial \varphi}{\partial \alpha}\right); \tag{8.14}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial \alpha\, \partial \beta} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi}{\partial \beta} + a_{44}\frac{\partial \psi}{\partial \alpha}\right). \tag{8.15}$$

In the discussed formulation the equations of solution in the combined method (7.44)-(7.47) take the form

$$\nabla_r F - \frac{h^3}{12}\left(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\right) = Z; \tag{8.16}$$

$$L_2(C_{ik})\, F - \nabla_r w = 0; \tag{8.17}$$

$$L_{13}(D_{ik})\, w - \frac{h^3}{10}\left[a_{55}L_{11}(D_{ik})\,\varphi + a_{44}L_{12}(D_{ik})\,\psi\right] + \frac{h^3}{12}\varphi = 0; \tag{8.18}$$

$$L_{23}(D_{ik})\, w - \frac{h^2}{10}\left[a_{44}L_{22}(D_{ik})\,\psi + a_{55}L_{12}(D_{ik})\,\varphi\right] + \frac{h^3}{12}\psi = 0. \tag{8.19}$$

These equations ((8.16)-(8.19)), as the similar equations in the general case, constitute a complete system of four differential equations in terms of four unknown functions.

In the case of the combined method we have for the design quantities (7.43), (8.7)-(8.9), (8.12)-(8.15). From (7.48) and (9.49) we have for the tangential displacements

$$\frac{\partial u}{\partial \alpha}=\frac{C_{22}}{\Omega}\frac{\partial^2 F}{\partial \beta^2}-\frac{C_{12}}{\Omega}\frac{\partial^2 F}{\partial \alpha^2}-k_1 w; \tag{8.20}$$

$$\frac{\partial v}{\partial \beta}=\frac{C_{11}}{\Omega}\frac{\partial^2 F}{\partial \alpha^2}-\frac{C_{12}}{\Omega}\frac{\partial^2 F}{\partial \beta^2}-k_2 w \tag{8.21}$$

The system of equations of solution in the combined method (8.16)-(8.19) may be reduced to one tenth-order equation of solution in terms of a certain unknown function* $\Phi=\Phi(\alpha,\beta)$. Assuming

$$\varphi=\left[\frac{h^2}{10}a_{44}D_{66}\frac{\partial}{\partial \alpha}L_2'(D_{ik})-\frac{h^3}{12}L_{13}(D_{ik})\right]L_2(C_{ik})\Phi; \tag{8.22}$$

$$\psi=\left[\frac{h^2}{10}a_{55}D_{66}\frac{\partial}{\partial \beta}L_2'(D_{ik})-\frac{h^3}{12}L_{23}(D_{ik})\right]L_2(C_{ik})\Phi; \tag{8.23}$$

$$w=\left\{\frac{h^4}{100}a_{55}a_{44}D_{66}L_2'(D_{ik})-\frac{h^5}{120}\left[(a_{44}D_{22}+a_{55}D_{66})\frac{\partial^2}{\partial \beta^2}+\right.\right.$$
$$\left.\left.+(a_{55}D_{11}+a_{44}D_{66})\frac{\partial^2}{\partial \alpha^2}\right]+\frac{h^6}{144}\right\}L_2(C_{ik})\Phi; \tag{8.24}$$

$$F=\left\{\frac{h^4}{100}a_{55}a_{44}D_{66}L_2'(D_{ik})-\frac{h^5}{120}\left[(a_{44}D_{22}+a_{55}D_{66})\frac{\partial^2}{\partial \beta^2}+\right.\right.$$
$$\left.\left.+(a_{55}D_{11}+a_{44}D_{66})\frac{\partial^2}{\partial \alpha^2}\right]+\frac{h^6}{144}\right\}\nabla_r\Phi, \tag{8.25}$$

we identically satisfy the three equations of solution in the system (8.17)-(8.19) and from the fourth equation (8.16) we obtain the differential equation of solution for the unknown function

$$\left[\frac{h^5}{120}D_{66}\left(a_{55}\frac{\partial^2}{\partial \beta^2}+a_{44}\frac{\partial^2}{\partial \alpha^2}\right)L_2'(D_{ik})-\frac{h^6}{144}L_1(D_{ik})\right]L_2(C_{ik})\Phi-$$
$$-\left\{\frac{h^6}{144}+\frac{h^4}{100}a_{55}a_{44}D_{66}L_2'(D_{ik})-\frac{h^5}{120}\left[(a_{44}D_{22}+a_{55}D_{66})\frac{\partial^2}{\partial \beta^2}+\right.\right.$$
$$\left.\left.+(a_{55}D_{11}+a_{44}D_{66})\frac{\partial^2}{\partial \alpha^2}\right]\right\}\nabla_r^2\Phi=-Z, \tag{8.26}$$

where, as usual, we have for the linear operators

$$L_2(C_{ik})=\frac{C_{11}}{\Omega}\frac{\partial^4}{\partial \alpha^4}+\frac{C_{22}}{\Omega}\frac{\partial^4}{\partial \beta^4}+\left(\frac{1}{C_{66}}-2\frac{C_{12}}{\Omega}\right)\frac{\partial^4}{\partial \alpha^2\partial \beta^2}; \tag{8.27}$$

$$L_2'(D_{ik})=D_{11}\frac{\partial^4}{\partial \alpha^4}+D_{22}\frac{\partial^4}{\partial \beta^4}+\left(\frac{D_{11}D_{22}-D_{12}^2}{D_{66}}-2D_{12}\right)\frac{\partial^4}{\partial \alpha^2\partial \beta^2}; \tag{8.28}$$

$$L_1(D_{ik})=D_{11}\frac{\partial^4}{\partial \alpha^4}+D_{22}\frac{\partial^4}{\partial \beta^4}+2(D_{12}+2D_{66})\frac{\partial^4}{\partial \alpha^2\partial \beta^2}. \tag{8.29}$$

by means of formulas (8.7)-(8.15), (8.20)-(8.25) and (7.43) all design quantities are easily represented through the sought function $\Phi(\alpha,\beta)$. However, considering the elementary nature of these constructions and the unwieldiness of the resulting formulas, we shall not present them here.

---

*S.A. Ambartsymyan, D.V. Peshtmaldzhyan, see footnote ***, p. 315.

SECTION 9. ANOTHER APPROXIMATE THEORY FOR AN EXTREMELY SHALLOW SHELL

For an extremely shallow shell, as in the case of a plate (Section 6), we assume:*

a) the distances along the normal (along γ) between two points of the shell remain unchanged before and after deformation;

b) the normal stresses σ_γ at areas parallel to the median surface of the shell may be disregarded in comparison with other stresses;

c) in determining deformations $e_{\alpha\gamma}$ and $e_{\beta\gamma}$ we consider that the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ do not differ from the corresponding stresses obtained from the hypothesis of nondeformal normals.

By virtue of these premises we have for the stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$

(5.8.26)

$$\tau_{\alpha\gamma}=\frac{1}{2}\left(\frac{h^2}{4}-\gamma^2\right)\varphi_0,\quad \tau_{\beta\gamma}=\frac{1}{2}\left(\frac{h^2}{4}-\gamma^2\right)\psi_0, \tag{9.1}$$

where

$$\varphi_0=-\left[B_{11}\frac{\partial^3 w_0}{\partial\alpha^3}+(B_{12}+2B_{66})\frac{\partial^3 w_0}{\partial\alpha\,\partial\beta^2}\right]; \tag{9.2}$$

$$\psi_0=-\left[B_{22}\frac{\partial^3 w_0}{\partial\beta^3}+(B_{12}+2B_{66})\frac{\partial^3 w_0}{\partial\beta\,\partial\alpha^2}\right]; \tag{9.3}$$

ω_0 is the normal displacement of the corresponding orthotropic shell as determined from the hypothesis of nondeformable normals.

Proceeding as in Section 7, for the displacements of any point of the shell we obtain

$$u_\alpha=u-\gamma\frac{\partial w}{\partial\alpha}+\gamma\frac{a_{55}}{2}\left(\frac{h^2}{4}-\frac{\gamma^2}{3}\right)\varphi_0; \tag{9.4}$$

$$u_\beta=v-\gamma\frac{\partial w}{\partial\beta}+\gamma\frac{a_{44}}{2}\left(\frac{h^2}{4}-\frac{\gamma^2}{3}\right)\psi_0; \tag{9.5}$$

$$u_\gamma=w=w(\alpha,\beta), \tag{9.6}$$

where, as usual, u, v and w are the components of the total displacement of the median surface of the shell.

---

*Section 5 of this chapter lists the literature for this topic.

For the stresses we have

$$\sigma_\alpha = B_{11}\frac{\partial u}{\partial \alpha} + B_{12}\frac{\partial v}{\partial \beta} - \gamma\left(B_{11}\frac{\partial^2 w}{\partial \alpha^2} + B_{12}\frac{\partial^2 w}{\partial \beta^2}\right) + (k_1 B_{11} + k_{12} B_{12}) w +$$
$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{11} a_{55}\frac{\partial \varphi_0}{\partial \alpha} + B_{12} a_{44}\frac{\partial \psi_0}{\partial \beta}\right); \qquad (9.7)$$

$$\sigma_\beta = B_{22}\frac{\partial v}{\partial \beta} + B_{12}\frac{\partial u}{\partial \alpha} - \gamma\left(B_{22}\frac{\partial^2 w}{\partial \beta^2} + B_{12}\frac{\partial^2 w}{\partial \alpha^2}\right) + (k_2 B_{22} + k_1 B_{12}) w +$$
$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{22} a_{44}\frac{\partial \psi_0}{\partial \beta} + B_{12} a_{55}\frac{\partial \varphi_0}{\partial \alpha}\right); \qquad (9.8)$$

$$\tau_{\alpha\beta} = B_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right) - 2B_{66}\gamma\frac{\partial^2 w}{\partial \alpha\,\partial \beta} +$$
$$+ \frac{\gamma}{2}\left(\frac{h^2}{4} - \frac{\gamma^2}{3}\right)\left(B_{66} a_{55}\frac{\partial \varphi_0}{\partial \beta} + B_{66} a_{44}\frac{\partial \psi_0}{\partial \alpha}\right). \qquad (9.9)$$

By means of expressions (9.1), (9.7)-(9.9), proceeding as usual, we obtain for the internal forces and moments

$$T_1 = C_{11}\left(\frac{\partial u}{\partial \alpha} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial \beta} + k_2 w\right); \qquad (9.10)$$

$$T_2 = C_{22}\left(\frac{\partial v}{\partial \beta} + k_2 w\right) + C_{12}\left(\frac{\partial u}{\partial \alpha} + k_1 w\right); \qquad (9.11)$$

$$S = C_{66}\left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right); \qquad (9.12)$$

$$N_1 = \frac{h^3}{12}\varphi_0, \qquad N_2 = \frac{h^3}{12}\psi_0; \qquad (9.13)$$

$$M_1 = -D_{11}\frac{\partial^2 w}{\partial \alpha^2} - D_{12}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^2}{10}\left(a_{55} D_{11}\frac{\partial \varphi_0}{\partial \alpha} + a_{44} D_{12}\frac{\partial \psi_0}{\partial \beta}\right); \qquad (9.14)$$

$$M_2 = -D_{22}\frac{\partial^2 w}{\partial \beta^2} - D_{12}\frac{\partial^2 w}{\partial \alpha^2} + \frac{h^2}{10}\left(a_{44} D_{22}\frac{\partial \psi_0}{\partial \beta} + a_{55} D_{12}\frac{\partial \varphi_0}{\partial \alpha}\right); \qquad (9.15)$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial \alpha\,\partial \beta} + \frac{h^2}{10} D_{66}\left(a_{55}\frac{\partial \varphi_0}{\partial \beta} + a_{44}\frac{\partial \psi_0}{\partial \alpha}\right). \qquad (9.16)$$

Eliminating N_1 and N_2 from equilibrium equations (7.33) - (7.35), we obtain

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = 0, \qquad \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = 0; \qquad (9.17)$$

$$(k_1 T_1 + k_2 T_2) - \frac{\partial^2 M_1}{\partial \alpha^2} - 2\frac{\partial^2 H}{\partial \alpha\,\partial \beta} - \frac{\partial^2 M_2}{\partial \beta^2} = Z. \qquad (9.18)$$

Inserting the values of internal forces and moments from (9.10)-(9.16) into equilibrium equations (9.17) and (9.18), we obtain the following system of differential equations of solution of the problem:

$$L_{11}(C_{ik}) u + L_{12}(C_{ik}) v + (k_1 C_{11} + k_2 C_{12})\frac{\partial w}{\partial \alpha} = 0; \qquad (9.19)$$

$$L_{22}(C_{ik}) v + L_{12}(C_{ik}) u + (k_2 C_{22} + k_1 C_{12})\frac{\partial w}{\partial \beta} = 0; \qquad (9.20)$$

$$\begin{aligned}(k_1C_{11}+k_2C_{12})\frac{\partial u}{\partial \alpha}+(k_2C_{22}+k_1C_{12})\frac{\partial v}{\partial \beta}+\\ +(k_1^2C_{11}+2k_1k_2C_{12}+k_2^2C_{22})w+L_1(D_{ik})w=\\ =Z+\frac{h^2}{10}\left\{a_{55}\left[D_{11}\frac{\partial^3\varphi_0}{\partial \alpha^3}+(D_{12}+2D_{66})\frac{\partial^3\varphi_0}{\partial \alpha\,\partial \beta^2}\right]+\right.\\ \left.+a_{44}\left[D_{22}\frac{\partial^3\psi_0}{\partial \beta^3}+(D_{12}+2D_{66})\frac{\partial^3\psi_0}{\partial \beta\,\partial \alpha^2}\right]\right\}.\end{aligned} \tag{9.21}$$

Thus, in the case of the proposed variant of the theory of extremely shallow shells, with consideration of the influence of transverse shear we reduce the problem to determination of only three components of total displacement of the median surface of the shell: $u(\alpha, \beta)$, $v(\alpha, \beta)$, and $w(\alpha, \beta)$. Functions $\varphi_0(\alpha, \beta)$ and $\psi_0(\alpha, \beta)$ are determined from the theory of extremely shallow shells as presented in Chapter VI, which is based on the hypothesis of nondeformable normals.

The sought functions of u, v and w must satisfy: differential equations (9.19)-(9.21) (which, as was to be expected, do not differ in their left members from the left members of the corresponding equations of system (6.6.30)-(6.6.32)) and the boundary conditions presented in Chapter II (Section 10).

Let us present the equations in the combined method. Assuming ((7.43))

$$T_1=\frac{\partial^2 F}{\partial \beta^2},\quad T_2=\frac{\partial^2 F}{\partial \alpha^2},\quad S=-\frac{\partial^2 F}{\partial \alpha\,\partial \beta}, \tag{9.22}$$

we identically satisfy the first two equilibrium equations (9.17) and from the third equilibrium equation (9.18) and the equation of compatibility of deformation (7.42), by virtue of (9.14), (9.16) and (9.22) and $A_1 = A_2 = 0$, we obtain the following system of differential equations in the combined method:

$$\left.\begin{aligned}L_2(C_{ik})F-\nabla_k w=0;\\ L_1(D_{ik})w+\nabla_k F=Z^*,\end{aligned}\right\} \tag{9.23}$$

where

$$\begin{aligned}Z^*=Z+\frac{h^2}{10}\left\{a_{55}\left[D_{11}\frac{\partial^3\varphi_0}{\partial \alpha^3}+(D_{12}+2D_{66})\frac{\partial^3\varphi_0}{\partial \alpha\,\partial \beta^2}\right]+\right.\\ \left.+a_{44}\left[D_{22}\frac{\partial^3\psi_0}{\partial \beta^3}+(D_{12}+2D_{66})\frac{\partial^3\psi_0}{\partial \beta\,\partial \alpha^2}\right]\right\}.\end{aligned} \tag{9.24}$$

The system of differential equations in the combined method (9.23) differs from the corresponding system in (6.6.43) only in the load term (9.24), which in the variant of the theory proposed herein contains not only the surface load Z but also the known differential operations on the known functions $\varphi_0(\alpha, \beta)$ and $\psi_0(\alpha, \beta)$. In this case we have for the design quantities: (9.7)-(9.9), (9.13)-(9.16) and (9.22), while the tangential displacements may be determined from (8.20) and (6.21).

As usual, system (9.23) may be reduced to a single eighth-order equation of solution in terms of a certain sought function $\Phi = \Phi(\alpha, \beta)$. Assuming ((6.6.20))

$$w = L_2(C_{ik})\Phi, \qquad F = \nabla_r\Phi, \tag{9.25}$$

we identically satisfy the first equation of the system in (9.23) and from the second equation we obtain the differential equation of solution of the problem

$$\begin{aligned} P_1\frac{\partial^8\Phi}{\partial\alpha^8} + P_3\frac{\partial^8\Phi}{\partial\alpha^6\,\partial\beta^2} + P_5\frac{\partial^8\Phi}{\partial\alpha^4\,\partial\beta^4} + \\ + P_4\frac{\partial^8\Phi}{\partial\alpha^2\,\partial\beta^6} + P_2\frac{\partial^8\Phi}{\partial\beta^8} + \nabla_r^2\Phi = Z^*, \end{aligned} \tag{9.26}$$

where, as usual, for coefficients P_i we have

$$P_1 = \frac{C_{11}}{\Omega}D_{11}; \qquad P_2 = \frac{C_{22}}{\Omega}D_{22}; \tag{9.27}$$

$$P_3 = D_{11}\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) + 2\frac{C_{11}}{\Omega}(D_{12} + 2D_{66}); \tag{9.28}$$

$$P_4 = D_{22}\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) + 2\frac{C_{22}}{\Omega}(D_{12} + 2D_{66}); \tag{9.29}$$

$$P_5 = \frac{C_{22}}{\Omega}D_{11} + 2(D_{12} + 2D_{66})\left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right) + \frac{C_{11}}{\Omega}D_{22}. \tag{9.30}$$

Equation of solution (9.26), as system (9.23), differs from the corresponding equation of solution (6.6.48) only in the load term (9.24). For the design quantities we have

$$T_1 = \frac{\partial^2}{\partial\beta^2}\nabla_r\Phi; \qquad T_2 = \frac{\partial^2}{\partial\alpha^2}\nabla_r\Phi; \qquad S = -\frac{\partial^2}{\partial\alpha\,\partial\beta}\nabla_r\Phi; \tag{9.31}$$

$$\begin{aligned} M_1 = -\left(D_{11}\frac{\partial^2}{\partial\alpha^2} + D_{12}\frac{\partial^2}{\partial\beta^2}\right)L_2(C_{ik})\Phi + \\ + \frac{h^2}{10}\left(a_{55}D_{11}\frac{\partial\varphi_0}{\partial\alpha} + a_{44}D_{12}\frac{\partial\psi_0}{\partial\beta}\right); \end{aligned} \tag{9.32}$$

$$\begin{aligned} M_2 = -\left(D_{22}\frac{\partial^2}{\partial\beta^2} + D_{12}\frac{\partial^2}{\partial\alpha^2}\right)L_2(D_{ik})\Phi + \\ + \frac{h^2}{10}\left(a_{44}D_{22}\frac{\partial\psi_0}{\partial\beta} + a_{55}D_{12}\frac{\partial\varphi_0}{\partial\alpha}\right); \end{aligned} \tag{9.33}$$

$$H = -2D_{66}\frac{\partial^2}{\partial\alpha\,\partial\beta}L_2(D_{ik})\Phi + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial\varphi_0}{\partial\beta} + a_{44}\frac{\partial\psi_0}{\partial\alpha}\right); \tag{9.34}$$

$$\begin{aligned} u = -\left\{\left(\frac{C_{11}}{\Omega}k_1 + \frac{C_{12}}{\Omega}k_2\right)\frac{\partial^3}{\partial\alpha^3} + \right. \\ \left. + \left[\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right)k_1 - \frac{C_{22}}{\Omega}k_2\right]\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right\}\Phi; \end{aligned} \tag{9.35}$$

$$v = -\left\{\left(\frac{C_{22}}{\Omega}k_2 + \frac{C_{12}}{\Omega}k_1\right)\frac{\partial^3}{\partial\beta^3} + \right.$$

$$\left. + \left[\left(\frac{1}{C_{66}} - \frac{C_{12}}{\Omega}\right)k_2 - \frac{C_{11}}{\Omega}k_1\right]\frac{\partial^3}{\partial\beta\,\partial\alpha^2}\right\}\Phi; \tag{9.36}$$

$$w = \left[\frac{C_{11}}{\Omega}\frac{\partial^4}{\partial\alpha^4} + \frac{C_{22}}{\Omega}\frac{\partial^4}{\partial\beta^4} + \left(\frac{1}{C_{66}} - 2\frac{C_{12}}{\Omega}\right)\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}\right]\Phi. \tag{9.37}$$

Having the values of u, v and w, the stresses may be determined by means of formulas (9.7)-(9.9).

Proceeding from the results obtained in the fifth and sixth sections of this chapter, the theory proposed herein may also be generalized for the case of consideration of the normal stress σ_γ. For this purpose, in the design

formulas and in the equation of solution (in the load term) it is necessary to consider terms with coefficients A_1 and A_2.

SECTION 10. EXAMPLE OF CALCULATION OF A SHELL

Let an extremely shallow, transversely isotropic shell which is rectangular in the plane be freely supported over its entire contour and sustain a normally applied load which is distributed over the surface of the shell in accordance with the following equation (Fig. 80)

$$Z = q\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \tag{10.1}$$

where q is the intensity of the load at the center of the shell $(\alpha = a/2,\ \beta = b/2)$

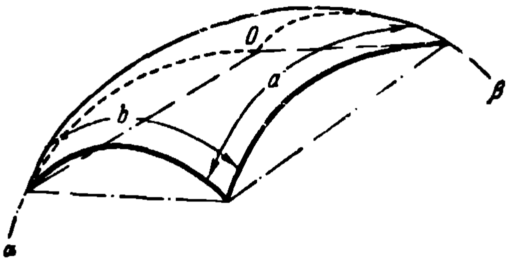

Figure 80

It is assumed that at each point of the shell the plane of isotropy is parallel to the median surface of the shell. Then, for the elastic constants of the shell materials we obviously have (6.1).

Assuming

$$\Phi = A \sin \frac{\pi\alpha}{a} \sin \frac{\pi\beta}{b}, \qquad (10.2)$$

we satisfy the hinge-support conditions and from equation (8.26), by virtue of (6.1), (10.1) and (10.2), we obtain by the usual method the sought coefficient A and thereby the sought function $\Phi(\alpha, \beta)$. Inserting the obtained value of Φ into (8.24), we finally obtain for the normal displacement $w(\alpha, \beta)$ of the shell center

$$w = w_0 [1 + h^*], \qquad (10.3)$$

where

$$h^* = \frac{\frac{h^4}{120} \frac{E}{G'} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^5}{\frac{h^2}{12}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^4 + (1 - \nu^2)\left(k_2 \frac{\pi^2}{a^2} + k_1 \frac{\pi^2}{b^2}\right)^2 \left[1 + \frac{h^2}{10} \frac{E}{G'}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)\right]}, \qquad (10.4)$$

w_0 is the normal displacement of the shell center as determined by means of the theory described in Chapter 6, that is, the theory which is based on the hypothesis of nondeformable normals.

In the special case where a = b, for the normal displacement of the shell center we obtain

$$w = w_0 \left[1 + \frac{\pi^2 \overline{E} h^2}{5G' a^2} \frac{4\pi^4 h^2}{4\pi^4 h^2 + 3a^4 (1 - \nu^2)(k_1 + k_2)^2 \left(1 + \frac{\pi^2 \overline{E} h^2}{5G' a^2}\right)}\right]. \qquad (10.5)$$

The same problem may be solved by means of the theory discussed in Section 9 of this chapter. For the normal displacement of the shell w_0 (that is, in the case when the hypothesis of nondeformable normals is valid) we have

$$w_0 = \frac{12q}{\overline{E} h^3} \frac{\pi^4 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 \sin \frac{\pi\alpha}{a} \sin \frac{\pi\beta}{b}}{\pi^8 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^4 + \frac{12\pi^4}{h^2} (1 - \nu^2) \left(\frac{k_2}{a^2} + \frac{k_1}{b^2}\right)^2}. \qquad (10.6)$$

Inserting the value of ω_0 from (10.6) into (9.2) and (9.3) and then inserting the resulting values of φ_0 and ψ_0 into (9.24) from (9.26), we obtain the following equation of solution of the problem:

$$\frac{h^2}{12(1-\nu^2)}\left(\frac{\partial^8\Phi}{\partial\alpha^8}+4\frac{\partial^8\Phi}{\partial\alpha^6\,\partial\beta^2}+6\frac{\partial^8\Phi}{\partial\alpha^4\,\partial\beta^4}+4\frac{\partial^8\Phi}{\partial\alpha^2\,\partial\beta^6}+\frac{\partial^8\Phi}{\partial\beta^8}\right)+\nabla_r^2\Phi=$$

$$=-Z+q\frac{h^2}{10}\frac{\bar{E}}{G'}\frac{\pi^{10}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^5\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}}{\pi^8\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^4+\frac{12\pi^4}{h^2}(1-\nu^2)\left(\frac{k_2}{a^2}+\frac{k_1}{b^2}\right)^2}. \quad (10.7)$$

As before, assuming

$$\Phi=A\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \quad (10.8)$$

we satisfy the conditions of hinge support and from equation of solution (10.7), in virtue of (6.1), (10.1) and (10.8), we obtain by the usual method the sought coefficient A and thereby the sought function $\Phi(\alpha,\beta)$. Inserting the resulting value of Φ into (9.26), we obtain for the normal displacement of the shell center

$$w=w_0\left[1+\frac{\frac{h^4}{120}\frac{\bar{E}}{G'}\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)^5}{\frac{h^2}{12}\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)^4+(1-\nu^2)\left(k_2\frac{\pi^2}{a^2}+k_1\frac{\pi^2}{b^2}\right)^2}\right], \quad (10.9)$$

where, as usual, w_0 is the normal displacement of the shell center as obtained by the theory based on the hypothesis of nondeformable normals. In the special case where a = b we obtain for w

$$w=w_0\left[1+\frac{\pi^2\bar{E}h^2}{5G'a^2}\,\frac{4\pi^4h^2}{4\pi^4h^2+3a^4(1-\nu^2)(k_1+k_2)^2}\right]. \quad (10.10)$$

Examining formulas (10.3)-(10.5) and (10.9), (10.10), we easily see that with an increase in rise of the shell (that is, with an increase in the ratios $\frac{a}{R_1}$, $\frac{b}{R_2}$) the admissible error in the hypothesis of nondeformable normals decreases. In the case of a plate $(k_1=k_2=0)$ this error reaches its maximum value. The fact is that with an increase in the rise of the shell the influence of bending parameters on the stress of the shell decreases, which also signifies a decrease in the influence of intersecting forces N_1 and N_2, that is, the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$, which also cause the transverse shear.

Comparing (10.3) and (10.5) with (10.9) and (10.10), respectively, we note* that the approximate theory of shells discussed in Section 9 of this chapter may with sufficient accuracy be used for calculation of essentially anisotropic shells. We will note that this occurs also in the case of a plate (see the end of Section 6). Here, in discussing the example of an extremely shallow shell we have limited ourselves to determination of only the normal stress w; the remaining design quantities may be determined in elementary form by means of formulas (8.7)-(8.15), (8.20)-(8.25).

---

*We do not present the results of numerical comparisons here.

As in the case of a plate (Section 6), the example of an extremely shallow shell may also be solved by means of the theory discussed in Section 7 of this chapter, that is, by means of the theory which takes into account effects associated with the normal stress σ_γ.

In examining the results obtained for anisotropic shells (Sections 7-10) we note that the theory of anisotropic shells as based on the hypothesis of nondeformable normals in the case of substantial (with regard to ratios of type B_{ik}/B_{55}, etc.) anisotropy of the shell material also requires correction.

Without going into detail we will state that all the remarks concerning anisotropic plate derived at the end of Section 6 of this chapter also remain in force for an anisotropic shell.

However, in the case of a shell the problem is somewhat complicated, since here the curvature of the coordinate surface of the shell $(\gamma = 0)$ begins to play a significant role. As was stated above, this circumstance plays a positive role from the point of view of broadening the limits of applicability of the hypothesis of nondeformable normals and is explained in the following general rule: the smaller the influence of bending phenomena on the stress of the shell, the smaller the correction in the classical theory of shells, that is, the theory based on the hypothesis of nondeformable normals. In the limiting case where we have a momentless stress the above mentioned correction is equal to zero.

Here, as in the case of a plate, for specific determination of the limits of applicability of the hypothesis of nondeformable normals it is necessary to perform test calculations; these may be performed by means of the theory described in Section 9 of this chapter. However, as a rough approximation it may be considered that the hypothesis of nondeformable normals is applicable with high accuracy if in the case of an isotropic or a slightly anisotropic shell $\varepsilon = 0.1$ (see Chapter II, Section 1), and in the case of a substantially anisotropic shell $\varepsilon = 0.05$, that is, we may adopt the limits of applicability of the hypothesis of nondeformable normals as established for a plate (see the end of Section 6).

We will note that in the case of shells there are no limitations on anisotropy in the surface $\gamma = 0$.

SECTION 11. THEORY OF EXTREMELY SHALLOW LAMINAR ORTHOTROPIC SHELLS

Among multi-layer shells, from the standpoint of the phenomena of transverse shear, great interest attaches to three-layer and two-layer shells. In this section we shall discuss only the theory of three-layer orthotropic shells with consideration of transverse shear.* The theory of two-layer orthotropic shells,**,*** is obtained as a special case in the theory of three-layer shells. Let us discuss a thin three-layer shell consisting of homogeneous orthotropic layers. Assume that one of the planes of elastic symmetry at each point of each layer is parallel to the external parallel surfaces of the shell and that the remaining planes of elastic symmetry are perpendicular to the coordinate lines.

Let us take the median surface of the center layer as the coordinate surface. Assume, as usual, that α and β are curvilinear orthogonal coordinates coinciding with the lines of principal curvature of the coordinate surface, γ is the distance along the normal from point (α, β) of the coordinate surface to point (α, β, γ) of the shell. As in Section 7 of this chapter, we assume that the coordinate system is so chosen that in the case of the discussed extremely shallow shell the coefficients of the first quadratic form $A(\alpha, \beta)$ and $B(\alpha, \beta)$, as well as the principal curvatures of the coordinate surface $k_1(\alpha, \beta)$ and $k_2(\alpha, \beta)$ upon differentiation behave as constants. In particular,

we may assume (7.2). We also assume that after deformation the layers remain elastic and function simultaneously without slipping.

Proceeding from the results in the previous sections, the theory proposed herein is based on the following assumptions:

a) at each layer of the shell the distances along the normal (along γ) between two points of the layer before and after deformation remain unchanged;

b) the normal stresses σ_γ^i at areas parallel to the median surface

of the shell may be disregarded in comparison with other stresses;****

---

*A.A. Khachatryan, Toward calculation of a three-layer orthotropic shell, Izvestiya AN ArmSSR (fiz.-mat. nauki), Vol. XII, No. 5, 1959.

**S.A. Ambartsumyan, On two methods of calculation of two-layer orthotropic shells, Izvestiya AN ArmSSR (fiz.-mat. nauki), Vol. X, No. 2, 1957.

***S. A. Ambartsumyan, Toward calculation of two-layer orthotropic shells, Izvestiya OTN AN SSSR, No. 7, 1957.

****In the ensuing discussion the superscripts will denote the layers, wherein 0, I, II (E^0, E^I, E^{II}) refer respectively to the middle, lower and upper layers of the shell.

c) in determining deformations $e^i_{\alpha\gamma}$ and $e^i_{\beta\gamma}$ we assume that the shearing stresses $\tau^i_{\alpha\gamma}$ and $\tau^i_{\beta\gamma}$ do not differ from the corresponding stresses as found on the assumption of validity of the hypothesis of nondeformable normals as given for the entire stack of the shell as a whole.

Proceeding in the usual manner (Chapter II, Section 4), we obtain for shearing stresses $\tau^i_{\alpha\gamma}$ and $\tau^i_{\beta\gamma}$, on the assumption of validity of the hypothesis of nondeformable normals*

$$\left.\begin{aligned}\tau^0_{\alpha\gamma} &= -(\gamma+\delta_0)R^0_1+\frac{1}{2}(\gamma^2-\delta_0^2)R^0_3-R_5;\\ \tau^0_{\beta\gamma} &= -(\gamma+\delta_0)R^0_2+\frac{1}{2}(\gamma^2-\delta_0^2)R^0_4-R_6;\end{aligned}\right\} \tag{11.1}$$

$$\left.\begin{aligned}\tau^I_{\alpha\gamma} &= -(\delta_1+\gamma)R^I_1-\frac{1}{2}(\delta_1^2-\gamma^2)R^I_3;\\ \tau^I_{\beta\gamma} &= -(\delta_1+\gamma)R^I_2-\frac{1}{2}(\delta_1^2-\gamma^2)R^I_4;\end{aligned}\right\} \tag{11.2}$$

$$\left.\begin{aligned}\tau^{II}_{\alpha\gamma} &= (\delta_2-\gamma)R^{II}_1-\frac{1}{2}(\delta_2^2-\gamma^2)R^{II}_3;\\ \tau^{II}_{\beta\gamma} &= (\delta_2-\gamma)R^{II}_2-\frac{1}{2}(\delta_2^2-\gamma^2)R^{II}_4.\end{aligned}\right\} \tag{11.3}$$

where

$$R^i_1 = B^i_{11}\frac{\partial^2 u_0}{\partial\alpha^2}+B^i_{66}\frac{\partial^2 u_0}{\partial\beta^2}+(B^i_{12}+B^i_{66})\frac{\partial^2 v_0}{\partial\alpha\,\partial\beta}+(k_1B^i_{11}+k_2B^i_{12})\frac{\partial w_0}{\partial\alpha}\ ^{**} \tag{11.4}$$

$$R^i_2 = B^i_{22}\frac{\partial^2 v_0}{\partial\beta^2}+B^i_{66}\frac{\partial^2 v_0}{\partial\alpha^2}+(B^i_{12}+B^i_{66})\frac{\partial^2 u_0}{\partial\alpha\,\partial\beta}+(k_2B^i_{22}+k_1B^i_{12})\frac{\partial w_0}{\partial\beta}; \tag{11.5}$$

$$R^i_3 = B^i_{11}\frac{\partial^3 w_0}{\partial\alpha^3}+(B^i_{12}+2B^i_{66})\frac{\partial^3 w_0}{\partial\alpha\,\partial\beta^2}; \tag{11.6}$$

$$R^i_4 = B^i_{22}\frac{\partial^3 w_0}{\partial\beta^3}+(B^i_{12}+2B^i_{66})\frac{\partial^3 w_0}{\partial\beta\,\partial\alpha^2}; \tag{11.7}$$

$$R^i_5 = (\delta_1-\delta_0)R^I_1+\frac{1}{2}(\delta_1^2-\delta_0^2)R^I_3; \tag{11.8}$$

$$R^i_6 = (\delta_1-\delta_0)R^I_2+\frac{1}{2}(\delta_1^2-\delta_0^2)R^I_4; \tag{11.9}$$

---

*Here, proceeding from the basic premises of the cited works of the author (see p. 376), the theory of extremely shallow, three-layer shells is given after A. A. Khachatryan (see p. 376) in generalizing the author's results for three-layer shells and plates.

**Other notation was previously used for certain of the operators R^i_j.

$2\delta_0$, $(\delta_1-\delta_0)$, $(\delta_2-\delta_0)$ are the thicknesses of the middle, lower and upper layers of the shell, respectively (Figure 81); B_{jk}^i are the known coefficients of elasticity of the individual layers (2.12.2); $u_0(\alpha, \beta)$, $v_0(\alpha, \beta)$, and $w_0(\alpha, \beta)$ are the components of displacement of points of the coordinate surface on the assumption of the hypothesis of nondeformable normals as given for the entire stack of the shell as a whole. Thus, the problem of determining stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ reduces to determination of displacements u_0, v_0, and w_0.

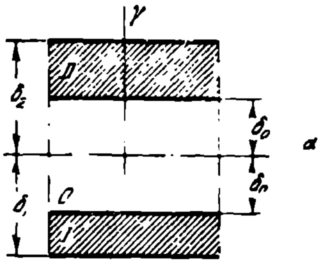

Figure 81

Examining formulas (11.1)-(11.3) and considering the equilibrium equations (6.5.40) and (6.5.42) (where X = 0, Y = 0), we note that the shearing stresses $\tau_{\alpha\gamma}^i$ and $\tau_{\beta\gamma}^i$ satisfy the following conditions (Chapter II, Section 5):

when $\gamma=-\delta_1$ $\qquad \tau_{\alpha\gamma}^{I}=\tau_{\beta\gamma}^{I}=0;$ (11.10)

when $\gamma=\delta_2$ $\qquad \tau_{\alpha\gamma}^{II}=\tau_{\beta\gamma}^{II}=0;$ (11.11)

when $\gamma=-\delta_0$ $\qquad \tau_{\alpha\gamma}^{0}=\tau_{\alpha\gamma}^{I}, \ \tau_{\beta\gamma}^{0}=\tau_{\beta\gamma}^{I};$ (11.12)

when $\gamma=\delta_0$ $\qquad \tau_{\alpha\gamma}^{0}=\tau_{\alpha\gamma}^{II}, \ \tau_{\beta\gamma}^{0}=\tau_{\beta\gamma}^{II}.$ (11.13)

On the other hand, in virtue of assumption (b), we obtain from the generalized Hooke's Law for the stresses in the layers

$$\left.\begin{array}{ll} \sigma_\alpha^i=B_{11}^i e_\alpha^i+B_{12}^i e_\beta^i; & \\ & \tau_{\alpha\beta}^i=B_{66}^i e_{\alpha\beta}^i; \\ \sigma_\beta^i=B_{12}^i e_\alpha^i+B_{12}^i e_\beta^i; & \\ \tau_{\alpha\gamma}^i=B_{55}^i e_{\alpha\gamma}^i; & \tau_{\beta\gamma}^i=B_{44}^i e_{\beta\gamma}^i. \end{array}\right\} \qquad (11.14)$$

The deformation components, as usual, are represented by the formulas

$$\left.\begin{aligned} e_\alpha &= \frac{\partial u_\alpha^l}{\partial\alpha} + k_1 u_\gamma^l; & e_\beta^l &= \frac{\partial u_\beta^l}{\partial\beta} + k_2 u_\gamma^l; \\ e_\gamma^l &= \frac{\partial u_\gamma^i}{\partial\gamma}; & e_{\alpha\beta}^l &= \frac{\partial u_\alpha^l}{\partial\beta} + \frac{\partial u_\beta^l}{\partial\alpha}; \\ e_{\alpha\gamma}^i &= \frac{\partial u_\alpha^l}{\partial\gamma} + \frac{\partial u_\gamma^i}{\partial\alpha}, & e_{\beta\gamma}^l &= \frac{\partial u_\beta^l}{\partial\gamma} + \frac{\partial u_\gamma^l}{\partial\beta}. \end{aligned}\right\} \tag{11.15}$$

In virtue of these assumptions and on the basis of the presented relationships and formulas (11.1)-(11.9), (11.14) and (11.15), we obtain for the displacement components of any point of any layer (for a detailed discussion of the procedure of calculation see the preceding sections of this chapter)

$$u_\gamma^0 = u_\gamma^{\rm I} = u_\gamma^{\rm II} = w(\alpha, \beta); \tag{11.16}$$

$$u_\alpha^0 = u - \gamma\frac{\partial w}{\partial\alpha} - \frac{1}{B_{55}^0}\left[\left(\frac{\gamma^2}{2} + \delta_0\gamma\right)R_1^0 - \frac{1}{2}\left(\frac{\gamma^3}{3} - \delta_0^2\gamma\right)R_3^0 + \gamma R_5\right]; \tag{11.17}$$

$$u_\beta^0 = v - \gamma\frac{\partial w}{\partial\beta} - \frac{1}{B_{44}^0}\left[\left(\frac{\gamma^2}{2} + \delta_0\gamma\right)R_2^0 - \frac{1}{2}\left(\frac{\gamma^3}{3} - \delta_0^2\gamma\right)R_4^0 + \gamma R_6\right]; \tag{11.18}$$

$$\begin{aligned} u_\alpha^{\rm I} = u - \gamma\frac{\partial w}{\partial\alpha} + \frac{1}{B_{55}^0}R_7 - \frac{1}{B_{55}^{\rm I}}\Big\{\Big[\delta_1(\gamma + \delta_0) + \frac{1}{2}(\gamma^2 - \delta_0^2)\Big]R_1^{\rm I} + \\ + \Big[\frac{\delta_1^2}{2}(\gamma + \delta_0) - \frac{1}{6}(\gamma^3 + \delta_0^3)\Big]R_3^{\rm I}\Big\}; \end{aligned} \tag{11.19}$$

$$\begin{aligned} u_\beta^{\rm I} = v - \gamma\frac{\partial w}{\partial\beta} + \frac{1}{B_{44}^0}R_8 - \frac{1}{B_{44}^{\rm I}}\Big\{\Big[\delta_1(\gamma + \delta_0) + \frac{1}{2}(\gamma^2 - \delta_0^2)\Big]R_2^{\rm I} + \\ + \Big[\frac{\delta_1^2}{2}(\gamma + \delta_0) - \frac{1}{6}(\gamma^3 + \delta_0^3)\Big]R_4^{\rm I}\Big\}; \end{aligned} \tag{11.20}$$

$$\begin{aligned} u_\alpha^{\rm II} = u - \gamma\frac{\partial w}{\partial\alpha} - \frac{1}{B_{55}^0}(\delta_0^2 R_1^0 + R_7) + \frac{1}{B_{55}^{\rm II}}\Big\{\Big[\delta_2(\gamma - \delta_0) - \\ - \frac{1}{2}(\gamma^2 - \delta_0^2)\Big]R_1^{\rm II} - \Big[\frac{\delta_2^2}{2}(\gamma - \delta_0) - \frac{1}{6}(\gamma^3 - \delta_0^3)\Big]R_1^{\rm II}\Big\}; \end{aligned} \tag{11.21}$$

$$\begin{aligned} u_\beta^{\rm II} = v - \gamma\frac{\partial w}{\partial\beta} - \frac{1}{B_{44}^0}(\delta_0^2 R_2^0 + R_8) + \frac{1}{B_{44}^{\rm II}}\Big\{\Big[\delta_2(\gamma - \delta_0) - \\ - \frac{1}{2}(\gamma^2 - \delta_0^2)\Big]R_2^{\rm II} - \Big[\frac{\delta_2^2}{2}(\gamma - \delta_0) - \frac{1}{6}(\gamma^3 - \delta_0^3)\Big]R_1^{\rm II}\Big\}, \end{aligned} \tag{11.22}$$

where $u(\alpha, \beta)$, $v(\alpha, \beta)$, and $w(\alpha, \beta)$ are the displacement components of the points of the coordinate surface

$$\left.\begin{aligned} R_7 &= \frac{\delta_0^2}{2}R_1^0 + \frac{\delta_0^3}{3}R_3^0 + \delta_0 R_5; \\ R_8 &= \frac{\delta_0^2}{2}R_2^0 + \frac{\delta_0^3}{3}R_4^0 + \delta_0 R_6. \end{aligned}\right\} \tag{11.23}$$

In deriving formulas (11.6)-(11.22) we applied the geometric conditions for contact of adjacent layers of the shell, which for the problem under discussion are written as follows (Chapter II, Section 5);

when $\gamma = -\delta_0$

when $\gamma = \delta_0$ $$u^0_\alpha = u^I_\alpha, \quad u^0_\beta = u^I_\beta, \quad u^0_\gamma = u^I_\gamma; \tag{11.24}$$

$$u^0_\alpha = u^{II}_\alpha, \quad u^0_\beta = u^{II}_\beta, \quad u^0_\gamma = u^I_\gamma. \tag{11.25}$$

Inserting the values of displacements from (11.16)-(11.22) into (11.15) and inserting the resulting values of deformation into (11.4), we obtain for the design stresses

$$\begin{aligned}\sigma^0_\alpha = {} & B^0_{11}\frac{\partial u}{\partial\alpha} + B^0_{12}\frac{\partial v}{\partial\beta} + \left(k_1B^0_{11} + k_2B^0_{12}\right)w - \\ & -\gamma\left(B^0_{11}\frac{\partial^2}{\partial\alpha^2} + B^0_{12}\frac{\partial^2}{\partial\beta^2}\right)w - \frac{B^0_{11}}{B^0_{55}}\frac{\partial}{\partial\alpha}\Delta^0_1 - \frac{B^0_{12}}{B^0_{44}}\frac{\partial}{\partial\beta}\Delta^0_2.\end{aligned} \tag{11.26}$$

$$\begin{aligned}\sigma^0_\beta = {} & B^0_{22}\frac{\partial v}{\partial\beta} + B^0_{12}\frac{\partial u}{\partial\alpha} + \left(k_2B^0_{22} + k_1B^0_{12}\right)w - \\ & -\gamma\left(B^0_{22}\frac{\partial^2}{\partial\beta^2} + B^0_{12}\frac{\partial^2}{\partial\alpha^2}\right)w - \frac{B^0_{22}}{B^0_{44}}\frac{\partial}{\partial\beta}\Delta^0_2 - \frac{B^0_{12}}{B^0_{55}}\frac{\partial}{\partial\alpha}\Delta^0_1;\end{aligned} \tag{11.27}$$

$$\tau^0_{\alpha\beta} = B^0_{66}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} - 2\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right) - \frac{B^0_{66}}{B^0_{55}}\frac{\partial}{\partial\beta}\Delta^0_1 - \frac{B^0_{66}}{B^0_{44}}\frac{\partial}{\partial\alpha}\Delta^0_2; \tag{11.28}$$

$$\begin{aligned}\sigma^I_\alpha = {} & B^I_{11}\frac{\partial u}{\partial\alpha} + B^I_{12}\frac{\partial v}{\partial\beta} + \left(k_1B^I_{11} + k_2B^I_{12}\right)w - \gamma\left(B^I_{11}\frac{\partial^2}{\partial\alpha^2} + B^I_{12}\frac{\partial^2}{\partial\beta^2}\right)w + \\ & + \frac{B^I_{11}}{B^0_{55}}\frac{\partial R_7}{\partial\alpha} + \frac{B^I_{12}}{B^0_{44}}\frac{\partial R_8}{\partial\beta} - \frac{B^I_{11}}{B^I_{55}}\frac{\partial}{\partial\alpha}\Delta^I_1 - \frac{B^I_{12}}{B^I_{44}}\frac{\partial}{\partial\beta}\Delta^I_2;\end{aligned} \tag{11.29}$$

$$\begin{aligned}\sigma^I_\beta = {} & B^I_{22}\frac{\partial v}{\partial\beta} + B^I_{12}\frac{\partial u}{\partial\alpha} + \left(k_2B^I_{22} + k_1B^I_{12}\right)w - \gamma\left(B^I_{22}\frac{\partial^2}{\partial\beta^2} + B^I_{12}\frac{\partial^2}{\partial\alpha^2}\right)w + \\ & + \frac{B^I_{22}}{B^0_{44}}\frac{\partial R_8}{\partial\beta} + \frac{B^I_{12}}{B^0_{55}}\frac{\partial R_7}{\partial\alpha} - \frac{B^I_{22}}{B^I_{44}}\frac{\partial}{\partial\beta}\Delta^I_2 - \frac{B^I_{12}}{B^I_{55}}\frac{\partial}{\partial\alpha}\Delta^I_1;\end{aligned} \tag{11.30}$$

$$\begin{aligned}\tau^I_{\alpha\beta} = {} & B^I_{66}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} - 2\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right) + \frac{B^I_{66}}{B^0_{55}}\frac{\partial R_7}{\partial\beta} + \\ & + \frac{B^I_{66}}{B^0_{44}}\frac{\partial R_8}{\partial\alpha} - \frac{B^I_{66}}{B^I_{55}}\frac{\partial}{\partial\beta}\Delta^I_1 - \frac{B^I_{66}}{B^I_{44}}\frac{\partial}{\partial\alpha}\Delta^I_2.\end{aligned} \tag{11.31}$$

$$\begin{aligned}\sigma^{II}_\alpha = {} & B^{II}_{11}\frac{\partial u}{\partial\alpha} + B^{II}_{12}\frac{\partial v}{\partial\beta} + \left(k_1B^{II}_{11} + k_2B^{II}_{12}\right)w - \gamma\left(B^{II}_{11}\frac{\partial^2}{\partial\alpha^2} + B^{II}_{12}\frac{\partial^2}{\partial\beta^2}\right)w \\ & - \frac{B^{II}_{11}}{B^0_{55}}\frac{\partial}{\partial\alpha}\left(\delta_0^2R^0_1 + R_7\right) - \frac{B^{II}_{12}}{B^0_{44}}\frac{\partial}{\partial\beta}\left(\delta_0^2R^0_2 + R_8\right) + \\ & + \frac{B^{II}_{11}}{B^{II}_{55}}\frac{\partial}{\partial\alpha}\Delta^{II}_1 + \frac{B^{II}_{12}}{B^{II}_{44}}\frac{\partial}{\partial\beta}\Delta^{II}_2;\end{aligned} \tag{11.32}$$

$$\sigma_\beta^{II} = B_{22}^{II}\frac{\partial v}{\partial\beta} + B_{12}^{II}\frac{\partial u}{\partial\alpha} + \left(k_2B_{22}^{II} + k_1B_{12}^{II}\right)w - \gamma\left(B_{22}^{II}\frac{\partial^2}{\partial\beta^2} + B_{12}^{II}\frac{\partial^2}{\partial\alpha^2}\right)w -$$

$$- \frac{B_{22}^{II}}{B_{44}^{0}}\frac{\partial}{\partial\beta}\left(\delta_0^2R_2^0 + R_8\right) - \frac{B_{12}^{II}}{B_{55}^{0}}\frac{\partial}{\partial\alpha}\left(\delta_0^2R_1 + R_7\right) +$$

$$+ \frac{B_{22}^{II}}{B_{44}^{II}}\frac{\partial}{\partial\beta}\Delta_2^{II} + \frac{B_{12}^{II}}{B_{55}^{II}}\frac{\partial}{\partial\alpha}\Delta_1^{II}; \tag{11.33}$$

$$\tau_{\alpha\beta}^{II} = B_{66}^{II}\left(\frac{\partial u}{\partial\beta} + \frac{\partial v}{\partial\alpha} - 2\gamma\frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right) - \frac{B_{66}^{II}}{B_{55}^{0}}\frac{\partial}{\partial\beta}\left(\delta_0^2R_1^0 + R_7\right) -$$

$$- \frac{B_{66}^{II}}{B_{44}^{0}}\frac{\partial}{\partial\alpha}\left(\delta_0^2R_2^0 + R_8\right) + \frac{B_{66}^{II}}{B_{55}^{II}}\frac{\partial}{\partial\beta}\Delta_1^{II} + \frac{B_{66}^{II}}{B_{11}^{II}}\frac{\partial}{\partial\alpha}\Delta_2^{II}. \tag{11.34}$$

where we have the following additional notation:

$$\Delta_1^0 = \left(\frac{\gamma^2}{2} + \delta_0\gamma\right)R_1^0 - \frac{1}{2}\left(\frac{\gamma^3}{3} - \delta_0^2\gamma\right)R_3^0 + \gamma R_5^I; \tag{11.35}$$

$$\Delta_2^0 = \left(\frac{\gamma^2}{2} + \delta_0\gamma\right)R_2^0 - \frac{1}{2}\left(\frac{\gamma^3}{3} - \delta_0^2\gamma\right)R_4^0 + \gamma R_6^I; \tag{11.36}$$

$$\Delta_1^I = \left[\delta_1(\gamma + \delta_0) + \frac{1}{2}(\gamma^2 - \delta_0^2)\right]R_1^I + \left[\frac{\delta_1^2}{2}(\gamma + \delta_0) - \frac{1}{6}(\gamma^3 + \delta_0^3)\right]R_3^I; \tag{11.37}$$

$$\Delta_2^I = \left[\delta_1(\gamma + \delta_0) + \frac{1}{2}(\gamma^2 - \delta_0^2)\right]R_2^I + \left[\frac{\delta_1^2}{2}(\gamma + \delta_0) - \frac{1}{6}(\gamma^3 + \delta_0^3)\right]R_4^I; \tag{11.38}$$

$$\Delta_1^{II} = \left[\delta_2(\gamma - \delta_0) - \frac{1}{2}(\gamma^2 - \delta_0^2)\right]R_1^{II} - \left[\frac{\delta_2^2}{2}(\gamma - \delta_0) - \frac{1}{6}(\gamma^3 - \delta_0^3)\right]R_3^{II}; \tag{11.39}$$

$$\Delta_2^{II} = \left[\delta_2(\gamma - \delta_0) - \frac{1}{2}(\gamma^2 - \delta_0^2)\right]R_2^{II} - \left[\frac{\delta_2^2}{2}(\gamma - \delta_0) - \frac{1}{6}(\gamma^3 - \delta_0^3)\right]R_4^{II}. \tag{11.40}$$

Considering the presented values of stresses in (11.26)-(11.34), we obtain for the internal forces and moments

$$T_1 = C_{11}\frac{\partial u}{\partial\alpha} + C_{12}\frac{\partial v}{\partial\beta} + (k_1C_{11} + k_2C_{12})w -$$

$$- \left(K_{11}\frac{\partial^2}{\partial\alpha^2} + K_{12}\frac{\partial^2}{\partial\beta^2}\right)w + T_1^*; \tag{11.41}$$

$$T_2 = C_{22}\frac{\partial v}{\partial\beta} + C_{12}\frac{\partial u}{\partial\alpha} + (k_2C_{22} + k_1C_{12})w -$$

$$- \left(K_{22}\frac{\partial^2}{\partial\beta^2} + K_{12}\frac{\partial^2}{\partial\alpha^2}\right)w + T_2^*; \tag{11.42}$$

$$S = C_{66}\frac{\partial u}{\partial\beta} + C_{66}\frac{\partial v}{\partial\alpha} - 2K_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + S; \tag{11.43}$$

$$M_1 = -\left(D_{11}\frac{\partial^2}{\partial\alpha^2} + D_{12}\frac{\partial^2}{\partial\beta^2}\right)w + K_{11}\frac{\partial u}{\partial\alpha} +$$

$$+ K_{12}\frac{\partial v}{\partial\beta} + (k_1K_{11} + k_2K_{12})w + M_1^*; \tag{11.44}$$

$$M_2 = -\left(D_{22}\frac{\partial^2}{\partial\beta^2} + D_{12}\frac{\partial^2}{\partial\alpha^2}\right)w + K_{22}\frac{\partial v}{\partial\beta} + \\ + K_{12}\frac{\partial u}{\partial\alpha} + (k_2K_{22} + k_1K_{12})\,w + M_2^*; \tag{11.45}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + K_{66}\frac{\partial u}{\partial\beta} + K_{66}\frac{\partial v}{\partial\alpha} + H^*, \tag{11.46}$$

where for the stiffnesses we have the following values:

$$C_{jk} = 2\delta_0 B^0_{jk} + (\delta_1 - \delta_0)\,B^I_{jk} + (\delta_2 - \delta_0)\,B^{II}_{jk}; \tag{11.47}$$

$$K_{jk} = \frac{1}{2}\left[(\delta_2^2 - \delta_0^2)\,B^{II}_{jk} - (\delta_1^2 - \delta_0^2)\,B^I_{jk}\right]; \tag{11.48}$$

$$D_{jk} = \frac{1}{3}\left[2\delta_0^3 B^0_{jk} + (\delta_1^3 - \delta_0^3)\,B^I_{jk} + (\delta_2^3 - \delta_0^3)\,B^{II}_{jk}\right], \tag{11.49}$$

while for the internal forces due to transverse shear we have:

$$T_1 = \frac{\partial}{\partial\alpha}\left[-\frac{\delta_0^3}{3}\frac{B^0_{11}}{B^0_{55}}R_1^0 + (\delta_1 - \delta_0)\frac{B^I_{11}}{B^0_{55}}R_7 - (\delta_2 - \delta_0)\times\right. \\ \times\frac{B^{II}_{11}}{B^0_{55}}(\delta_0^2 R_1 + R_7) + \frac{1}{3}(\delta_1 - \delta_0)^3\frac{B^I_{11}}{B^I_{55}}\left(R_1^I + \frac{5\delta_1 + 3\delta_0}{8}R_3^I\right) + \\ \left. + \frac{1}{3}(\delta_2 - \delta_0)^3\frac{B^{II}_{11}}{B^{II}_{55}}\left(R_1^{II} - \frac{5\delta_2 + 3\delta_0}{8}R_3^{II}\right)\right] + \\ + \frac{\partial}{\partial\beta}\left[-\frac{\delta_0^3}{3}\frac{B^0_{12}}{B^0_{44}}R_2^0 + (\delta_1 - \delta_0)\frac{B^I_{12}}{B^0_{44}}R_8 - (\delta_2 - \delta_0)\frac{B^{II}_{12}}{B^0_{44}}(\delta_0^2 R_2^0 + R_8) + \right. \\ + \frac{1}{3}(\delta_1 - \delta_0)^3\frac{B^I_{12}}{B^I_{44}}\left(R_2^I + \frac{5\delta_1 + 3\delta_0}{8}R_4^I\right) + \\ \left. + \frac{1}{3}(\delta_2 - \delta_0)^3\frac{B^{II}_{12}}{B^{II}_{44}}\left(R_2^{II} - \frac{5\delta_2 + 3\delta_0}{8}R_4^{II}\right)\right]; \tag{11.50}$$

$$T_2 = \frac{\partial}{\partial\beta}\left[-\frac{\delta_0^3}{3}\frac{B^0_{22}}{B^0_{44}}R_2^0 + (\delta_1 - \delta_0)\frac{B^I_{22}}{B^0_{44}}R_8 - (\delta_2 - \delta_0)\times\right. \\ \times\frac{B^{II}_{22}}{B^0_{44}}(\delta_0^2 R_2^0 + R_8) + \frac{1}{3}(\delta_1 - \delta_0)^3\frac{B^I_{22}}{B^I_{44}}\left(R_2^I + \frac{5\delta_1 + 3\delta_0}{8}R_1^I\right) + \\ \left. + \frac{1}{3}(\delta_2 - \delta_0)^3\frac{B^{II}_{22}}{B^{II}_{44}}\left(R_2^{II} - \frac{5\delta_2 + 3\delta_0}{8}R_4^{II}\right)\right] + \frac{\partial}{\partial\alpha}\times \\ \times\left[-\frac{\delta_0^3}{3}\frac{B^0_{12}}{B^0_{55}}R_1^0 + (\delta_1 - \delta_0)\frac{B^I_{12}}{B^0_{55}}R_7 - (\delta_2 - \delta_0)\frac{B^{II}_{12}}{B^0_{55}}(\delta_0^2 R_1^0 + R_7) + \right. \\ + \frac{1}{3}(\delta_1 - \delta_0)^3\frac{B^I_{12}}{B^I_{55}}\left(R_1^I + \frac{5\delta_1 + 3\delta_0}{8}R_3^I\right) + \\ \left. + \frac{1}{3}(\delta_2 - \delta_0)^3\frac{B^{II}_{12}}{B^{II}_{55}}\left(K_1^{II} - \frac{5\delta_2 + 3\delta_0}{8}R_3^{II}\right)\right]; \tag{11.51}$$

$$
\begin{aligned}
S^{*} = \frac{\partial}{\partial\beta}\Bigg[& -\frac{\delta_0^3}{3}\frac{B^0_{66}}{B^0_{55}}R_1^0 + (\delta_1-\delta_0)\frac{B^{I}_{66}}{B^0_{55}}R_7 - (\delta_2-\delta_0)\times \\
& \times\frac{B^{II}_{66}}{B^0_{55}}(\delta_0^2R_1^0+R_7) + \frac{1}{3}(\delta_1-\delta_0)^3\frac{B^{I}_{66}}{B^{I}_{55}}\left(R_1^{I}+\frac{5\delta_1+3\delta_0}{8}R_3^{I}\right) + \\
& + \frac{1}{3}(\delta_2-\delta_0)^3\frac{B^{II}_{66}}{B^{II}_{55}}\left(R_1^{II}-\frac{5\delta_2+3\delta_0}{8}R_3^{II}\right)\Bigg] + \\
+ \frac{\partial}{\partial\alpha}\Bigg[& -\frac{\delta_0^3}{3}\frac{B^0_{66}}{B^0_{44}}R_2^0 + (\delta_1-\delta_0)\frac{B^{I}_{66}}{B^0_{44}}R_8 - (\delta_2-\delta_0)\frac{B^{II}_{66}}{B^0_{44}}(\delta_0^2R_2^0+R_8) + \\
& + \frac{1}{3}(\delta_1-\delta_0)^3\frac{B^{I}_{66}}{B^{I}_{44}}\left(R_2^{I}+\frac{5\delta_1+3\delta_0}{8}R_4^{I}\right) + \\
& + \frac{1}{3}(\delta_2-\delta_0)^3\frac{B^{II}_{66}}{B^{II}_{44}}\left(R_2^{II}-\frac{5\delta_2+3\delta_0}{8}R_4^{II}\right)\Bigg];
\end{aligned}
\tag{11.52}
$$

$$
\begin{aligned}
M_1^{*} = \frac{\partial}{\partial\alpha}\Bigg[& -\frac{2\delta_0^3}{3}\frac{B^0_{11}}{B^0_{55}}\left(\delta_0R_1^0+\frac{2\delta_0^2}{5}R_3^0+R_5\right) - \\
& - \frac{\delta_1^2-\delta_0^2}{2}\frac{B^{I}_{11}}{B^0_{55}}R_7 - \frac{(\delta_1-\delta_0)^3}{12}\frac{B^{I}_{11}}{B^{I}_{55}}\left(\frac{5\delta_1+3\delta_0}{2}R_1^{I} + \right. \\
& \left. + \frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R_3^{I}\right) - \frac{\delta_2^2-\delta_0^2}{2}\frac{B^{II}_{11}}{B^0_{55}}(\delta_0^2R_1^0+R_7) + \\
& + \frac{(\delta_2-\delta_0)^3}{12}\frac{B^{II}_{11}}{B^{II}_{55}}\left(\frac{5\delta_2+3\delta_0}{2}R_1^{II}-\frac{8\delta_2^2+9\delta_2\delta_0+3\delta_0^2}{5}R_3^{II}\right)\Bigg] + \\
+ \frac{\partial}{\partial\beta}\Bigg[& -\frac{2\delta_0^3}{3}\frac{B^0_{12}}{B^0_{44}}\left(\delta_0R_2^0+\frac{2\delta_0^2}{5}R_4^0+R_6\right) - \\
& - \frac{\delta_1^2-\delta_0^2}{2}\frac{B^{I}_{12}}{B^0_{44}}R_8 - \frac{(\delta_1-\delta_0)^3}{12}\frac{B^{I}_{12}}{B^{I}_{44}}\left(\frac{5\delta_1+3\delta_0}{2}R_2^{I} + \right. \\
& \left. + \frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R_4^{I}\right) - \frac{\delta_2^2-\delta_0^2}{2}\frac{B^{II}_{12}}{B^0_{44}}(\delta_0^2R_2^0+R_8) + \\
& + \frac{(\delta_2-\delta_0)^3}{12}\frac{B^{II}_{12}}{B^{II}_{44}}\left(\frac{5\delta_2+3\delta_0}{2}R_2^{II}-\frac{8\delta_2^2+9\delta_2\delta_0+3\delta_0^2}{5}R_4^{II}\right)\Bigg];
\end{aligned}
\tag{11.53}
$$

$$
\begin{aligned}
M_2^{*} = \frac{\partial}{\partial\beta}\Bigg[& -\frac{2\delta_0^3}{3}\frac{B^0_{22}}{B^0_{44}}\left(\delta_0R_2^0+\frac{2\delta_0^2}{5}R_4^0+R_6\right) - \\
& - \frac{\delta_1^2-\delta_0^2}{2}\frac{B^{I}_{22}}{B^0_{44}}R_8 - \frac{(\delta_1-\delta_0)^3}{12}\frac{B^{I}_{22}}{B^{I}_{44}}\left(\frac{5\delta_1+3\delta_0}{2}R_2^{I} + \right. \\
& \left. + \frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R_4^{I}\right) - \frac{\delta_2^2-\delta_0^2}{2}\frac{B^{II}_{22}}{B^0_{44}}(\delta_0^2R_2^0+R_8) +
\end{aligned}
$$

$$+\frac{(\delta_2-\delta_0)^3}{12}\frac{B^{11}_{22}}{B^{11}_{44}}\left(\frac{5\delta_2+3\delta_0}{2}R^{11}_2-\frac{8\delta_2^2+9\delta_2\delta_0+3\delta_0^2}{5}R^{11}_4\right)\Bigg]+$$

$$+\frac{\partial}{\partial\alpha}\left[-\frac{2\delta_0^3}{3}\frac{B^0_{12}}{B^0_{55}}\left(\delta_0R^0_1+\frac{2\delta_0^2}{5}R^0_3+R_5\right)-\right.$$

$$-\frac{\delta_1^2-\delta_0^2}{2}\frac{B^1_{12}}{B^0_{66}}R_7-\frac{(\delta_1-\delta_0)^3}{12}\frac{B^1_{12}}{B^1_{55}}\left(\frac{5\delta_1+3\delta_0}{2}R^1_1+\right.$$

$$\left.+\frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R^1_3\right)-\frac{\delta_2^2-\delta_0^2}{2}\frac{B^{11}_{12}}{B^0_{66}}(\delta_0^2R^0_1+R_7)+$$

$$\left.+\frac{(\delta_2-\delta_0)^3}{12}\frac{B^{11}_{12}}{B^{11}_{66}}\left(\frac{5\delta_2+3\delta_0}{2}R^{11}_1+\frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R^{11}_3\right)\right]; \qquad (11.54)$$

$$H^*=\frac{\partial}{\partial\beta}\left[-\frac{2\delta_0^3}{3}\frac{B^0_{66}}{B^0_{55}}\left(\delta_0R^0_1+\frac{2\delta_0^2}{5}R^0_3+K_5\right)-\right.$$

$$-\frac{\delta_1^2-\delta_0^2}{2}\frac{B^1_{66}}{B^0_{55}}R_7-\frac{(\delta_1-\delta_0)^3}{12}\frac{B^1_{66}}{B^1_{55}}\left(\frac{5\delta_1+3\delta_0}{2}R^1_1+\right.$$

$$\left.+\frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R^1_3\right)-\frac{\delta_2^2-\delta_0^2}{2}\frac{B^{11}_{66}}{B^0_{55}}(\delta_0^2R^0_1+R_7)+$$

$$\left.+\frac{(\delta_2-\delta_0)^3}{12}\frac{B^{11}_{66}}{B^{11}_{55}}\left(\frac{5\delta_2+3\delta_0}{2}R^{11}_1-\frac{8\delta_2^2+9\delta_2\delta_0+3\delta_0^2}{5}R^{11}_3\right)\right]+$$

$$+\frac{\partial}{\partial\alpha}\left[-\frac{2\delta_0^3}{3}\frac{B^0_{66}}{B^0_{44}}\left(\delta_0R^0_2+\frac{2\delta_0^2}{5}R^0_4+R_6\right)-\right.$$

$$-\frac{\delta_1^2-\delta_0^2}{2}\frac{B^1_{66}}{B^0_{44}}R_8-\frac{(\delta_1-\delta_0)^3}{12}\frac{B^1_{66}}{B^1_{44}}\left(\frac{5\delta_1+3\delta_0}{2}R^1_2+\right.$$

$$\left.+\frac{8\delta_1^2+9\delta_1\delta_0+3\delta_0^2}{5}R^1_4\right)-\frac{\delta_2^2-\delta_0^2}{2}\frac{B^{11}_{66}}{B^0_{44}}(\delta_0^2R^0_2+R_8)+$$

$$\left.+\frac{(\delta_2-\delta_0)^3}{12}\frac{B^{11}_{66}}{B^{11}_{44}}\left(\frac{5\delta_2+3\delta_0}{2}R^{11}_2-\frac{8\delta_2^2+9\delta_2\delta_0+3\delta_0^2}{5}R^{11}_4\right)\right]. \qquad (11.55)$$

Inserting the values for internal forces and moments into the equilibrium equations (9.17) and (9.18), which in this case also remain unchanged, we obtain the following system of differential equations of solution in terms of the three sought displacements $u=u(\alpha,\ \beta)$, $v=v(\alpha,\ \beta)$, and $w=w(\alpha,\ \beta)$:

$$\left(C_{11}\frac{\partial^2}{\partial\alpha^2}+C_{66}\frac{\partial^2}{\partial\beta^2}\right)u+(C_{12}+C_{66})\frac{\partial^2v}{\partial\alpha\,\partial\beta}+$$

$$+(k_1C_{11}+k_2C_{12})\frac{\partial w}{\partial\alpha}-\left[K_{11}\frac{\partial^3}{\partial\alpha^3}+(K_{12}+2K_{66})\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right]w=$$

$$=-\frac{\partial T^*_1}{\partial\alpha}-\frac{\partial S^*}{\partial\beta}; \qquad (11.56)$$

$$\left(C_{22}\frac{\partial^2}{\partial\beta^2}+C_{66}\frac{\partial^2}{\partial\alpha^2}\right)v+(C_{12}+C_{66})\frac{\partial^2 u}{\partial\alpha\,\partial\beta}+$$
$$+(k_2C_{22}+k_1C_{12})\frac{\partial w}{\partial\beta}-\left[K_{22}\frac{\partial^3}{\partial\beta^3}+(K_{12}+2K_{66})\frac{\partial^3}{\partial\beta\,\partial\alpha^2}\right]w=$$
$$=-\frac{\partial T_2^*}{\partial\beta}-\frac{\partial S^*}{\partial\alpha}; \qquad (11.57)$$

$$(k_1C_{11}+k_2C_{12})\frac{\partial u}{\partial\alpha}+(k_2C_{22}+k_1C_{12})\frac{\partial v}{\partial\beta}+$$
$$+\left(k_1^2C_{11}+2k_1k_2C_{12}+k_2^2C_{22}\right)w+D_{11}\frac{\partial^4 w}{\partial\alpha^4}+$$
$$+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\,\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}-\left[K_{11}\frac{\partial^3}{\partial\alpha^3}+\right.$$
$$\left.+(K_{12}+2K_{66})\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right]u+\left[(K_{12}+2K_{66})\frac{\partial^3}{\partial\beta\,\partial\alpha^2}+\right.$$
$$\left.+K_{22}\frac{\partial^3}{\partial\beta^3}\right]v-2\left[(k_1K_{11}+k_2K_{12})\frac{\partial^2}{\partial\alpha^2}+\right.$$
$$\left.+(k_2K_{22}+k_1K_{12})\frac{\partial^2}{\partial\beta^2}\right]w=Z-\left(k_1T^*+k_2T_2^*\right)+$$
$$+\left(\frac{\partial^2M_1^*}{\partial\alpha^2}+2\frac{\partial^2H^*}{\partial\alpha\,\partial\beta}+\frac{\partial^2M_2^*}{\partial\beta^2}\right). \qquad (11.58)$$

Examining the differential equations of solution in (11.56)-(11.58), we note that, as was to be expected, in their left members they coincide with the corresponding left members of the equations of solution in (6.5.40)-(6.5.42) as derived on the basis of the hypothesis of nondeformable normals as given for the entire stack of the shell as a whole. The additional terms due to consideration of the phenomena of transverse shear enter into the right members of the derived equations. These terms, being known (we will remember that proceeding from the theory based on the hypothesis of nondeformable normals as given for the entire stack as a whole, the additional terms are described by means of formulas (11.50)-(11.55), (11.4)-(11.9) and (11.23)), are the load terms. The boundary conditions then are the same as the boundary conditions listed in Section 10, Chapter 6.

In the special case of a two-layer shell* where the coordinate surface coincides with the contact surface of the layers (Figure 82), in order to obtain the equations of solution of the design formulas it is necessary only to assume that $\delta_0 = 0$. In view of the elementary nature of the transformations we shall not present them here.

Proceeding from the results obtained in Sections 2, 5 and 7 of this chapter, the proposed theory may be somewhat simplified, that is, we may

---

*The details of the theory of two-layer anisotropic shells may be found in the author's works (see footnotes **,***, on p. 376).

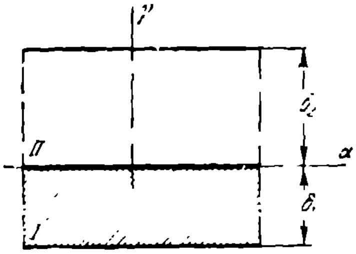

Figure 82

omit assumption (b) (the assumption concerning the negligibility of σ^i_γ in comparison with other stresses.

The theory for calculation of three-layer shells without consideration of the hypothesis of nondeformable normals has been discussed in the works of many authors, for example, E. Reissner*, E.I. Grigolyuk**, et al.; however, we shall not evaluate these works here.

SECTION 12. EXAMPLES OF CALCULATION OF LAMINAR SHELLS AND PLATES

For illustration of the procedure for calculation of laminar shells and plates let us discuss two examples.

1. A rectangular ($a \times b$) orthotropic plate whose layers are symmetrically arranged relative to the coordinate surface and is hinge-supported over its entire contour sustains a normally applied load with distribution over its surface in accordance with the equation (Figure 83)

$$Z = q \sin \frac{\pi\alpha}{a} \sin \frac{\pi\beta}{b}. \tag{12.1}$$

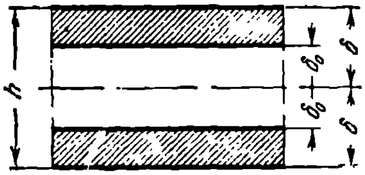

Figure 83

---

*E. Reissner, Contributions to the problem of structural analysis of sandwich-type plates and shells, Theory and Practice of Sandwich Construction in Aircraft, A Symposium, Preprint, No. 165, 1948.

**E.I. Grigolyuk, Finite flexures of three-layer shells with stiff filling, Izvestiya OTN AN SSSR, No. 1, 1958.

For this plate we have

$$\delta_1 = \delta_2 = \delta, \quad B^{I}_{jk} = B^{II}_{jk} = B_{jk}; \tag{12.2}$$

then from (11.47)-(11.49) we obtain for the stiffnesses

$$C_{jk} = 2\left[\delta_0 B^{0}_{jk} + (\delta - \delta_0) B_{jk}\right]; \tag{12.3}$$

$$K_{jk} = 0; \tag{12.4}$$

$$D_{jk} = \frac{2}{3}\left[\delta_0^3 B^{0}_{jk} + \left(\delta^3 - \delta_0^3\right) B_{jk}\right]. \tag{12.5}$$

Considering that for the discussed symmetrically assembled three-layer plate*,**.

$$u_0 = v_0 = 0, \qquad u = v = 0, \tag{12.6}$$

we obtain from (11.50-(11.52)

$$T_1^* = T_2^* = S^* = 0. \tag{12.7}$$

Then the first two equations of the system in (11.56) and (11.57) are identically satisfied and the third equation (11.58) is rewritten as follows:

$$D_{11}\frac{\partial^4 w}{\partial \alpha^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + D_{22}\frac{\partial^4 w}{\partial \beta^4} = = Z + \left(\frac{\partial^2 M_1^*}{\partial \alpha^2} + 2\frac{\partial^2 H^*}{\partial \alpha \partial \beta} + \frac{\partial^2 M_2^*}{\partial \beta^2}\right), \tag{12.8}$$

where for M_j^* and H* we have (11.53)-(11.55). Finally, these formulas become somewhat simplified but are nevertheless somewhat unwieldy.

Solution of the discussed problem in the case where we admit the hypothesis of nondeformable normals is represented as follows:***

$$w_0 = \frac{q \sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}}{\pi^4\left(\frac{D_{11}}{a^4} + 2\frac{D_{12} + D_{66}}{a^2 b^2} + \frac{D_{22}}{b^4}\right)}. \tag{12.9}$$

Inserting the value of w_0 from (12.9) into (11.53)-(11.55) with consideration of (12.6), we obtain the final values of the components of the load term, that is,

---

*S.G. Lekhnitskiy, see p. 241-271 of work cited in footnote on p. 328.

**A.A. Khachatryan, see footnote *, on p. 378.

***S.G. Lekhnitskiy, see pp. 282-283 of work cited in footnote on p. 328.

M_1^*, M_2^*, and H*. Then, inserting the values of M_1^*, M_2^*, and H* into (12.8), we obtain the final equation of solution of the problem (in view of the unwieldiness of the right member we shall not present this equation here).

Let us seek the solution of equation (12.8) in the form

$$w = A \sin \frac{\pi\alpha}{a} \sin \frac{\pi\beta}{b}, \qquad (12.10)$$

which satisfies the conditions for free support over the contour of the plate. Inserting the values of Z from (12.1) and w from (12.10) into equation (12.8) (the values of M_1^*, M_2^* and H* have already been inserted), we determine the value of the required coefficient A and then, upon inserting this into (12.10), we finally obtain from the normal displacement of the plate

$$\begin{aligned} w = w_0 \Big\{ 1 + \frac{\pi^2}{D^*} \Big[\frac{4\delta_0^5}{15} \left(\frac{t_{01}^2}{a^2 B_{55}^0} + \frac{t_{02}^2}{b^2 B_{44}^0} \right) + \\ + \frac{2\delta_0^3 (\delta^2 - \delta_0^2)}{3} \left(\frac{t_{01} t_{11}}{a^2 B_{55}^0} + \frac{t_{02} t_{22}}{b^2 B_{44}^0} \right) + \frac{(\delta - \delta_0)^3}{30} (8\delta^2 + 9\delta\delta_0 + 3\delta_0^2) \times \\ \times \left(\frac{t_{11}^2}{a^2 B_{66}} + \frac{t_{22}^2}{b^2 B_{44}} \right) + \frac{\delta_0 (\delta^2 - \delta_0^2)^2}{2} \left(\frac{t_{11}^2}{a^2 B_{66}^0} + \frac{t_{22}^2}{b^2 B_{44}^0} \right) \Big] \Big\}, \end{aligned} \qquad (12.11)$$

$$\left.\begin{aligned} t_{01} = \frac{B_{11}^0}{a^2} + \frac{B_{12}^0 + 2B_{66}^0}{b^2}, \quad t_{02} = \frac{B_{22}^0}{b^2} + \frac{B_{12}^0 + 2B_{66}^0}{a^2}; \\ t_{11} = \frac{B_{11}}{a^2} + \frac{B_{12} + 2B_{66}}{b^2}, \quad t_{22} = \frac{B_{22}}{b^2} + \frac{B_{12} + 2B_{66}}{a^2}; \end{aligned}\right\} \qquad (12.12)$$

$$D^* = \frac{D_{11}}{a^4} + \frac{2(D_{12} + 2D_{66})}{a^2 b^2} + \frac{D_{22}}{b^4}. \qquad (12.13)$$

Examining formula (12.11), we note that it has the usual structure, that is, the first term is the normal displacement of the discussed plate as determined from the classical theory of laminar plates (based upon the hypothesis of nondeformable normals as given for the entire stack as given for the entire stack as a whole), while the remaining terms are the correction for the classical theory, that is, the terms resulting from consideration of transverse shear.

For sake of illustration let us discuss a special case. Let a square plate (a = b) consist of transversely isotropic layers in which the planes of isotropy are parallel to the coordinate plane of plane ($\gamma = 0$). For the sake of simplicity we shall also assume that $\nu = \nu_0$. Assuming

$$\frac{\delta_0}{\delta} = r, \quad \frac{E}{E_0} = n, \quad \frac{E}{G'} = s, \quad \frac{E_0}{G_0'} = s_0, \qquad (12.14)$$

where, E, ν and E_0, ν_0 are the elastic moduli and the Poisson ratios in the plane of isotropy of the outer layers and the middle layer, respectively; G' and G'_0 are the shear moduli (of the outer and middle layers, respectively) characterizing distortion of the angles between directions in the planes of isotropy and the direction perpendicular thereto, we obtain for the normal displacement w from (12.11)

$$w = w_0\left\{1 + \frac{3\pi^2}{4}\frac{h^2}{a^2}\frac{1}{r^3 + n(1-r^3)}\left[\frac{4}{15}r^5 s_0 + \right.\right.$$
$$+ \frac{2}{3}r^3(1-r^2)s_0 n + \frac{1}{2}r(1-r^2)^2 s_0 n^2 +$$
$$\left.\left. + \frac{1}{30}(1-r)^3(8+9r+3r^2)sn\right]\right\}. \qquad (12.15)$$

In the more general case when

$$\delta = 2\delta_0 = \frac{h}{2}, \quad s = s_0, \qquad (12.16)$$

we obtain for the normal displacement

$$w = w_0\left[1 + \frac{\pi^2}{160}\frac{h^2}{a^2}s\frac{8 + 113n + 135n^2}{1 + 7n}\right]. \qquad (12.17)$$

The results of a few calculations of the values of ratio w/w_0 for different values of n and s when $h/a = 0{,}1$ (remembering that h is the total thickness of the plate) are listed in the following table:

n \ s	2.0	5,0	10,0
10,0	1,254	1,636	2,272
5,0	1,135	1,338	1,677
2,0	1,064	1,159	1,318
1,0	1,039	1,099	1,198
0,5	1,027	1,067	1,135
0,2	1,019	1,046	1,093
0,1	1,015	1,037	1,075

(the column s = 2.0 refers to the case in which all layers of the plate are made of isotropic materials and line n = 1.0 refers to the case of a homogeneous (single-layer) plate).

Examining formula (12.17) and its corresponding numerical values as listed in the table, we note that with an increase in $n = E/E_0$ there is an increase in the difference between w and w_0 and, the greater this difference, the greater $s = E/G' = E_0/G_0'$. With a decrease in n, even at large values of s, w differs but little from w_0. This indicates that if the material of the middle layer is stiffer than the material of the outer layers, the classical theory of laminar plates as based on the hypothesis of nondeformable normals yields sufficiently good results. In the case where the material of the middle layer is considerably weaker than the material of the outer layers and the plate is sufficiently thin it is necessary to take into account the corrections for the classical theory.

In conjunction with the discussed example, without additional calculations let us derive the final formula for determining the normal displacement of an extremely shallow shell:

$$w = w_0 \left[1 + \frac{\pi^2}{160} \frac{h^2}{a^2} s \frac{8 + 113n + 135n^2}{(1 + 7n) + \frac{3a^4}{\pi^4 h^2} (k_1 + k_2)^2 (1 + n)} \right], \tag{12.18}$$

where k_1 and k_2 are the principal curvatures of the coordinate surface of the shell.

Examining formula (12.18), we confirm that in the case of extremely shallow laminar shells or laminar plates the correction for the classical theory, due to consideration of transverse shear, decreases.

2. A three-layer, circular cylindrical shell with axially symmetrical loading is freely supported at its ends and sustains a distributed, normally applied load which varies over the surface of the shell according to the equation

$$Z = q \sin \frac{\pi \alpha}{l}, \tag{12.19}$$

where l is the length of the shell and q is the intensity of load at the central cross-section of the shell (that is, when $\alpha = l/2$) (Figure 84).

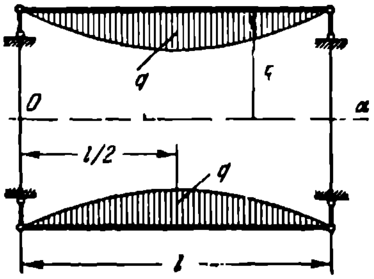

Figure 84

In view of the complete axial symmetry, it is evident that

$$v = v_0 = 0, \quad S = S^* = 0, \quad H = H^* = 0; \tag{12.20}$$

then the system of equations of solution (11.56), (11.58) is rewritten as follows:

$$C_{11}\frac{d^2u}{d\alpha^2} + \left(\frac{1}{R}C_{12}\frac{d}{d\alpha} - K_{11}\frac{d^3}{d\alpha^3}\right)w = -\frac{dT_1^*}{d\alpha}; \tag{12.21}$$

$$\left(\frac{1}{R}C_{12}\frac{d}{d\alpha} - K_{11}\frac{d^3}{d\alpha^3}\right)u + \left(\frac{1}{R^2}C_{22} - \frac{2}{R}K_{12}\frac{d^2}{d\alpha^2} + D_{11}\frac{d^4}{d\alpha^4}\right)w =$$
$$= Z - \frac{1}{R}T_2^* + \frac{d^2M_1^*}{d\alpha^2}, \tag{12.22}$$

where, from the general formulas for T_1^*, T_2^*, and M_1^* we have

$$T_1^* = a_1\frac{d^3u_0}{d\alpha^3} + \left(b_1\frac{d^2}{d\alpha^2} + b_2\frac{d^4}{d\alpha^4}\right)w_0; \tag{12.23}$$

$$\frac{T_2^*}{R} = a_2\frac{d^3u_0}{d\alpha^3} + \left(b_3\frac{d^2}{d\alpha^2} + b_4\frac{d^4}{d\alpha^4}\right)w_0; \tag{12.24}$$

$$M_1^* = -a_3\frac{d^3u_0}{d\alpha^3} - \left(b_5\frac{d^2}{d\alpha^2} + b_6\frac{d^4}{d\alpha^4}\right)w_0. \tag{12.25}$$

In these formulas

$$a_1 = -\frac{\delta_0^2B_{11}^0}{6B_{55}^0}\left[2\delta_0B_{11}^0 - 3(\delta_1 - \delta_0)B_{11}^{I} + 9(\delta_2 - \delta_0)B_{11}^{II}\right] +$$
$$+ \delta_0(\delta_1 - \delta_0)\frac{B_{11}^{I}}{B_{55}^0}\left[(\delta_1 - \delta_0)B_{11}^{I} - (\delta_2 - \delta_0)B_{11}^{II}\right] +$$
$$+ (\delta_1 - \delta_0)^3\frac{(B_{11}^{I})^2}{3B_{55}^{I}} + (\delta_2 - \delta_0)^3\frac{(B_{11}^{II})^2}{3B_{55}^{II}}; \tag{12.26}$$

$$b_1 = -\frac{\delta_0^2B_{12}^0}{6RB_{55}^0}\left[2\delta_0B_{11}^0 - 3(\delta_1 - \delta_0)B_{11}^{I} + 9(\delta_2 - \delta_0)B_{11}^{II}\right] +$$
$$+ \delta_0(\delta_1 - \delta_0)\frac{B_{12}^{I}}{RB_{55}^0}\left[(\delta_1 - \delta_0)B_{11}^{I} - (\delta_2 - \delta_0)B_{11}^{II}\right] +$$
$$+ (\delta_1 - \delta_0)^3\frac{B_{11}^{I}B_{12}^{I}}{3RB_{55}^{I}} + (\delta_2 - \delta_0)^3\frac{B_{11}^{II}B_{12}^{II}}{3RB_{55}^{II}}; \tag{12.27}$$

$$b_2 = \frac{\delta_0^3B_{11}^0}{3B_{55}^0}\left[(\delta_1 - \delta_0)B_{11}^{I} - (\delta_2 - \delta_0)B_{11}^{II}\right] +$$
$$+ \delta_0(\delta_1^2 - \delta_0^2)\frac{B_{11}^{I}}{2B_{55}^0}\left[(\delta_1 - \delta_0)B_{11}^{I} - (\delta_2 - \delta_0)B_{11}^{II}\right] +$$
$$+ (\delta_1 - \delta_0)^3(5\delta_1 + 3\delta_0)\frac{(B_{11}^{I})^2}{24B_{55}^{I}} - (\delta_2 - \delta_0)^3(5\delta_2 + 3\delta_0)\frac{(B_{11}^{II})^2}{24B_{55}^{II}}; \tag{12.28}$$

$$\begin{aligned} a_2 = &- \frac{\delta_0^2 B_{11}^{0}}{6B_{55}^{0}R}\left[2\delta_0 B_{12}^{0} - 3(\delta_1 - \delta_0) B_{12}^{\mathrm{I}} + 9(\delta_2 - \delta_0) B_{12}^{\mathrm{II}}\right] + \\ &+ \delta_0(\delta_1 - \delta_0)\frac{B_{11}^{\mathrm{I}}}{B_{55}^{0}R}\left[(\delta_1 - \delta_0) B_{12}^{\mathrm{I}} - (\delta_2 - \delta_0) B_{12}^{\mathrm{II}}\right] + \\ &+ (\delta_1 - \delta_0)^3 \frac{B_{11}^{\mathrm{I}} B_{12}^{\mathrm{I}}}{3B_{55}^{\mathrm{I}}R} + (\delta_2 - \delta_0)^3 \frac{B_{11}^{\mathrm{II}} B_{12}^{\mathrm{II}}}{3B_{55}^{\mathrm{II}}R}; \end{aligned} \tag{12.29}$$

$$\begin{aligned} b_3 = &- \frac{\delta_0^2 B_{12}^{0}}{6R^2 B_{55}^{0}}\left[2\delta_0 B_{12}^{0} - 3(\delta_1 - \delta_0) B_{12}^{\mathrm{I}} + 9(\delta_2 - \delta_0) B_{12}^{\mathrm{II}}\right] + \\ &+ \delta_0(\delta_1 - \delta_0)\frac{B_{12}^{\mathrm{I}}}{R^2 B_{55}^{0}}\left[(\delta_1 - \delta_0) B_{12}^{\mathrm{I}} - (\delta_2 - \delta_0) B_{12}^{\mathrm{II}}\right] + \\ &+ (\delta_1 - \delta_0)^3 \frac{(B_{12}^{\mathrm{I}})^2}{3R^2 B_{55}^{\mathrm{I}}} + (\delta_2 - \delta_0)^3 \frac{(B_{12}^{\mathrm{II}})^2}{3R^2 B_{55}^{\mathrm{II}}}; \end{aligned} \tag{12.30}$$

$$\begin{aligned} b_4 = &\frac{\delta_0^3 B_{11}^{0}}{3RB_{55}}\left[(\delta_1 - \delta_0) B_{12}^{\mathrm{I}} - (\delta_2 - \delta_0) B_{12}^{\mathrm{II}}\right] + \\ &+ \delta_0(\delta_1^2 - \delta_0^2)\frac{B_{11}^{\mathrm{I}}}{2B_{55}^{0}R}\left[(\delta_1 - \delta_0) B_{12}^{\mathrm{I}} - (\delta_2 - \delta_0) B_{12}^{\mathrm{II}}\right] + \\ &+ (\delta_1 - \delta_0)^3(5\delta_1 + 3\delta_0)\frac{B_{11}^{\mathrm{I}} B_{12}^{\mathrm{I}}}{24B_{55}^{\mathrm{I}}R} - (\delta_2 - \delta_0)^3(5\delta_2 + 3\delta_0)\frac{B_{11}^{\mathrm{II}} B_{12}^{\mathrm{II}}}{24B_{55}^{\mathrm{II}}R}; \end{aligned} \tag{12.31}$$

$$\begin{aligned} a_3 = &\frac{\delta_0^2 B_{11}^{0}}{12B_{55}^{0}}\left[8\delta_0^2 B_{11}^{0} + 3(\delta_1^2 - \delta_0^2) B_{11}^{\mathrm{I}} + 9(\delta_2^2 - \delta_0^2) B_{11}^{\mathrm{II}}\right] + \\ &+ \delta_0(\delta_1 - \delta_0)\frac{B_{11}^{\mathrm{I}}}{6B_{55}^{0}}\left[4\delta_0^2 B_{11}^{0} + 3(\delta_1^2 - \delta_0^2) B_{11}^{\mathrm{I}} + 3(\delta_2^2 - \delta_0^2) B_{11}^{\mathrm{II}}\right] + \\ &+ (\delta_1 - \delta_0)^3(5\delta_1 + 3\delta_0)\frac{(B_{11}^{\mathrm{I}})^2}{24B_{55}^{\mathrm{I}}} - (\delta_2 - \delta_0)^3(5\delta_2 + 3\delta_0)\frac{(B_{11}^{\mathrm{II}})^2}{24B_{55}^{\mathrm{II}}}; \end{aligned} \tag{12.32}$$

$$\begin{aligned} b_5 = &\frac{\delta_0^2 B_{12}^{0}}{12RB_{55}^{0}}\left[8\delta_0^2 B_{11}^{0} + 3(\delta_1^2 - \delta_0^2) B_{11}^{\mathrm{I}} + 9(\delta_2^2 - \delta_0^2) B_{11}^{\mathrm{I}}\right] + \\ &+ \delta_0(\delta_1 - \delta_0)\frac{B_{12}^{\mathrm{I}}}{6RB_{55}^{0}}\left[4\delta_0^2 B_{11}^{0} + 3(\delta_1^2 - \delta_0^2) B_{11}^{\mathrm{I}} + 3(\delta_2^2 - \delta_0^2) B_{11}^{\mathrm{II}}\right] + \\ &+ (\delta_1 - \delta_0)^3(5\delta_1 + 3\delta_0)\frac{B_{11}^{\mathrm{I}} B_{12}^{\mathrm{I}}}{24RB_{55}^{\mathrm{I}}} - (\delta_2 - \delta_0)^3(5\delta_2 + 3\delta_0)\frac{B_{11}^{\mathrm{II}} B_{12}^{\mathrm{II}}}{24RB_{55}^{\mathrm{II}}}; \end{aligned} \tag{12.33}$$

$$\begin{aligned} b_6 = &\frac{\delta_0^4 B_{11}^{0}}{30B_{55}^{0}}\left[8\delta_0^2 B_{11}^{0} + 5(\delta_1^2 - \delta_0^2) B_{11}^{\mathrm{I}} + 5(\delta_2^2 - \delta_0^2) B_{11}^{\mathrm{II}}\right] + \\ &+ \delta_0(\delta_1^2 - \delta_0^2)\frac{B_{11}^{\mathrm{I}}}{12B_{55}^{0}}\left[4\delta_0^2 B_{11}^{0} + 3(\delta_1^2 - \delta_0^2) B_{11}^{\mathrm{I}} + 3(\delta_2^2 - \delta_0^2) B_{11}^{\mathrm{II}}\right] + \\ &+ (\delta_1 - \delta_0)^3(8\delta_1 + 9\delta_1\delta_0 + 3\delta_0^2)\frac{(B_{11}^{\mathrm{I}})^2}{60B_{55}^{\mathrm{I}}} + \\ &+ (\delta_2 - \delta_0)^3(8\delta_2^2 + 9\delta_2\delta_0 + 3\delta_0^2)\frac{(B_{11}^{\mathrm{II}})^2}{60B_{55}^{\mathrm{II}}}. \end{aligned} \tag{12.34}$$

With consideration of the hypothesis of nondeformable normals the solution of this problem is presented in the form

$$w_0 = \frac{d_{11}}{d_{11}d_{22} - d_{12}} q \sin \frac{\pi x}{l}, \tag{12.35}$$

$$u_0 = \frac{d_{12}}{d_{11}d_{22} - d_{12}} q \cos \frac{\pi x}{l}, \tag{12.36}$$

where

$$\left.\begin{aligned} d_{11} &= \frac{\pi^2}{l^2} C_{11}, \\ d_{12} &= \frac{\pi}{l}\left(\frac{C_{12}}{R} + \frac{\pi^2}{l^2} K_{11}\right), \\ d_{22} &= \frac{C_{22}}{R^2} + \frac{2\pi^2}{Rl^2} K_{12} + \frac{\pi^4}{l^4} D_{11}. \end{aligned}\right\} \tag{12.37}$$

As in the solution of system (12.21), let us seek (12.22) in the form

$$\begin{aligned} w &= A \sin \frac{\pi x}{l}, \\ u &= B \cos \frac{\pi x}{l}, \end{aligned} \tag{12.38}$$

which satisfies the conditions for free support at the ends.

Inserting the values of Z from (12 19), w and u from (12.38), T_1^*, T_2^* and M_1^* from (12.23), (12.24) and (12.25), respectively, and considering (12.35)-(12.37), we obtain for the sought displacements

$$u = u_0\left[1 + \frac{d_{12}(F_2 d_{12} - F_1 d_{22}) + d_{11}(G_1 d_{22} - G_2 d_{12})}{d_{12}\left(d_{11}d_{22} - d_{12}^2\right)}\right]; \tag{12.39}$$

$$w = w_0\left[1 + \frac{d_{12}(F_2 d_{11} - F_1 d_{12}) + d_{11}(G_1 d_{12} - G_2 d_{11})}{d_{11}\left(d_{11}d_{22} - d_{12}^2\right)}\right], \tag{12.40}$$

where

$$\left.\begin{aligned} F_1 &= \frac{\pi^4}{l^4} a_1, \\ G_2 &= \frac{\pi^2}{l^2}\left[b_3 - \frac{\pi^2}{l^2}(b_4 + b_5) + \frac{\pi^4}{l^4} b_6\right]; \\ F_2 &= \frac{\pi^3}{l^3}\left(a_2 - \frac{\pi^2}{l^2} a_3\right); \\ G_1 &= \frac{\pi^3}{l^3}\left(b_1 - \frac{\pi^2}{l^2} b_2\right). \end{aligned}\right\} \tag{12.41}$$

In the special case where the layers of the shell are symmetrically arranged relative to the coordinate surface (that is, when $\delta_1 = \delta_2 = \delta$, $B'_{ik} = B''_{ik} = B_{ik}$ (Figure 84)), we obtain from (12.39) and (12.40) for the sought displacements

$$u = u_0\left\{1 + \frac{\pi^2}{l^2}\left[\frac{1}{\Delta}\left(\frac{\pi^4}{l^4}b_7 + b_8\right) + b_9\right]\right\}; \tag{12.42}$$

$$w = w_0\left[1 + \frac{\pi^2}{l^2}\frac{1}{\Delta}\left(\frac{\pi^4}{l^4}b_7 + b_8\right)\right], \tag{12.43}$$

where

$$u_0 = \frac{q}{\Delta}\frac{l}{\pi R}\frac{C_{12}}{C_{11}}\cos\frac{\pi x}{l}; \tag{12.44}$$

$$w_0 = \frac{q}{\Delta}\sin\frac{\pi x}{l}; \tag{12.45}$$

$$\Delta = \frac{1}{R^2}C_{22} + \frac{\pi^4}{l^4}D_{11} - \frac{1}{R^2}\frac{C_{12}^2}{C_{11}}; \tag{12.46}$$

$$b_7 = \frac{\delta_0}{B_{55}^0}\left[\frac{4\delta_0^4}{15}(B_{11}^0)^2 + \frac{2}{3}\delta_0^2(\delta^2 - \delta_0^2)B_{11}^0 B_{11} + \frac{1}{2}(\delta^2 - \delta_0^2)^2(B_{11})^2\right] + $$
$$+ (\delta - \delta_0)^3(8\delta^2 + 9\delta\delta_0 + 3\delta_0^2)\frac{(B_{11})^2}{30B_{55}}; \tag{12.47}$$

$$b_8 = \frac{8\delta_0^2(\delta - \delta_0)^2}{3R^2C_{11}^2}(B_{11}^0 B_{12} - B_{11}B_{12}^0)^2\left(\frac{\delta - \delta_0}{B_{55}} + \frac{\delta_0}{B_{55}^0}\right); \tag{12.48}$$

$$b_9 = \frac{2\delta_0(\delta - \delta_0)}{C_{11}C_{12}}(B_{11}B_{12}^0 - B_{11}^0 B_{12})\left\{\frac{2}{3}(\delta - \delta_0)^2\frac{B_{11}}{B_{55}} + \right.$$
$$\left. + \frac{\delta_0}{B_{55}^0}\left[\frac{\delta_0}{3}B_{11}^0 + (\delta - \delta_0)B_{11}\right]\right\}. \tag{12.49}$$

Let us examine a numerical example. For the sake of simplicity in calculation we shall take the case in which the layers are symmetrically arranged relative to the coordinate surface of the shell and are made of transversally isotropic materials, the planes of isotropy of which at each point of each layer are parallel to the median surface of the shell. Assuming

$$\delta = 2\delta_0 = \frac{h}{2}, \quad \frac{E}{G'} = \frac{E_0}{G_0'} = s, \quad \frac{E}{E_0} = n, \quad \nu = \nu_0 = 0, \tag{12.50}$$

we obtain from (12.42) and (12.43)

$$\frac{u}{u_0} = \frac{w}{w_0} = 1 + \frac{\pi^2}{320}\frac{h^2}{l^2}\frac{s(8 + 113n + 135n^2)}{(1 + 7n) + \frac{48}{\pi^4}\frac{l^4}{R^2h^2}(1 + n)}. \tag{12.51}$$

The results of calculation of ratio w/w_0 (u/u_0) for different values of n and s when $h/l = 1/10$, and $l/R = 1$ are listed in the following table:

n \ s	2,0	5,0	10,0
0,5	1,0008	1,0019	1,0039
1,0	1,0015	1,0037	1,0074
2,0	1,0029	1,0073	1,0147
10,0	1,0147	1,0368	1,0736

Examining formula (12.51) and the table, we note that the results obtained for a laminar plate are also confirmed in the case of a laminar shell. In the special case of a shell, as was to be expected, even with large n and s (n = 10.0, s = 10.0) the correction for the classical theory with consideration of transverse shear is insignificant.

3. The theory of laminar shells and plates discussed herein was also used in the calculation of two-layer structure. The numerical results were compared with the corresponding results obtained by means of the precise theory of two-layer plates. This comparison revealed that the approximate theory of laminar shells and plates is wholly suitable even for calculation of somewhat thick two-layer plates.*

Let us give a few remarks concerning the limits of applicability of the hypothesis of nondeformable normals as given for the entire stack of the shell as a whole. It is evident to the reader that in this case the problem of determining the limits of applicability of the hypothesis of nondeformable normals for an arbitrary anisotropic laminar shell is still further complicated, since along with the previously mentioned (Sections 6 and 10) independent parameters (the sharp and relative thickness of the shell boundary conditions, type of loading, values of ratios of the type B_{ik}/B_{55}, etc.) there are other new and extremely important parameters representing the ratios of the elastic constants of individual layers.

Thus, in the case of an anisotropic laminar shell (or plate) for specific determination of the limits of applicability of the hypothesis of nondeformable normals it is also necessary to perform additional calculations. However, as calculations performed by us reveal, the hypothesis of nondeformable normals as given for the entire stack of the shell as a whole is applicable with sufficiently high accuracy:

---

*A. P. Melkonyan, see footnote ****, on p. 326.

a) for a shell with isotropic or insignificantly anisotropic layers (B_{1k}/B_{55}, etc. < 3): when $\epsilon = 0.05$, if $n \leqslant 10$; when $\epsilon = 0.02$, if $n \leqslant 100$; and when $\epsilon = 0.01$, if $n \leqslant 500$;

b) for a shell with substantially anisotropic layers (B_{1k}/B_{55} etc. ~ 10-15): when $\epsilon = 0.02$, if $n \leqslant 10$; when $\epsilon = 0.01$, if $n \leqslant 100$, and when $\epsilon = 0.005$, if $n \leqslant 500$.

As always, here again there are no limitations on anisotropicity in the surfaces $\gamma = \text{const}$.

In conclusion let it be pointed out that the hypothesis of nondeformable normals as given for the entire stack of an anisotropic laminar shell as a whole is applicable only in those cases when quantities of the order of $\lambda h^2/a^2$ negligibly small in comparison with unity. Here λ is a numerical coefficient which depends: on the shape and relative thickness of the shell, on the boundary conditions, on the type of loading, and on the physical and mechanical characteristics of the materials of the shell layers.*

---

*When this work was completed there appeared the reports by S. G. Lekhnitskiy, "Toward a theory of thick anisotropic plates", and Kh. M. Mushtari, "Theory of bending of plates of medium thickness" (Izvestiya OTN AN SSSR, mekhanika i mashinostroyeniye, No. 2, 1959), which in many respects confirmed the results obtained in this chapter.